Strong converse theorems using Rényi entropies

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Beyond IID, Banff

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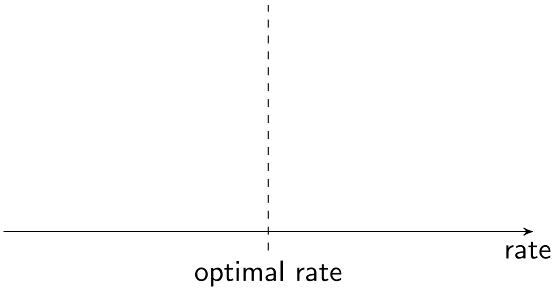

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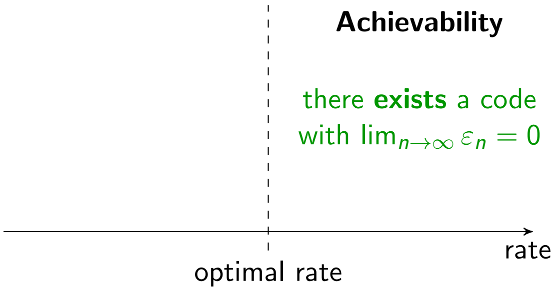

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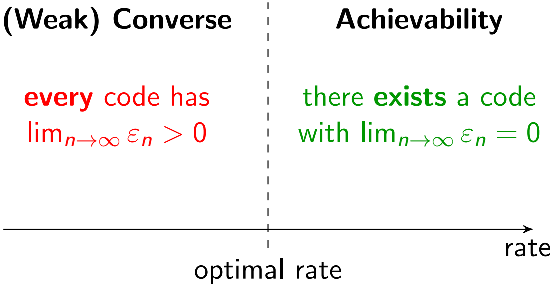

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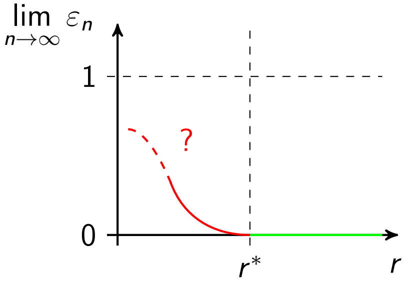

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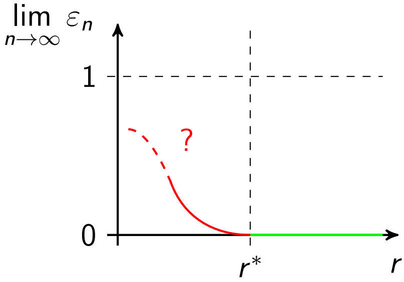

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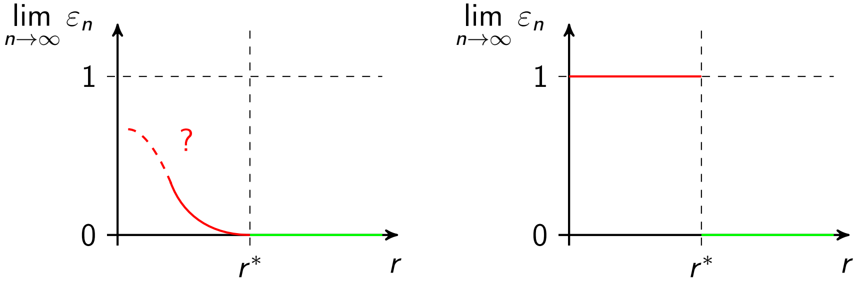

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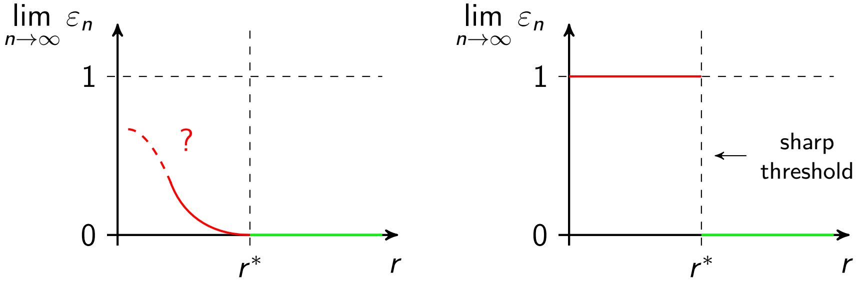

