A Proof of the Strong Converse Theorem for Gaussian Multiple Access Channels

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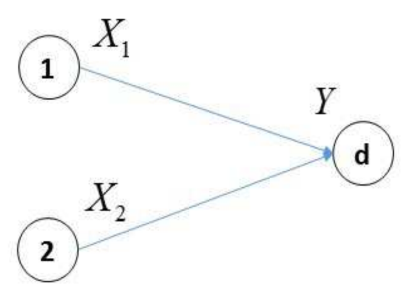
Multiple Access Channel (MAC)



- 2 sources transmit messages to a single destination.
- Each source transmits 1 message.
- The destination decodes all the messages.

Discrete Memoryless MAC (DM-MAC)

- Characterized by a transition matrix $q_{Y|X_1,X_2}$
- Capacity region $\mathcal{C}_{\text{AL}}$ was derived by [Ahlswede, 1971] and [Liao, 1972] in the early 1970s, which is the convex closure of

$$\bigcup_{p_{X_1}p_{X_2}} \left\{ \begin{matrix}(R_1, R_2) \\ \in \mathbb{R}_+^2\end{matrix} \middle| \begin{array}{l} R_1 \le I_{p_{X_1}p_{X_2}q_{Y|X_1,X_2}}(X_1; Y|X_2), \\ R_2 \le I_{p_{X_1}p_{X_2}q_{Y|X_1,X_2}}(X_2; Y|X_1), \\ R_1 + R_2 \le I_{p_{X_1}p_{X_2}q_{Y|X_1,X_2}}(X_1, X_2; Y) \end{array} \right\}$$

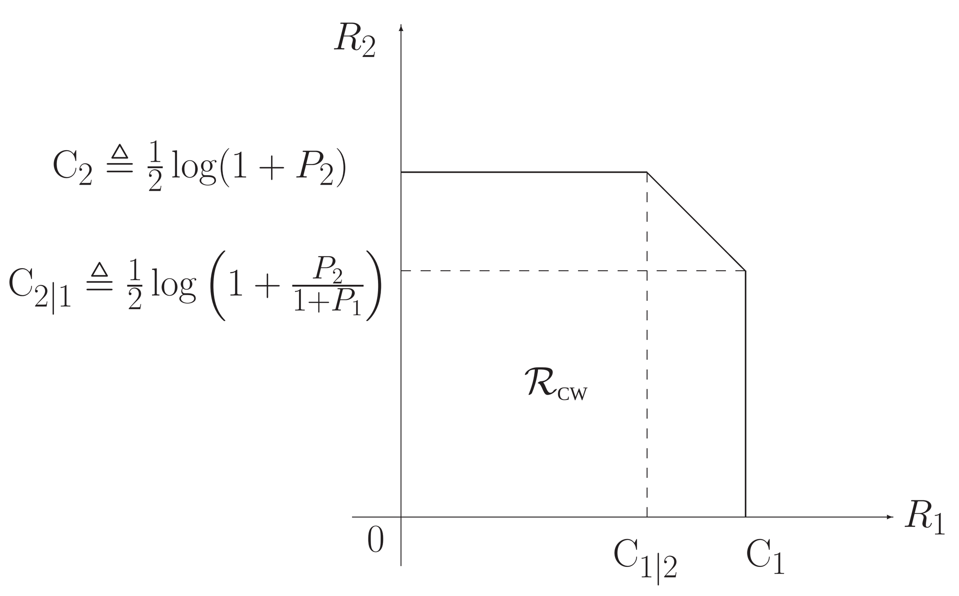
Weak Converse vs Strong Converse

- **Weak Converse**
  - For a fixed rate pair (R_1, R_2), if the decoding error probability *vanishes* as the blocklength increases, then $(R_1, R_2) \in \mathcal{C}_{\text{AL}}$.
  - If the rate pair falls outside $\mathcal{C}_{\text{AL}}$, then the decoding error is *bounded away from 0* as the blocklength increases.
- **Strong Converse**
  - For a fixed rate pair (R_1, R_2), if the decoding error probability is *upper bounded by some* $\varepsilon \in (0, 1)$ as the blocklength increases, then $(R_1, R_2) \in \mathcal{C}_{\text{AL}}$.
  - If the rate pair falls outside $\mathcal{C}_{\text{AL}}$, then the decoding error *tends to 1* as the blocklength increases.
- The original proofs in [Ahlswede, 1971] and [Liao, 1972] are weak converse results.
- Strong converse was proved by [Dueck, 1981] and [Ahlswede, 1982].

Gaussian MAC

- $Y = X_1 + X_2 + Z$ where Z is a standard Gaussian noise.
- The codewords X_1^n and X_2^n should satisfy $\|X_1^n\|^2 \le nP_1$ and $\|X_2^n\|^2 \le nP_2$ respectively.
- Capacity region was derived by [Cover, 1975] and [Wyner, 1974]:

$$\mathcal{R}_{\text{CW}} \triangleq \left\{ \begin{matrix}(R_1, R_2) \\ \in \mathbb{R}_+^2\end{matrix} \middle| \begin{array}{l} R_1 \le \frac{1}{2}\log(1 + P_1), \\ R_2 \le \frac{1}{2}\log(1 + P_2), \\ R_1 + R_2 \le \frac{1}{2}\log(1 + P_1 + P_2) \end{array} \right\}.$$



- The proofs in [Cover, 1975] and [Wyner, 1974] are weak converse results.
- Our contribution is a strong converse proof for the Gaussian MAC.

