Detecting quantum capacity

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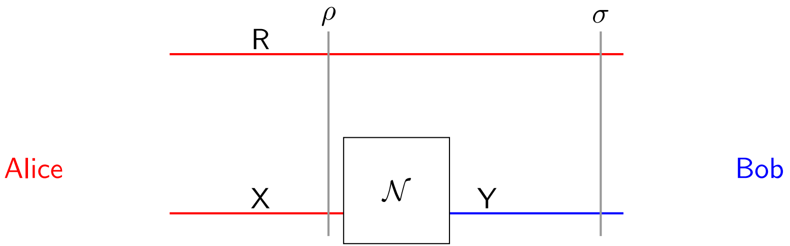
arXiv:1408.5115

Coherent information and quantum capacity



$$I_{coh}(\mathcal{N}) := \max\{I(\mathsf{R}\rangle\mathsf{Y})_\sigma : \rho_{\mathsf{RX}} \text{ pure }\},$$

where $I(\mathsf{R}\rangle\mathsf{Y})_\sigma$ is coherent information of R given Y for state σ:

$$I(\mathsf{R}\rangle\mathsf{Y})_\sigma := H(\mathsf{Y})_\sigma - H(\mathsf{RY})_\sigma = -H(\mathsf{R}|\mathsf{Y})_\sigma$$

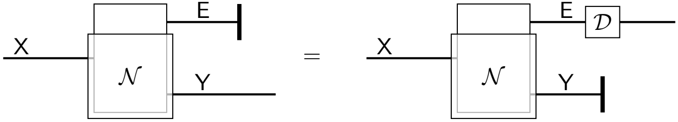
$$Q(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n} I_{coh}(\mathcal{N}^{\otimes n}).$$

[Lloyd 1997], [Shor 2002], [Devetak 2005]

Detecting quantum capacity

- **PPT** channels have $Q = 0$:
$\mathrm{T}^{\mathsf{Y}\leftarrow\mathsf{Y}}\mathcal{N}^{\mathsf{Y}\leftarrow\mathsf{X}}$ is a channel, where $\mathrm{T}^{\mathsf{Y}\leftarrow\mathsf{Y}} : |i\rangle\langle j|_{\mathsf{Y}} \to |j\rangle\langle i|_{\mathsf{Y}}$.

- **Anti-degradable** channels have $Q = 0$:
Exists channel $\mathcal{D}$ such that

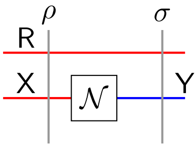


- Both conditions easily checked,
but there may be other zero Q channels

- LSD formula tells us $(\exists k \text{ s.t. } I_{coh}(\mathcal{N}^{\otimes k}) > 0) \iff Q(\mathcal{N}) > 0$
but can have $I_{coh}(\mathcal{N}) = 0$ and $Q(\mathcal{N}) > 0\ldots$

Superadditivity of coherent information

Superadditivity of I_{coh} [DiVincenzo+Smith+Smolin 1997]:
$\mathcal{N}$ is qubit depolarising channel with $0.8099 \leq f \leq 0.8107$.



- $I(\mathsf{B}\rangle\mathsf{R})_\sigma \leq 0$ for all pure ρ_{RA}, so $I_{coh}(\mathcal{N}) = 0$.
- However, for $|\psi\rangle_{\mathsf{RX_1X_2X_3}} = (|0\rangle|0\rangle|0\rangle|0\rangle + |1\rangle|1\rangle|1\rangle|1\rangle)/\sqrt{2}$, $I(\mathsf{R}\rangle\mathsf{Y_1Y_2Y_3})_\tau > 0$, so $I_{coh}(\mathcal{N}^{\otimes 3}) > 0$ and $Q(\mathcal{N}) > 0$.

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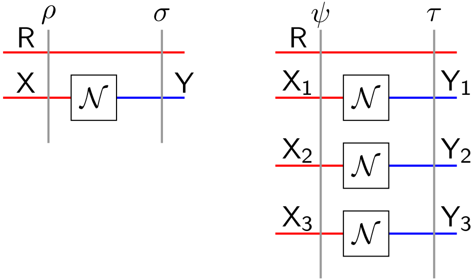

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Our result

In fact...

