Trace inequalities related to quantum entropies

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The von Neumann Entropy

For two density matrices ρ and σ (i.e., non-negative operators with trace one) their **von Neumann entropy** and **relative entropy** are defined as

$$S(\rho) = -\operatorname{Tr} \rho \ln \rho \qquad \text{and} \qquad D(\rho||\sigma) = \operatorname{Tr} \rho \ln \rho - \operatorname{Tr} \rho \ln \sigma \,.$$

Question: How do these quantities behave under the application of **quantum channels** (i.e., completely positive, trace preserving linear maps)?

A fundamental property: SSA (**Lieb, Ruskai 1973**). For any density matrix ρ_{123} on a tri-partite system $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$, (with notation $\rho_{12} = \operatorname{Tr}_3 \rho_{123}$, etc.)

$$\boxed{S(\rho_{12}) + S(\rho_{23}) \geq S(\rho_{123}) + S(\rho_2)} \quad \equiv \quad \boxed{S(\rho_{12}) + S(\rho_{23}) \geq S(\rho_1) + S(\rho_3)} \,.$$

Further properties: (**Lindblad 1974**) **Monotonicity under CPTP maps**

$$\boxed{D(\mathcal{E}(\rho)||\mathcal{E}(\sigma)) \leq D(\rho||\sigma)} \qquad \textbf{(MONO)}$$

Convexity of the relative entropy

$$\boxed{D((1-\theta)\rho_0 + \theta\rho_1||(1-\theta)\sigma_0 + \theta\sigma_1) \leq (1-\theta)D(\rho_0||\sigma_0) + \theta D(\rho_1||\sigma_1)} \qquad \textbf{(CONV)}$$

Properties of the von Neumann Entropy

Remarkably, (SSA), (MONO) and (CONV) are **equivalent** (in the sense that one of these implies other ones by simple and rather abstract arguments).

(CONV) $\Longrightarrow$ (MONO): (Due to **Lindblad**; we will use this argument later!) According to **Stinespring**, if $\mathcal{E}$ is CPTP,

$$\mathcal{E}(\tau) = \mathrm{Tr}_{\mathcal{K}}\, U(\tau \otimes |\psi\rangle\langle\psi|)U^* \,, \qquad U \text{ unitary}\,, \;\; \psi \in \mathcal{K} \text{ normalized}\,,$$

so with $du =$ Haar measure on unitaries on $\mathcal{K}$,

$$(\dim \mathcal{K})^{-1}\mathcal{E}(\tau) \otimes 1 = \int du\, (1 \otimes u)U(\tau \otimes |\psi\rangle\langle\psi|)U^*(1 \otimes u^*)\,.$$

Now (CONV) implies

$$\begin{aligned} D(\mathcal{E}(\rho)||\mathcal{E}(\sigma)) &= D(\int du\, uU(\rho \otimes |\psi\rangle\langle\psi|)U^*u^*|| \int du\, uU(\sigma \otimes |\psi\rangle\langle\psi|)U^*u^*) \\ &\leq \int du\, D(uU(\rho \otimes |\psi\rangle\langle\psi|)U^*u^*||uU(\sigma \otimes |\psi\rangle\langle\psi|)U^*u^*) = D(\rho||\sigma) \end{aligned}$$

Clearly, **(MONO) $\Longrightarrow$ (MONO')**, where (MONO') is monotonicity under partial traces

(MONO') $\Longrightarrow$ (CONV): Consider block matrices $\theta\rho^{(1)} \otimes |\uparrow\rangle\langle\uparrow| + (1-\theta)\rho^{(2)} \otimes |\downarrow\rangle\langle\downarrow|$

Properties of the von Neumann Entropy, cont'd

Now we want to prove **(MONO')**$\Longleftrightarrow$**(SSA)**.
Proof proceeds via **another property**, originally proved by **Lieb, Ruskai 1973**: **concavity of the conditional entropy** $S_{1|2}(\rho_{12}) = S(\rho_{12}) - S(\rho_1)$

$$\boxed{S_{1|2}((1-\theta)\rho_{12} + \theta\sigma_{12}) \geq (1-\theta)S_{1|2}(\rho_{12}) + \theta S_{1|2}(\sigma_{12})} \qquad \textbf{(CONC)}$$

(MONO') $\Longrightarrow$ **(SSA)** is immediate (in the right form of (SSA))

(SSA) $\Longrightarrow$ **(CONC):** Consider block matrices, as before

(CONC) $\Longrightarrow$ **(MONO'):** (due to **Lieb, Ruskai**) (CONC) is an equality at $\theta = 1$, so we can differentiate there. This gives (MONO').