Network Model

- Sources 1 and 2 transmit information to d in n time slots:
  - Each source i chooses W_i to transmit. Message W_i is uniform on $\{1, 2, \ldots, 2^{nR_i}\}$ where R_i denotes the rate of W_i.
- Each source i transmits $X_{i,k}$ in time slot k and node d receives

$$Y_k = X_{1,k} + X_{2,k} + Z_k\,,$$

  where Z^n are n i.i.d. standard normal random variables.
- Power constraint P_i: X_i^n should satisfy

$$\|X_i^n\|^2 \triangleq \sum_{k=1}^{n} X_{i,k}^2 \le nP_i\,.$$

ε-Capacity Region

- A length-n code operating at rate (R_1, R_2) is called an $(n, (R_1, R_2), \varepsilon_n)$*-code* if the *average* probability of decoding error is less than ε_n .
- (R_1, R_2) is ε-achievable if $\exists$ a sequence of $(n, (R_1, R_2), \varepsilon)$-codes such that $\limsup_{n\to\infty} \varepsilon_n \le \varepsilon$.
- Define ε-capacity region

$$\mathcal{C}_\varepsilon \triangleq \left\{ (R_1, R_2) \in \mathbb{R}_+^2 \,\middle|\, (R_1, R_2) \text{ is } \varepsilon\text{-achievable} \right\}$$

Main Result

Theorem: For each $\varepsilon \in [0, 1)$,

$$\mathcal{C}_\varepsilon = \mathcal{R}_{\text{CW}}. \tag{1}$$

Generalization to N sources: Let $\mathcal{I} \triangleq \{1, 2, \ldots, N\}$ denote the N sources. For each $\varepsilon \in [0, 1)$,

$$\mathcal{C}_\varepsilon = \bigcap_{T \subseteq \mathcal{I}} \left\{ (R_1, \ldots, R_N) \in \mathbb{R}_+^N \,\middle|\, \sum_{i\in T} R_i \le \frac{1}{2}\log\left(1 + \sum_{i\in T} P_i\right) \right\}$$

Major Challenge of Proving the Theorem

- The first step of the proof is to convert the given *average error code* to a *maximal error code* by expurgating appropriate codewords. After the expurgation step, we need a *wringing technique* to "wring out" the dependence between $X_{1,k}$ and $X_{2,k}$.
- After the wringing step, the proof can be completed by using Augustin's converse [Augustin, 1966], a well-known tool for establishing strong converses for the DMC as well as AWGN.
- For the *discrete-alphabet* case, Ahlswede established a wringing technique in [Ahlswede, 1982] to prove the strong converse of the DM-MAC.
- However for the Gaussian case where the alphabet is *continuous*, we cannot directly apply the wringing technique. Instead, we need to quantize the codewords judiciously so that the dependence between $\hat{X}_{1,k}$ and $\hat{X}_{2,k}$ (quantized versions) can be wringed out. This is the main challenge.

Codebook Expurgation [Dueck, 1981]

- Construct a *maximal error code* from the given *average error code* by expurgating a fraction $\frac{2\varepsilon}{1+\varepsilon}$ of codewords, resulting in *maximal probability of decoding error* less than $\frac{1+\varepsilon}{2}$. The messages of the maximal error code are uniformly distributed on some $\mathcal{A} \subset \mathcal{W}_1 \times \mathcal{W}_2$.
- Under the maximal error code,

$$|\mathcal{A}| \ge \left(\frac{1-\varepsilon}{2(1+\varepsilon)}\right) 2^{n(R_1+R_2)}$$

  and for each $(w_1, w_2) \in \mathcal{A}$,

$$p_{W_1,W_2}(w_1, w_2) = \frac{1}{|\mathcal{A}|} \le \frac{1}{2^{n(R_1+R_2)}} \cdot \left(\frac{2(1+\varepsilon)}{1-\varepsilon}\right)$$

Codeword Quantizer

Let $N > 0$ and $\Delta > 0$, and let

$$\mathbb{Z}_{N,\Delta} \triangleq \{-N\Delta, (-N+1)\Delta, \ldots, N\Delta\} \tag{2}$$

be a set of $2N+1$ quantization points where Δ specifies the quantization precision. A scalar quantizer with domain $[-N\Delta, N\Delta]$ and precision Δ is the mapping $\Omega_{N,\Delta} : [-N\Delta, N\Delta] \to \mathbb{Z}_{N,\Delta}$ such that

$$\Omega_{N,\Delta}(x) = \begin{cases} \lfloor x/\Delta \rfloor \Delta & \text{if } x \ge 0, \\ \lceil x/\Delta \rceil \Delta & \text{otherwise.} \end{cases}$$