For any n, $\exists\mathcal{N}$ with $I_{coh}(\mathcal{N}^{\otimes n}) = 0$ but $Q(\mathcal{N}) > 0$.
[Cubitt+Elkouss+Matthews+Pérez-García+Ozols+Strelchuk 2014]

Based on the [Smith+Yard 2008] superactivation result, and Oppenheim's description of it.

Ingredients:

- Erasure channels
- PPT approximate pbits
- Switched channels

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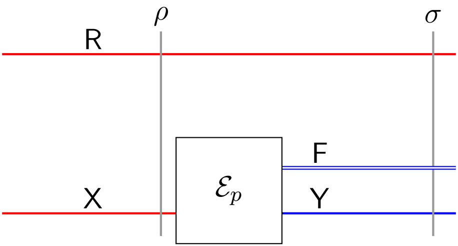

- $\mathcal{E}_p : \rho_{\mathsf{X}} \mapsto (1-p)\rho_{\mathsf{Y}} \otimes |0\rangle\langle 0|_{\mathsf{F}} + p|0\rangle\langle 0|_{\mathsf{Y}} \otimes |1\rangle\langle 1|_{\mathsf{F}}$.
- $V|\psi\rangle_{\mathsf{X}} = \sqrt{1-p}|0\rangle_{\mathsf{F}}|\psi\rangle_{\mathsf{Y}}|1\rangle_{\mathsf{F}'}|0\rangle_{\mathsf{B}'} + \sqrt{p}|1\rangle_{\mathsf{F}}|0\rangle_{\mathsf{Y}}|0\rangle_{\mathsf{F}'}|\psi\rangle_{\mathsf{B}'}$.
- Anti-degradable for $p \geq 1/2$.

$$\sigma_{\mathsf{RYF}} = \sum_f p(f)\sigma[f]_{\mathsf{RY}} \otimes |f\rangle\langle f|_{\mathsf{F}} \implies I(\mathsf{R}\rangle\mathsf{YF})_\sigma = \sum_f p(f) I(\mathsf{R}\rangle\mathsf{Y})_{\sigma[f]}.$$

e.g. $I(\mathsf{R}\rangle\mathsf{YF})_\sigma = (1-p)I(\mathsf{R}\rangle\mathsf{X})_\rho + pI(\mathsf{R}\rangle\mathsf{Y})_{\rho_{\mathsf{R}}\otimes|0\rangle\langle 0|_{\mathsf{Y}}} = (1-2p)H(\mathsf{X})_\rho$

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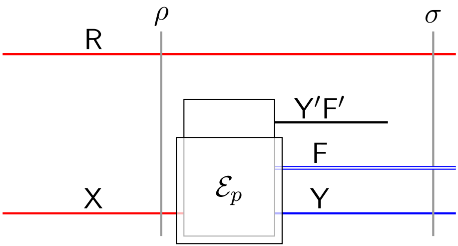

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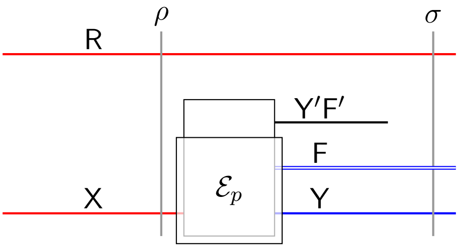

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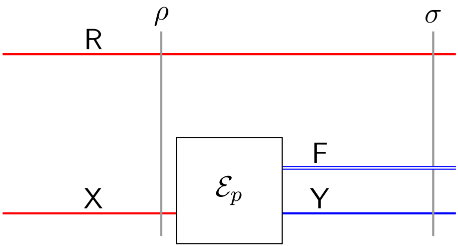

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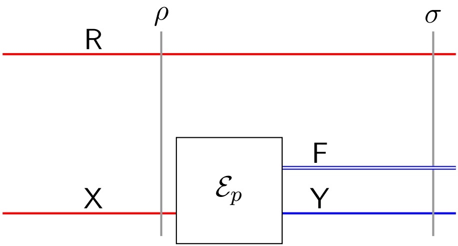

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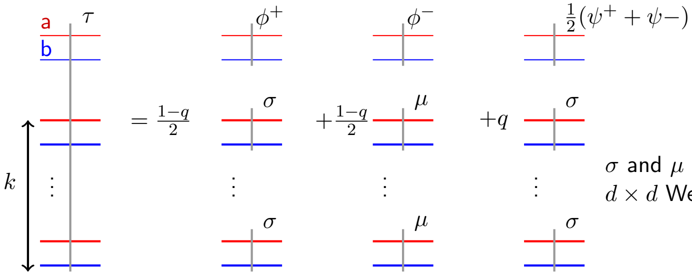
PPT approximate pbits [3×Horodecki+Oppenheim 2005]



σ and μ are $d \times d$ Werner states.