This completes the circle of equivalences!

But how do we enter the circle?

How to prove SSA?

One way to prove **SSA** is to show that $(A, B) \mapsto \operatorname{Tr} A \ln A - \operatorname{Tr} A \ln B$ is **convex** and to use the above **equivalence**. (We write (A, B) instead of (ρ, σ), since no normalization on the trace is needed, only $A > 0$ and $B > 0$.)

In fact, one has

Lieb concavity (1973) $\quad \boxed{(A, B) \mapsto \operatorname{Tr} A^\alpha B^{1-\alpha} \quad \text{is concave for } 0 \leq \alpha \leq 1}$

Ando convexity (1979) $\quad \boxed{(A, B) \mapsto \operatorname{Tr} A^\alpha B^{1-\alpha} \quad \text{is convex for } 1 \leq \alpha \leq 2}$

Each of these theorems implies, as $\alpha \to 1$, convexity of $(A, B) \mapsto \operatorname{Tr} A \ln A - \operatorname{Tr} A \ln B$.

There are alternative proofs, for instance, by **Epstein** and **Effros**.

There is also a proof of SSA using the non-commutative version of Minkowski's inequality:

Minkowski's Inequality for Traces

Theorem. (**Carlen–Lieb** (1999 and 2008)) *For* $1 \leq q \leq p \leq 2$*, and all positive operators* A *on* $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$*,*

$$\boxed{\mathrm{Tr}_3 \left(\mathrm{Tr}_2 \left[(\mathrm{Tr}_1 A^q)^{p/q} \right] \right)^{q/p} \leq \mathrm{Tr}_3 \left(\mathrm{Tr}_1 \left[(\mathrm{Tr}_2 A^p)^{q/p} \right] \right)} \tag{1}$$

For $0 \leq p \leq 1$*, and any* $q \geq p$*, this inequality reverses.*

Since (1) is an equality at $p = q = 1$, we can obtain inequalities by differentiating in p at $p = q = 1$. Since

$$\frac{\mathrm{d}}{\mathrm{d}p} \mathrm{Tr}(\rho^p) \big|_{p=1} = \mathrm{Tr}(\rho \ln \rho) \ ,$$

the resulting inequality will involve the entropies of various partial traces of ρ. In fact, we obtain the strong subadditivity of the quantum entropy (SSA):

$$S(\rho_{13}) + S(\rho_{23}) \geq S(\rho_{123}) + S(\rho_3) \ .$$

The Minkowski inequality must have other uses for entropy inequalities. Find them!

Another way to enter the circle

The **Triple Matrix Inequality** (**Lieb, 1973**) states that for operators $X, Y, Z > 0$,

$$\operatorname{Tr} e^{\ln X - \ln Y + \ln Z} \leq \int_0^\infty dt \, \operatorname{Tr} X(Y+t)^{-1} Z (Y+t)^{-1}$$

(The ordinary Golden–Thompson inequality is recovered if one of X, Y, Z is the identity.)
The original proof is somewhat complicated, but it is the only proof so far!
Find a better one.