In addition, define the scalar quantizer for a real-valued tuple as $\Omega_{N,\Delta}^{(n)} : [-N\Delta, N\Delta]^n \to \mathbb{Z}_{N,\Delta}^n$ such that

$$\Omega_{N,\Delta}^{(n)}(x^n) \triangleq (\Omega_{N,\Delta}(x_1), \Omega_{N,\Delta}(x_2), \ldots, \Omega_{N,\Delta}(x_n))$$

Wringing Technique

- Suppose we are given the $\left(n, \mathcal{A}, \frac{1+\varepsilon}{2}\right)_{\text{max}}$-code constructed after the expurgation step. Then, there exists an $\left(n, \mathcal{A}', \frac{1+\varepsilon}{2}\right)_{\text{max}}$-code with

$$|\mathcal{A}'| \ge n^{\frac{-8(1+3\varepsilon)}{(1-\varepsilon)}\sqrt{\frac{n}{\log n}}} \left(\frac{1-\varepsilon}{2(1+\varepsilon)}\right) 2^{n(R_1+R_2)}$$

  such that the following holds: Let $p_{W_1,W_2,X_1^n,X_2^n,Y^n}$ denote the distribution induced by the $\left(n, \mathcal{A}', \frac{1+\varepsilon}{2}\right)_{\text{max}}$-code. Let

$$\hat{X}_i^n = \Omega_{\lceil n\sqrt{nP_i}\rceil, \frac{1}{n}}(X_i^n),$$

  define the alphabet

$$\hat{\mathcal{X}}_i \triangleq \mathbb{Z}_{\lceil n\sqrt{nP_i}\rceil, \frac{1}{n}}$$

  for each $i \in \{1, 2\}$ and define

$$\begin{aligned} & p_{X_1^n,X_2^n,\hat{X}_1^n,\hat{X}_2^n}(x_1^n, x_2^n, \hat{x}_1^n, \hat{x}_2^n) \\ & \triangleq p_{X_1^n,X_2^n}(x_1^n, x_2^n) \prod_{i\in\{1,2\}} \mathbf{1}\left\{\hat{x}_i^n = \Omega_{\lceil n\sqrt{nP_i}\rceil, \frac{1}{n}}(x_i^n)\right\} \end{aligned}$$

  for all $(x_1^n, x_2^n, \hat{x}_1^n, \hat{x}_2^n) \in \mathcal{X}_1 \times \mathcal{X}_2 \times \hat{\mathcal{X}}_1^n \times \hat{\mathcal{X}}_2^n$. Then there exists a distribution $u_{\hat{X}_1^n,\hat{X}_2^n}$ such that for all $k \in \{1, 2, \ldots, n\}$ and for all $\hat{x}_{1,k}, \hat{x}_{2,k} \in \hat{\mathcal{X}}_1 \times \hat{\mathcal{X}}_2$, we have

$$\begin{aligned} & p_{\hat{X}_{1,k},X_{2,k}}(\hat{x}_{1,k}, \hat{x}_{2,k}) \\ & \le \max\left\{\left(1 + \sqrt{\frac{\log n}{n}}\right) \prod_{i\in\{1,2\}} u_{\hat{X}_{i,k}}(\hat{x}_{i,k}), \frac{1}{n^8}\right\}. \end{aligned}$$

- Therefore, we can approximate $p_{\hat{X}_{1,k},\hat{X}_{2,k}}$ by a product distribution $\prod_{i\in\{1,2\}} u_{\hat{X}_{i,k}}$ through an inequality, which wrings out the independence between $X_{1,k}$ and $X_{2,k}$.

References

[Ahlswede, 1971] Ahlswede, R. (1971). Multi-way communication channels. In *Proc. of IEEE International Symposium on Information Theory*, pages 23–52, Thakadsor, Armenian SSR.

[Ahlswede, 1982] Ahlswede, R. (1982). An elementary proof of the strong converse theorem for the multiple access channel. *Journal of Combinatorics, Information & System Sciences*, 7(3):216–230.

[Augustin, 1966] Augustin, U. (1966). Gedachtnisfreie kannale for diskrete zeit. *Z. Wahrscheinlichkeitstheorie verw. Gebiete*, pages 10–61.

[Cover, 1975] Cover, T. M. (1975). Some advances in broadcast channels. In Viterbi, A. J., editor, *Advances in Communication Systems*, volume 4, pages 229–260. Academic Press.

[Dueck, 1981] Dueck, G. (1981). The strong converse coding theorem for the multiple-access channel. *Journal of Combinatorics, Information & System Sciences*, 6(3):187–196.

[Liao, 1972] Liao, H. H. J. (1972). *Multiple access channels*. PhD thesis, University of Hawaii, Honolulu.

[Wyner, 1974] Wyner, A. D. (1974). Recent results in the Shannon theory. *IEEE Transactions on Information Theory*, 20:2–10.