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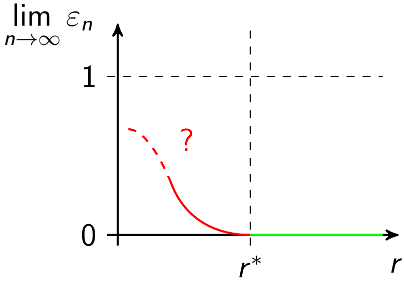
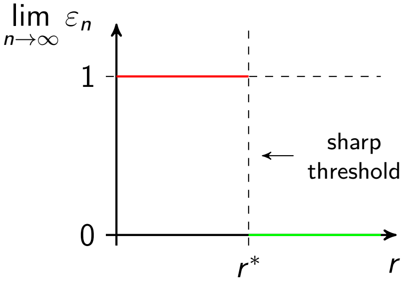
- Optimal rate satisfies *strong converse property*.





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- Rényi entropy approach [Arimoto 1973; Ogawa and Nagaoka 1999; Polyanskiy and Verdú 2010]

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Definition (Rényi entropy of order α)

Let $\alpha \in (0, \infty) \setminus \{1\}$ and $\rho \in \mathcal{D}(\mathcal{H})$, then

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- Convergence to von Neumann entropy:

$$S_\alpha(\rho) \xrightarrow{\alpha \to 1} S(\rho)$$

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$$\varepsilon_n \geq 1 - \exp\left\{-n\kappa(\alpha)\left[S_\alpha(\rho) - r\right]\right\}$$

with rate r, optimal rate $r^* = S(\rho)$, $\alpha > 1$, $\kappa(\alpha) > 0$.

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4. Setting $K := \kappa(\alpha_0)[S_{\alpha_0}(\rho) - r] > 0$, we obtain

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  - **Fidelity bound:** Let $\rho, \sigma \in \mathcal{D}(\mathcal{H})$, and for $\alpha \in (1/2, 1)$ define $\beta(\alpha) := \alpha/(2\alpha - 1)$, then

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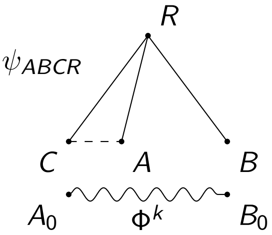

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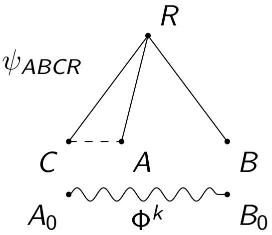

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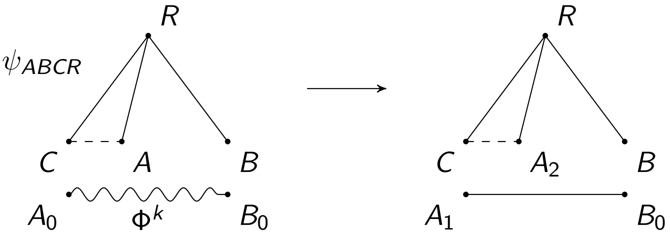

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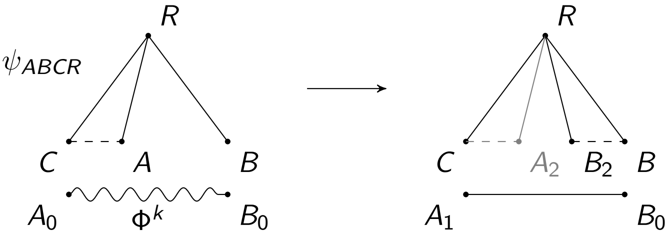

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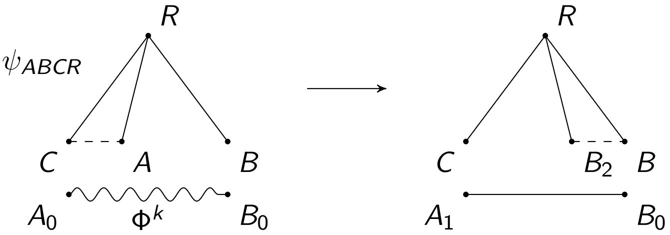

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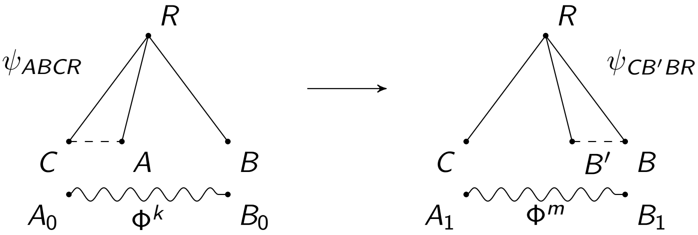