Setting $q = 1/3$, $k \geq 3m$:

- $\gamma_{\mathsf{abA}^{m;k}\mathsf{B}^{m;k}}$ is arbitrarily close to a pbit for sufficiently large m, and so $I(\mathsf{a}\rangle\mathsf{bBA})_\gamma \geq 1 - \Delta$, $I(\mathsf{a}\rangle\mathsf{bB})_\gamma \geq I(\mathsf{a}\rangle\mathsf{b})_\gamma \geq -\Delta$, where $\Delta \to 0$ as $m \to \infty$.
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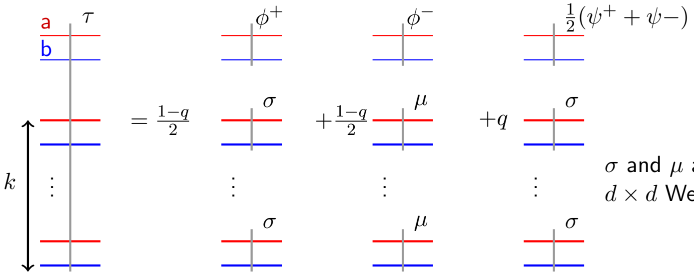

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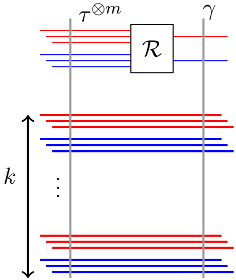

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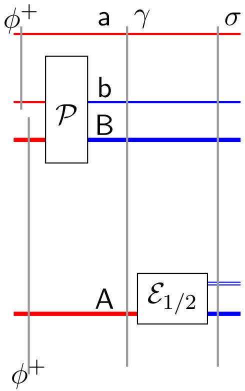
Superactivation of Q



[Smith+Yard 2008], [Oppenheim 2008]

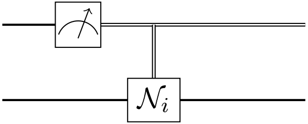
Given a PPT approx pbit γ_{abAB} let $\mathcal{P}$ be such that $\mathcal{P}\phi^+_{\mathsf{a'a}}\phi^+_{\mathsf{A'A}} = \gamma_{\mathsf{abAB}}$.

$\mathcal{P}$ is PPT, so $Q(\mathcal{P}) = 0$.

$Q(\mathcal{E}_{1/2}) = 0$.

$$I_{coh}(\mathcal{P} \otimes \mathcal{E}_{1/2}) \geq I(\mathsf{a}\rangle\mathsf{bBAF})_\sigma = \frac{1}{2} I(\mathsf{a}\rangle\mathsf{bBA})_{\sigma[0]} + \frac{1}{2} I(\mathsf{a}\rangle\mathsf{bBA})_{\sigma[1]}$$
$$= \frac{1}{2}(1 - \Delta) - \frac{1}{2}\Delta \longrightarrow \frac{1}{2}$$

Switched channels



- Switched channel: $\sum_i \mathcal{S}_i \otimes \mathcal{N}_i$ where $\mathcal{S}_i : \rho \mapsto |i\rangle\langle i|\rho|i\rangle\langle i|$.
- Many uses of switched channel is just a bigger switched channel, with a switch for each use.
- $I_{coh}\left(\sum_i \mathcal{S}_i \otimes \mathcal{N}_i\right) = \max_i I_{coh}(\mathcal{N}_i)$

