Asymptotic and Non-Asymptotic Fundamental Limits for Quantum Communication

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joint work with M. Wilde, A. Winter, M. Berta and J. Renes
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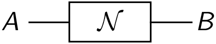
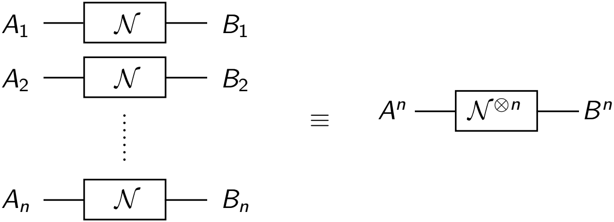
Quantum Coding: Channels

- **Quantum channel**: completely positive trace-preserving linear map $\mathcal{N} \equiv \mathcal{N}_{A \to B}$ from (states on) A to (states on) B.



  Assume A and B are finite-dimensional.

- The channel is memoryless:



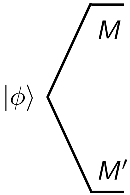
Quantum Coding: Encoder and Decoder

- **Entanglement transmission code** (for $\mathcal{N}^{\otimes n}$):

$$\mathcal{C}_n = \{d_n, \mathcal{E}_n, \mathcal{D}_n)\,.$$

1. code size d_n:
   - M, M', M'' of dimension d_n.
   - maximally entangled state

$$|\phi\rangle_{MM'} = \frac{1}{\sqrt{d_n}} \sum_{i=1}^{d_n} |i\rangle_M \otimes |i\rangle_{M'}\,.$$



2. encoder $\mathcal{E}_n$: quantum channel from M' to A^n.
3. decoder $\mathcal{D}_n$: quantum channel from B^n to M''.

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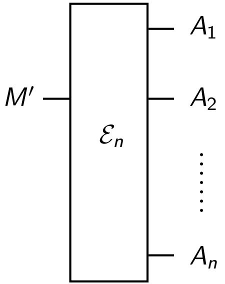

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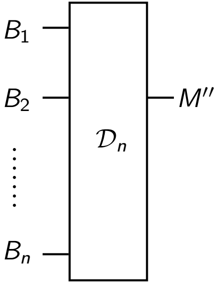
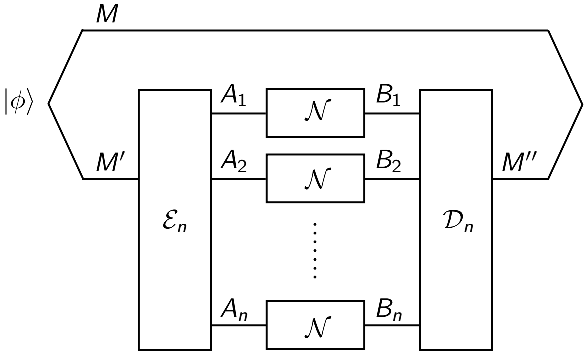


Quantum Coding: Entanglement Fidelity



- **Fidelity** with maximally entangled state:

$$F(\mathcal{C}_n, \mathcal{N}^{\otimes n}) = \operatorname{tr}\left((\mathcal{D}_n \circ \mathcal{N}^{\otimes n} \circ \mathcal{E}_n)(\phi_{MM'})\phi_{MM''}\right).$$

Quantum Capacity

- A triple (R, n, ε) is achievable on $\mathcal{N}$ if $\exists\, \mathcal{C}_n$ with

$$\frac{1}{n} \log d_n \geq R, \quad \text{and} \quad F(\mathcal{C}_n \mathcal{N}^{\otimes n}) \geq 1 - \varepsilon\,.$$

- Boundary of (non-asymptotic) achievable region:

$$\hat{R}(n; \varepsilon, \mathcal{N}) := \max \left\{ R : (R, n, \varepsilon) \text{ is achievable on } \mathcal{N} \right\}.$$

- The *quantum capacity*, $Q(\mathcal{N})$, is the rate at which qubits can be transmitted with fidelity approaching one asymptotically.

$$Q_\varepsilon(\mathcal{N}) := \lim_{n \to \infty} \hat{R}(n; \varepsilon, \mathcal{N}), \qquad \varepsilon \in (0, 1)$$
$$Q(\mathcal{N}) := \lim_{\varepsilon \to 0} Q_\varepsilon(\mathcal{N})\,.$$