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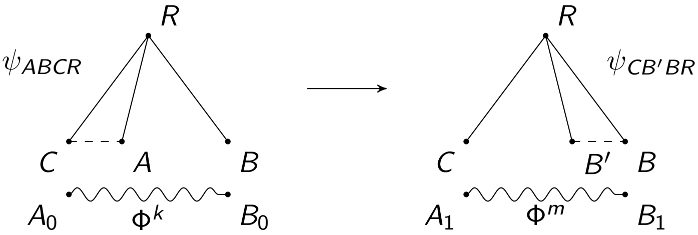

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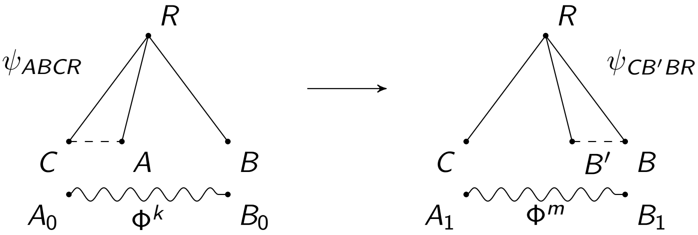

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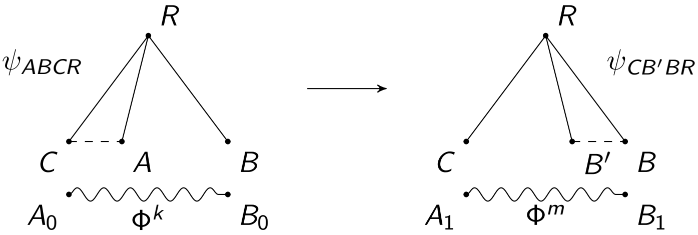

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State redistribution: protocol

Let Alice and Bob share n copies of the state $\Omega \equiv \psi_{ABCR} \otimes \Phi^k_{A_0 B_0}$.

- overall map given by $\Lambda_n \colon A^n C^n A_0^n \otimes B^n B_0^n \to C^n A_1^n \otimes B_1^n B'^n B^n$
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State redistribution: optimal rates

Definition

(e, q) is achievable: for $\rho \equiv \rho_{ABC}$ there is $\{(\rho^{\otimes n}, \Lambda_n)\}_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} F_n = 1$ and

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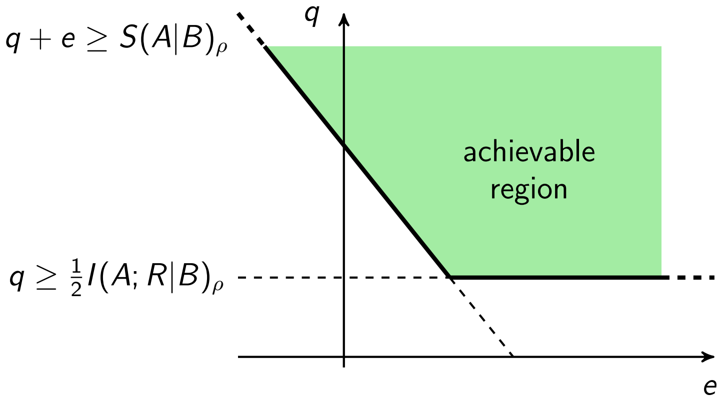

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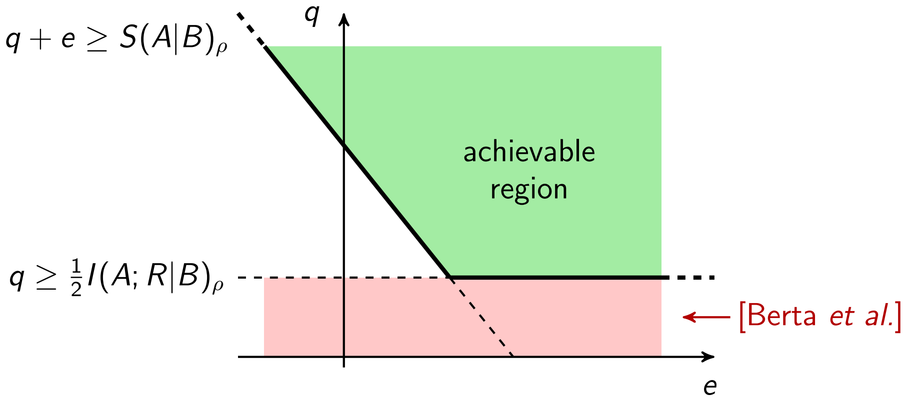