Therefore, the switched channel

$$\mathcal{N} := \mathcal{S}_0 \otimes \mathcal{P} + \mathcal{S}_1 \otimes \mathcal{E}_{1/2}$$

has $I_{coh}(\mathcal{N}) = 0$ but $I_{coh}(\mathcal{N}^{\otimes 2}) = I_{coh}(\mathcal{P} \otimes \mathcal{E}_{1/2}) > 0$.
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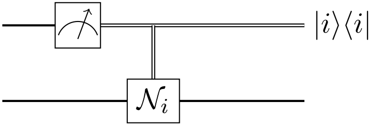

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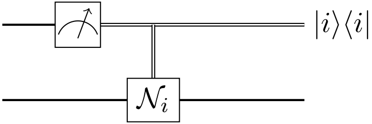

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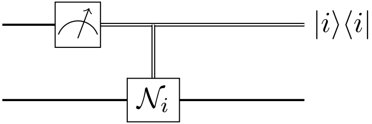

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Our construction

$$\mathcal{N} := \mathcal{S}_0 \otimes \mathcal{P}_\kappa^{\mathsf{YF}\leftarrow\mathsf{X}} + \mathcal{S}_1 \otimes \mathcal{E}_p^{\mathsf{YF}\leftarrow\mathsf{X}},$$

$$\text{where } \mathcal{P}_\kappa^{\mathsf{YF}\leftarrow\mathsf{X}} := \mathcal{E}_\kappa^{\mathsf{YF}\leftarrow\mathsf{Y}} \circ \mathcal{P}^{\mathsf{Y}\leftarrow\mathsf{X}}.$$

$\mathsf{X} \cong \mathsf{aA}^{\mathsf{m;Nr}}$, $\mathsf{Y} \cong \mathsf{bB}^{\mathsf{m;Nr}}$, and $\mathcal{P}\phi^+ = \gamma_{\mathsf{abA}^{\mathsf{m;Nr}}\mathsf{B}^{\mathsf{m;Nr}}}$.

$k = Nr$ so that shield divides into N parts of equal dimension. Chucking away any $N-1$ of these parts leaves $\gamma_{\mathsf{abA}^{\mathsf{m;r}}\mathsf{B}^{\mathsf{m;r}}}$.

For $q = 1/3$, $d = 2Nr$, $r = 3m$:

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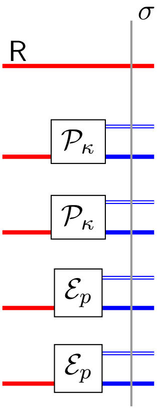

All erase (i.e. $f_1 = \cdots = f_n = 1$):
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All $\mathcal{E}_p$ erase:
$I(\mathrm{R}\rangle \mathrm{Y}^\mathrm{n})_{\sigma[f]} \leq 0$

Other cases:
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$$\begin{aligned} I(\mathrm{R}\rangle \mathrm{Y}^\mathrm{n}\mathrm{F}^\mathrm{n}) &\leq \{(1-p^{n-j})H(\mathrm{R}) - \kappa^j p^{n-j} H(\mathrm{R})\} \\ &\leq \{1-(1+\kappa^n)p^n\} H(\mathrm{R}). \end{aligned}$$

- No more than zero if $p \geq (1+\kappa^n)^{-1/n}$,

in which case $I_{coh}(\mathcal{N}^{\otimes n}) = 0$.
Input/output dimensions irrelevant.

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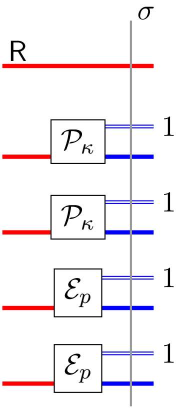

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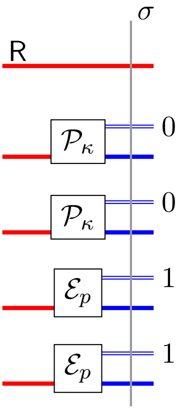
$$I_{coh}(\mathcal{N}^{\otimes n}) = \max_j I_j, \text{ where } I_j := I_{coh}(\mathcal{P}_\kappa^{\otimes j} \otimes \mathcal{E}_p^{\otimes n-j})$$



All erase (i.e. $f_1 = \cdots = f_n = 1$):
$I(\mathsf{R}\rangle \mathsf{Y}^\mathsf{n})_{\sigma[f]} = -H(\mathsf{R})$.