Let's see how this implies **SSA**. By the **Gibbs** variational principle,

$$S(\rho_{12}) + S(\rho_{23}) - S(\rho_{123}) - S(\rho_2) = \operatorname{Tr}_{123} H \rho_{123} - S(\rho_{123}) \geq - \ln \operatorname{Tr}_{123} e^{-H} \,,$$

with the 'Hamiltonian' $H = -\ln \rho_{12} + \ln \rho_2 - \ln \rho_{23}$. Thus, by **Triple Matrix Ineq.**,

$$\begin{aligned} \operatorname{Tr}_{123} e^{-H} &\leq \int_0^\infty dt \, \operatorname{Tr}_{123} \rho_{12} (\rho_2 + t)^{-1} \rho_{23} (\rho_2 + t)^{-1} \\ &= \int_0^\infty dt \, \operatorname{Tr}_2 \rho_2 (\rho_2 + t)^{-1} \rho_2 (\rho_2 + t)^{-1} = \operatorname{Tr}_2 \rho_2 = 1 \,. \quad \square \end{aligned}$$

Lower bound for SSA

Recall the inequality $S(\rho_{12}) + S(\rho_{23}) \geq S(\rho_1) + S(\rho_3)$ which is equivalent to SSA and which is deduced from it by *'purification'*.

Add to this the equally valid inequality $S(\rho_{13}) + S(\rho_{23}) \geq S(\rho_1) + S(\rho_2)$ and obtain

$$S(\rho_{12}) + S(\rho_{13}) + 2S(\rho_{23}) \geq 2S(\rho_1) + S(\rho_2) + S(\rho_3)$$

which, by purification, is equivalent to

$$S(\rho_{13}) + S(\rho_{34}) - S(\rho_{134}) - S(\rho_3)) \geq 2\big(S(\rho_1) - S(\rho_{14})\big).$$

By symmetry, we obtain, finally (**Carlen–Lieb, 2012**)

$$\boxed{S(\rho_{13}) + S(\rho_{34}) - S(\rho_{134}) - S(\rho_3)) \geq 2\max\left\{S(\rho_1) - S(\rho_{14}),\ S(\rho_4) - S(\rho_{14}),\ 0\right\}.}$$

Classically, $S(\rho_1) - S(\rho_{14}) \leq 0$, so this says, that when we are in the quantum regime ($S(\rho_1) - S(\rho_{14}) > 0$ or $S(\rho_4) - S(\rho_{14}) > 0$) we have a bound on SSA.
(**Christandl–Winter** (2004) had this earlier with the average $\frac{1}{2}(S(\rho_1) + S(\rho_4)) - S(\rho_{14})$, which can be very different.)

An aside: Entropy and Uncertainty

Here is a recent application of **Triple matrix** (**Frank–Lieb 2013**). If $\sum_j A_j^* A_j = \sum_k B_k^* B_k = 1$ on $\mathcal{H}_1$, then for any density matrix ρ_{12} on $\mathcal{H}_1 \otimes \mathcal{H}_2$,

$$\boxed{H(1^A|2) + H(1^B|2) \geq S(1|2) - 2\ln c_1}$$

where $c_1 := \sup_{j,k} \sqrt{\mathrm{Tr}_1\, B_k A_j^* A_j B_k^*}$, $S(1|2) := S(\rho_{12}) - S(\rho_2)$ and

$$H(1^A|2) := -\sum_j \mathrm{Tr}_2 \left(\mathrm{Tr}_1\, A_j \rho_{12} A_j^*\right) \ln \left(\mathrm{Tr}_1\, A_j \rho_{12} A_j^*\right) - S(\rho_2)$$

Under additional rank-one assumptions on A_j and B_k this is due to **Berta et al.**, **Coles et al.**, **Tomamichel–Renner**, 2010–2012. For trival $\mathcal{H}_2$, this is due to **Maassen–Uffink**. Our theorem is also valid for general POVMs (for instance Fourier transform).

To prove this, use **operator Jensen** (not needed for rank one) to bound

$$H(1^A|2) + H(1^B|2) - S(1|2) \geq \mathrm{Tr}_{12}\, H \rho_{12} - S(\rho_{12})$$

with 'Hamiltonian' $H = -\ln \sum_j A_j^* A_j \left(\mathrm{Tr}_1 A_j \rho_{12} A_j^*\right) - \ln \sum_k B_k^* B_k \left(\mathrm{Tr}_1 B_k \rho_{12} B_k^*\right) + \ln \rho_2$. Now apply **Gibbs** and **Triple matrix** **exactly** as in the proof of **SSA**.