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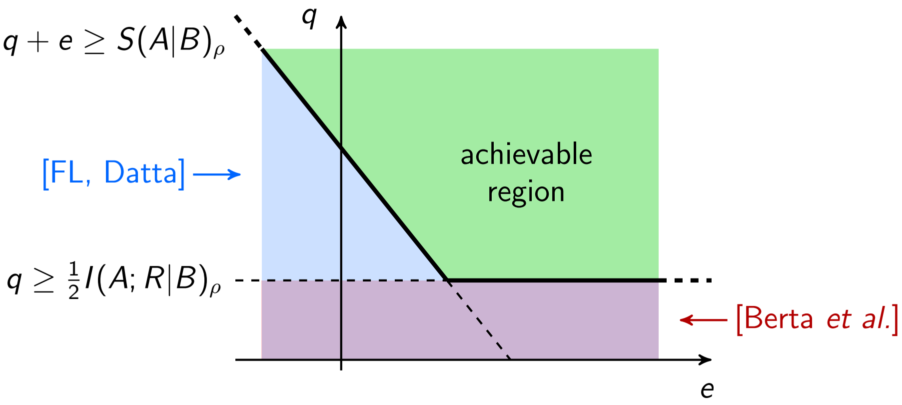

State redistribution: Strong converse theorem

- tripartite state $\rho \equiv \rho_{ABC}$, purification $\psi \equiv \psi_{ABCR}$
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Proof idea

Strong converse property for $q + e$

- Define state

$$|\omega_{CA_1A_2B_0BRE}\rangle := U_{\mathcal{E}}\left(|\psi_{ABCR}\rangle \otimes |\Phi^k_{A_0B_0}\rangle\right)$$

where $U_{\mathcal{E}}$ is a Stinespring isometry of Alice's encoding map $\mathcal{E} : ACA_0 \to CA_1A_2$ with environment E.

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- Subadditivity property yields

$$S_\alpha(CA_1RE)_\omega = S_\alpha(A_2B_0B)_\omega \leq \log|A_2| + \log|A_0| + S_\alpha(B)_\rho.$$

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- Fidelity bound for ω_{CA_1RE} and $\pi^m_{A_1} \otimes \rho_{CR} \otimes \phi_E$ yields

$$S_\alpha(CA_1RE)_\omega - S_\beta(CR)_\rho - \log|A_1| \geq \frac{2\alpha}{1-\alpha}\log F$$

  where $F = F(\psi \otimes \Phi^m, (\mathcal{D} \circ \mathcal{E})(\psi \otimes \Phi^k))$.

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- Both bounds together ($S_\beta(CR)_\rho = S_\beta(AB)_\rho$):

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- For q-bound, assume isometric encoding $\mathcal{E}$, then use fidelity bound twice to obtain

$$\frac{4\alpha}{1-\alpha} \log F_n \leq n \left[2q - S_\beta(AB)_\rho - S_\beta(BR)_\rho + S_\alpha(B)_\rho + S_\alpha(ABR)_\rho \right].$$

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1. Strong converse and Rényi entropies
2. State redistribution: definition and strong converse
3. Strong converse theorems for other protocols
4. Extensions of our results

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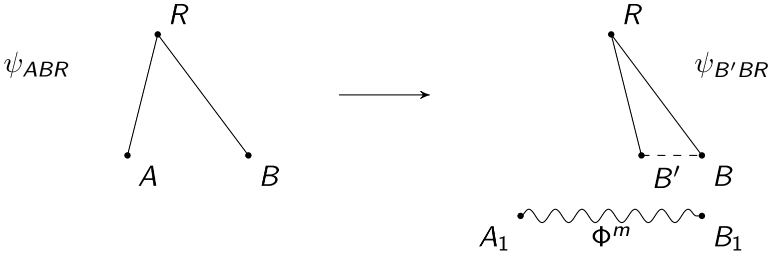
- Goal: Transfer A-part of bipartite system ρ_{AB} from Alice to Bob and create entanglement between them