All $\mathcal{E}_p$ erase:
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Other cases:
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$$\begin{aligned} I(\mathsf{R}\rangle \mathsf{Y}^\mathsf{n}\mathsf{F}^\mathsf{n}) &\leq \{(1-p^{n-j})H(\mathsf{R}) - \kappa^j p^{n-j} H(\mathsf{R})\} \\ &\leq \{1-(1+\kappa^n)p^n\} H(\mathsf{R}). \end{aligned}$$

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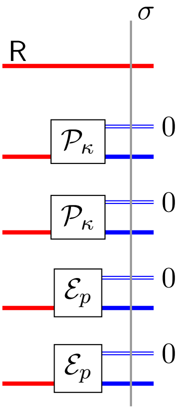

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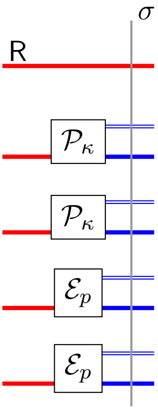

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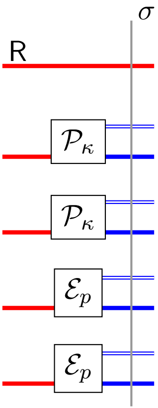

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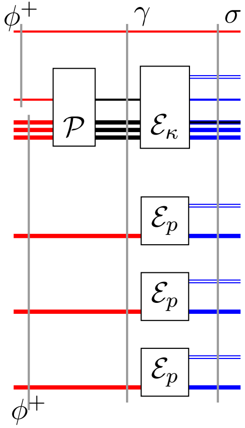
$Q(\mathcal{N}) > 0$



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Otherwise (prob. $(1-\kappa)(1-p^N)$) $I(\mathsf{a}\rangle\mathsf{Y}^{\mathsf{N+1}})_{\sigma[f]} \geq I(\mathsf{a}\rangle\mathsf{bB}^{\mathsf{m;r}}\mathsf{A}^{\mathsf{m;r}})_\gamma \geq 1-\Delta$.

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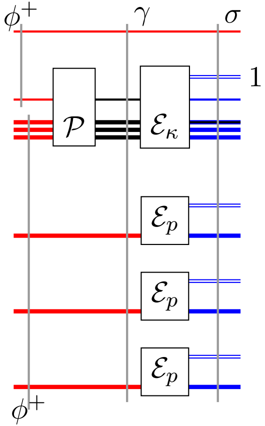

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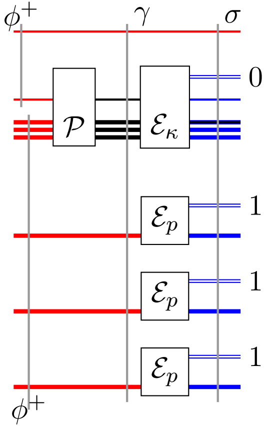

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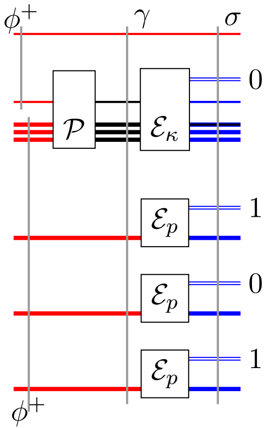

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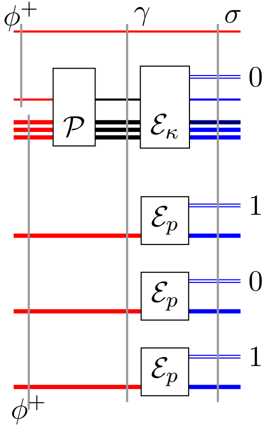

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Conclusion

For any n, letting $\kappa = 1/3$ and $p = (1+\kappa^n)^{-1/n} < 1$:
Then $I_{coh}(\mathcal{N}^{\otimes n}) = 0$, and for some N and γ, $I_{coh}(\mathcal{N}^{\otimes N+1}) > 0$, which establishes $Q(\mathcal{N}) > 0$.

- To make n larger, requires larger p, larger N and larger shield systems.
- Can we obtain a similar result for a channel with bounded input/output dimensions?
- Can we obtain a similar result for the private information?
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