Next topic: Other notions of entropy

Rényi entropy:

$$(\alpha - 1)^{-1} \ln \operatorname{Tr} \rho^{\alpha} \sigma^{1-\alpha}$$

This is **monotone under CPTP maps** if $\alpha \in [0, 2]$. (Follows from **Lieb convexity/Ando concavity** plus the argument (CONV) $\Longrightarrow$ (MONO) from before.)

Sandwiched Rényi entropy: (**Wilde et al., Müller-Lennert et al.**)

$$(\alpha - 1)^{-1} \ln \operatorname{Tr} \left(\sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}} \right)^{\alpha}$$

Frank–Lieb 2013 showed that this is **monotone under CPTP maps** if $\alpha \in [1/2, \infty)$. (Later alternative proofs in certain parameter regimes.) Our proof uses again **Lieb convexity/Ando concavity** plus (CONV) $\Longrightarrow$ (MONO), but also another ingredient, to be discussed below.

Even more general: $\alpha - z$ **entropies: Jaksic et al., Audenaert–Datta**

$$D_{\alpha,z}(\rho||\sigma) := (\alpha - 1)^{-1} \ln \operatorname{Tr} \left(\sigma^{\frac{1-\alpha}{2z}} \rho^{\frac{\alpha}{z}} \sigma^{\frac{1-\alpha}{2z}} \right)^{z}$$

Open problem: For which values of α and z is this **monotone under CPTP maps**?

The Audenaert–Datta conjecture

Conjecture: (**Audenaert–Datta**, 2015) **Monotonicity under CPTP maps** holds for

$$D_{\alpha,z}(\rho||\sigma) = (\alpha-1)^{-1} \ln \mathrm{Tr} \left(\sigma^{\frac{1-\alpha}{2z}} \rho^{\frac{\alpha}{z}} \sigma^{\frac{1-\alpha}{2z}} \right)^z$$

iff $0 < \alpha < 1$, $z \geq \max\{\alpha, 1-\alpha\}$ or $1 < \alpha < \infty$, $\max\{\alpha/2, \alpha-1\} \leq z \leq \alpha$.

Hiai (2013) has shown that these conditions are necessary for monotonicity and has shown monotonicity for $0 < \alpha < 1$. ($0 < \alpha < 1$ and $z = 1$ is **Lieb concavity**.)
Monotonicity holds for $1 < \alpha \leq 2$ and $z = 1$ by **Ando convexity**. Moreover, by **Frank–Lieb** (2013) monotonicity holds for $1 < \alpha < \infty$ and $z = \alpha$.
Carlen–Frank–Lieb (2014) have shown monotonicity for $1 < \alpha \leq 2$ and $z = \alpha/2$.
Thus, for $\alpha > 1$ the conjecture has been proved on all the 'endpoint' lines, but there remains a lot of work to be done...

All **monotonicity** results come from **convexity/concavity** results for

$$(\rho, \sigma) \mapsto \mathrm{Tr} \left(\sigma^{\frac{1-\alpha}{2z}} \rho^{\frac{\alpha}{z}} \sigma^{\frac{1-\alpha}{2z}} \right)^z .$$

This leads us to...

NEW CONVEXITY/CONCAVITY THEOREMS

The **Audenaert–Datta conjecture** stimulated work on convexity/concavity properties of more general trace functionals

$$(A,B) \mapsto \operatorname{Tr}\left(B^{\frac{q}{2}} A^p B^{\frac{q}{2}}\right)^s \qquad (*)$$

Theorem 1 (Concavity). *Let $p,q \in \mathbb{R}$ and $s>0$. Then $(*)$ is concave iff $0 \leq p,q \leq 1$ and $s \leq 1/(p+q)$.*

Theorem 2 (Convexity). *If*

$$p \in \{1,2\},\ -1 \leq q \leq 0 \text{ and } s \geq 1/(p+q)$$

or

$$p \in (1,2),\ -1 \leq q \leq 0 \text{ and } s \geq \min\{1/(p-1), 1/(q+1)\},$$

then $(*)$ is convex.