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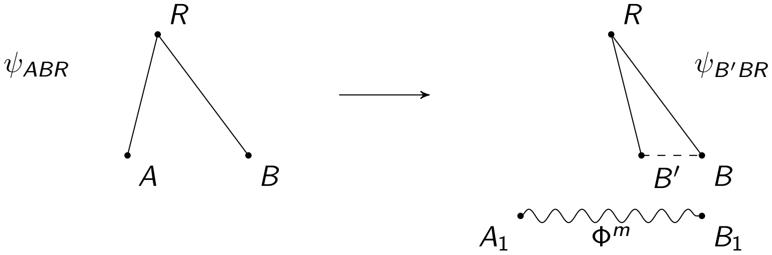

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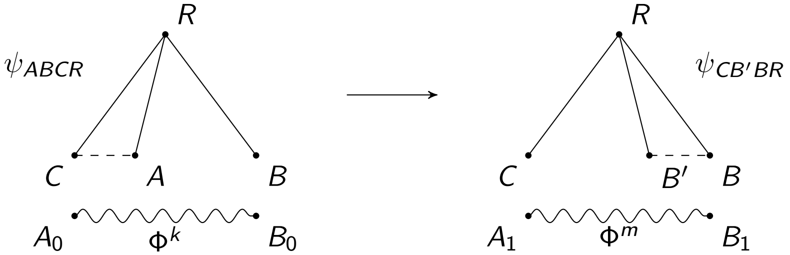

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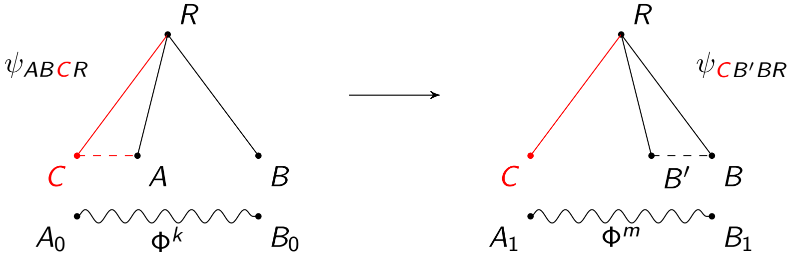

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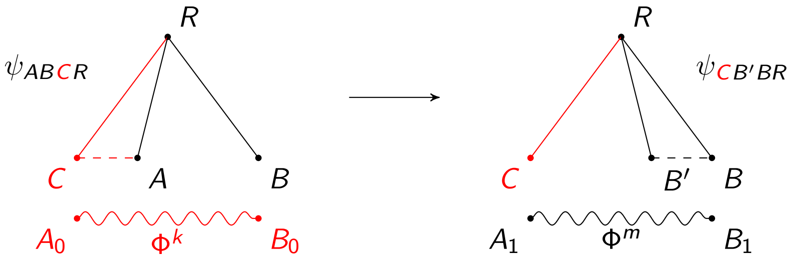

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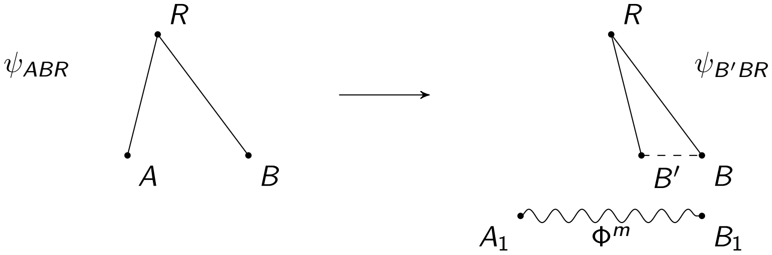

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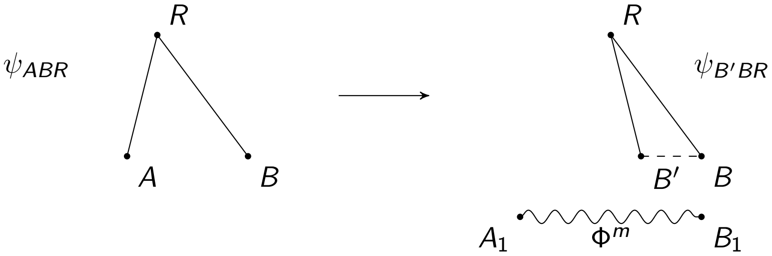