Remarks. (1) Most of Thm. 1 is due to **Hiai**. **Carlen–Frank–Lieb** removed his restriction $s \geq \frac{1}{2}$.
(2) Thm. 2 is due to **Carlen–Frank–Lieb**. From **Hiai** we know that for $p \in [1,2]$, the condition $s \geq 1/(p+q)$ is necessary, so for $p \in \{1,2\}$ we have an '**iff**' result. For $p \in (1,2)$ convexity is **conjectured** also for $1/(p+q) \leq s < \min\{1/(p-1), 1/(q+1)\}$.
(3) Convexity is **conjectured** for $-1 \leq p,q < 0$ and $0 < s < 1/2$.

A sample proof

Claim. $(A,B) \mapsto \mathrm{Tr}\left(B^{\frac{q}{2}} A^p B^{\frac{q}{2}}\right)^s$ is convex for $p \in [1,2]$, $q \in (-1,0)$, $s \geq 1/(1+q)$.

Proof. Following **Carlen–Lieb** (2008) we use the 'quasi-linearization formula'

$$\mathrm{Tr}\left(B^{\frac{q}{2}} A^p B^{\frac{q}{2}}\right)^s = \sup_{Z \geq 0} \left(s\, \mathrm{Tr}\, B^{\frac{q}{2}} A^p B^{\frac{q}{2}} Z - (s-1)\, \mathrm{Tr}\, Z^{\frac{s}{s-1}} \right) .$$

Now change variables $D^2 = B^{\frac{q}{2}} Z B^{\frac{q}{2}}$ to get

$$\mathrm{Tr}\left(B^{\frac{q}{2}} A^p B^{\frac{q}{2}}\right)^s = \sup_{D \geq 0} \left(s\, \mathrm{Tr}\, D A^p D - (s-1)\, \mathrm{Tr}\left(D B^{-q} D\right)^{\frac{s}{s-1}} \right) .$$

Since a **supremum** of convex functions is convex, it suffices to prove that (1) $A \mapsto \mathrm{Tr}\, D A^p D$ is convex and (2) $B \mapsto \mathrm{Tr}\left(D B^{-q} D\right)^{\frac{s}{s-1}}$ is concave, for each fixed $D \geq 0$. Note that we have **decoupled** A and B!

(1) follows since $A \mapsto A^p$ is operator convex for $1 \leq p \leq 2$.

(2) follows from **Hiai**'s extension of theorems of **Epstein** and **Carlen–Lieb**. (Here $-qs/(s-1) \leq 1$, that is, $s \geq 1/(1+q)$, is used.) □

Final topic: Joint operator convexity

For the proof of convexity for $s \geq 1/(p-1)$ we need to study **operator convexity/concavity** properties of

$$(A,B) \mapsto B^{\frac{q}{2}} A^p B^{\frac{q}{2}} \qquad (**)$$

Lemma 3. *Let* $p,q \in \mathbb{R} \setminus \{0\}$. *Then*

- $(**)$ *is convex iff* $-1 \leq p < 0$ *and* $q = 2$
- $(**)$ *is* not *concave*

This lemma is closely connected to **triple convexity/concavity** properties of

$$(A,B,C) \mapsto \operatorname{Tr} C^{\frac{r}{2}} B^{\frac{q}{2}} A^p B^{\frac{q}{2}} C^{\frac{r}{2}} \qquad (***)$$

Lemma 4. *Let* $p,q,r \in \mathbb{R} \setminus \{0\}$. *Then*

- $(***)$ *is convex iff* $q = 2$, $p,r < 0$ *and* $p + r \geq -1$
- $(***)$ *is* not *concave*

The positive results are in **Lieb** ('73) and the negative results in **Carlen–Frank–Lieb** ('14). These lemmas are somewhat disappointing, but they show how subtle matrix inequalities are!

THANK YOU FOR YOUR ATTENTION!