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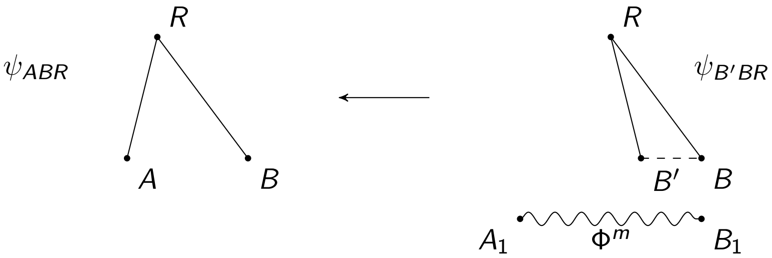

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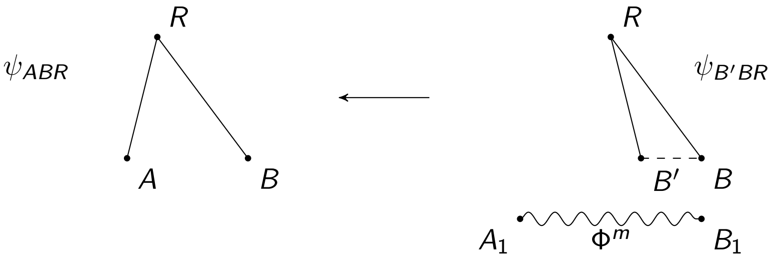

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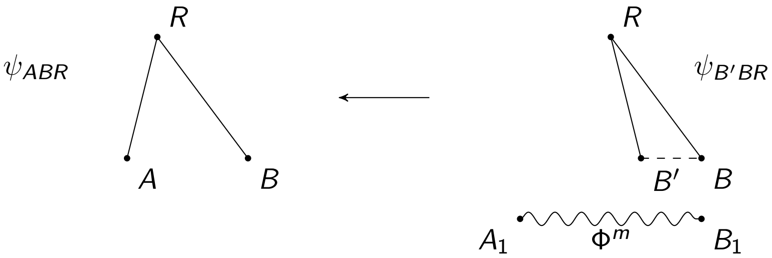

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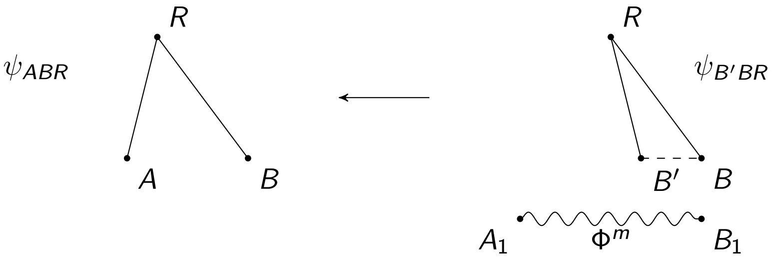

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  - Fidelity bound on Rényi entropies yields strong converse bound.

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2 State redistribution: definition and strong converse

3 Strong converse theorems for other protocols

4 Extensions of our results

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Open question:

- Establish above quantities as strong converse exponents, e.g. prove existence of a code such that fidelity satisfies

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References

Arimoto, S. (1973). *IEEE Transactions on Information Theory* 19.3, pp. 357–359.

Berta, M. et al. (2014). *arXiv preprint arXiv:1409.4338*.

Leditzky, F. and N. Datta (2015). *arXiv preprint arXiv:1506.02635*.

Luo, Z. and I. Devetak (2009). *IEEE Transactions on Information Theory* 55.3, pp. 1331–1342.

Ogawa, T. and H. Nagaoka (1999). *IEEE Transactions on Information Theory* 45, pp. 2486–2489.

Polyanskiy, Y. and S. Verdú (2010). *48th Annual Allerton Conf. on Comm., Ctrl., and Comp.* Pp. 1327–1333.

Renes, J. M. (2010). *Proc. of the Royal Soc. of London A: Math., Phys. and Eng. Sciences*. The Royal Society, rspa20100445.

Schumacher, B. (1995). *Physical Review A* 51.4, p. 2738.

Sharma, N. (2014). *arXiv preprint arXiv:1404.5940*.

Winter, A. (1999). *IEEE Transactions on Information Theory* 45.7, pp. 2481–2485.

Wolfowitz, J. (1961). 31. Springer Berlin-Göttingen-Heidelberg.

Yard, J. T. and I. Devetak (2009). *IEEE Transactions on Information Theory* 55.11, pp. 5339–5351.

van Dam, W. and P. Hayden (2002). *arXiv preprint quant-ph/0204093*.

Thank you very much!