Quantum Capacity Theorem

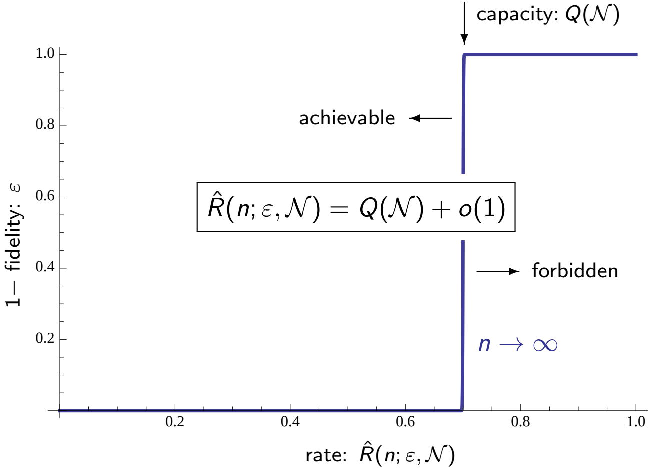
- Barnum, Nielsen and Schumacher (1996-2000) as well as Lloyd, Shor and Devetak (1997-2005) established

$$Q(\mathcal{N}) = \lim_{\ell\to\infty} \frac{1}{\ell} I_{\mathrm{c}}\big(\mathcal{N}^{\otimes\ell}\big), \qquad \text{where}$$
$$I_{\mathrm{c}}(\mathcal{N}) = \max_{\rho_A} \{-H(A|B)_\omega\}, \quad \omega_{AB} = \mathcal{N}_{A'\to B}(\psi^{\rho}_{A'A})$$

- This result is unsatisfactory for many reasons:
  1. It is not a single-letter formula.
  2. The limit $\ell \to \infty$ is necessary in general (Cubitt *et al.*'14).
  3. It cannot be calculated except for e.g. degradable channels which satisfy $I_{\mathrm{c}}\big(\mathcal{N}^{\otimes n}\big) = nI_{\mathrm{c}}(\mathcal{N})$.
  4. We do not know anything about $Q_\varepsilon(\mathcal{N})$.

Capacity and Strong Converse

- What we would like to know:



State of the Art

- **Until this work, the strong converse property could only be established for some channels with trivial capacity.**
- Morgan and Winter showed that *degradable quantum channels* satisfy a "pretty strong" converse:

$$Q_\varepsilon(\mathcal{N}) = Q(\mathcal{N}) \quad \text{for all } \varepsilon \in \left(0, \frac{1}{2}\right)$$

  (Extending their proof to all $\varepsilon \in (0, 1)$ appears difficult.)
- Strong converse rates are known, for example the entanglement-assisted capacity established via channel simulation (Bennett *et al.*)
- However, they are not tight except for trivial channels.

A lot of (fundamental) work still needs to be done!

Result 1: Rains Entropy is Strong Converse Rate

- The *Rains relative entropy* of the channel is defined as

$$R(\mathcal{N}) := \max_{\rho_A} \min_{\sigma_{AB} \in \mathrm{Rains}(A:B)} D\big(\mathcal{N}_{A' \to B}(\psi^{\rho}_{A'A}) \,\big\|\, \sigma_{AB}\big) \, .$$

- A state $\sigma_{AB} \in \mathrm{Rains}(A : B)$ (cf. Rains'99) satisfies

$$\operatorname{tr}\big(\phi_{AB}\sigma_{AB}\big) \leq \frac{1}{d} \quad \forall \text{ maximally entangled } \phi_{AB}.$$

Theorem

For any channel $\mathcal{N}$, communication at a rate exceeding $R(\mathcal{N})$ implies (exponentially) vanishing fidelity.

- Key Idea: Consider correlations σ_{AB} that are useless for quantum communication. Classically:

$$C(W) = \max_{P_X} \min_{Q_X, Q_Y} D\big(P_X \times W_{Y|X} \big\| Q_X \times Q_Y\big) \, .$$

Step 1: Arimoto-Type (One-Shot) Converse Bound

- Following Sharma–Warsi'13 ...
- Consider $\mathcal{C} = \{d, \mathcal{E}, \mathcal{D}\}$ for $\mathcal{N}$ with $F(\mathcal{C}, \mathcal{N}) \geq 1 - \varepsilon$.
- Test if a state is $\phi_{MM''}$, or not:

$$\mathcal{T}(\cdot) = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|, \quad p = \mathrm{tr}(\phi_{MM''} \cdot).$$

- Let $\rho_{AM} = \mathcal{E}(\phi_{MM'})$. Due to data-processing, we have

$$\begin{aligned} \widetilde{D}_\alpha(\mathcal{N}(\rho_{AM}) \| \sigma_{BM}) &\geq D_\alpha\left(\mathcal{T} \circ \mathcal{D} \circ \mathcal{N}(\rho_{AM}) \| \mathcal{T} \circ \mathcal{D}(\sigma_{RB})\right) \\ &\geq \log d + \frac{\alpha}{\alpha - 1} \log(1 - \varepsilon), \end{aligned}$$

  for Rényi divergence with $\alpha > 1$
- Sandwiched Rényi divergence (Lennert *et al.*, Wilde *et al.*'13):

$$\widetilde{D}_\alpha(\rho \| \sigma) = \frac{1}{\alpha - 1} \log \mathrm{tr}\left(\left(\sigma^{\frac{1-\alpha}{2\alpha}} \rho \sigma^{\frac{1-\alpha}{2\alpha}}\right)^\alpha\right).$$

Step 2: Asymptotics

- Minimizing $\sigma_{AB} \in \mathrm{Rains}(A : B)$ and optimizing over codes:

Lemma

We have the following one-shot converse:

$$\hat{R}(1; \varepsilon, \mathcal{N}) \leq \max_{\rho_A} \min_{\sigma_{AB}} \widetilde{D}_\alpha(\mathcal{N}_{A' \to B}(\psi^\rho_{AA'}) \| \sigma_{AB}) + \frac{\alpha \log \frac{1}{1-\varepsilon}}{\alpha - 1}$$

- This yields an upper bound on the ε-capacity:

$$Q_\varepsilon(\mathcal{N}) \leq \limsup_{n \to \infty} \frac{1}{n} \underbrace{\max_{\rho_{A^n}} \min_{\sigma_{A^n B^n}} \widetilde{D}_\alpha\left(\mathcal{N}^{\otimes n}(\psi^\rho_{A^n A'^n}) \middle\| \sigma_{A^n B^n}\right)}_{\widetilde{R}_\alpha(\mathcal{N}^{\otimes n})} .$$

- It remains to show that $\widetilde{R}_\alpha(\mathcal{N})$ satisfies an asymptotic sub-additivity property, i.e. $\widetilde{R}_\alpha(\mathcal{N}^{\otimes n}) \leq n\widetilde{R}_\alpha(\mathcal{N}) + o(n)$.

Step 3: Covariant Channels and Permutations

- Covariance group of the channel $\mathcal{N}$: Group G with unitary representations U_A and V_B such that

$$\mathcal{N}_{A\to B}(U_A(g)(\,\cdot\,)U_A^\dagger(g)) = V_B(g)\mathcal{N}_{A\to B}(\,\cdot\,)V_B^\dagger(g) \quad \forall g \in G$$

Lemma (Channel Covariance)

Let G be a covariance group of $\mathcal{N}$. Then,

$$\widetilde{R}_\alpha(\mathcal{N}) = \max_{\bar{\rho}_A} \min_{\sigma_{AB}} \widetilde{D}_\alpha\big(\mathcal{N}_{A'\to B}(\psi^{\bar{\rho}}_{AA'}) \,\big\|\, \sigma_{AB}\big)$$

where $\bar{\rho}_A = U_A(g)\bar{\rho}_A U_A^\dagger(g)$, i.e. $\bar{\rho}_A$ is invariant under G.

- Covariance group of $\mathcal{N}^{\otimes n}$ always contains permutations S_n.
- Thus, we can restrict the optimization in $\widetilde{R}_\alpha(\mathcal{N}^{\otimes n})$ to permutation invariant states $\bar{\rho}_{A^n}$.

Step 4: Asymptotic Sub-Additivity

- Employ the fact that $\psi^{\bar{\rho}}_{A^n A'^n}$ is in the symmetric subspace:

$$\psi^{\bar{\rho}}_{AA'} \leq P^{\mathrm{symm}}_{A^n R^n} \leq n^{|A|^2} \int \mathrm{d}\mu(\theta)\, \theta^{\otimes n}_{AR}\,.$$

- The quantum way to restrict to product states in the converse.
- This allows us to show (skipping a few technical steps) that

$$\widetilde{R}_\alpha(\mathcal{N}^{\otimes n}) \leq n\widetilde{R}_\alpha(\mathcal{N}) + O(\log(n)).$$

- Hence, $Q_\varepsilon(\mathcal{N}) \leq \widetilde{R}_\alpha(\mathcal{N})$ for all $\alpha > 1$.
- And, thus, by continuity as $\alpha \to 1$, we find $Q_\varepsilon(\mathcal{N}) \leq R(\mathcal{N})$.
- A more detailed analysis reveals that the fidelity converges exponentially fast to 0 for any $d > R(\mathcal{N})$.

Result 2: Dephasing Channels Satisfy Strong Converse

- For all quantum channels we thus have

$$I_{\mathrm{c}}(\mathcal{N}) \leq Q(\mathcal{N}) \leq Q_{\varepsilon}(\mathcal{N}) \leq R(\mathcal{N})$$

for all $\varepsilon \in (0,1)$.

Theorem

For generalized dephasing channels $\mathcal{Z}$, we have $I_c(\mathcal{Z}) = R(\mathcal{Z})$.

- The inequalities collapse and $Q_{\varepsilon}(\mathcal{Z}) = Q(\mathcal{Z})$.
- Includes qubit dephasing channel:

$$\mathcal{Z}_{\lambda} : \rho \mapsto (1-\lambda)\rho + \lambda Z \, \rho \, Z \, .$$

$$\begin{pmatrix} a & c \\ c^{\dagger} & b \end{pmatrix} \mapsto \begin{pmatrix} a & (1-2\lambda)c \\ (1-2\lambda)c^{\dagger} & b \end{pmatrix}$$

Result 3: Second Order Converse

Theorem

If the covariance group of $\mathcal{N}$ is irreducible on A, then

$$\hat{R}(n;\varepsilon,\mathcal{N}) \leq R(\mathcal{N}) + \sqrt{\frac{V(\mathcal{N})}{n}}\Phi^{-1}(\varepsilon) + O\left(\frac{\log n}{n}\right)$$

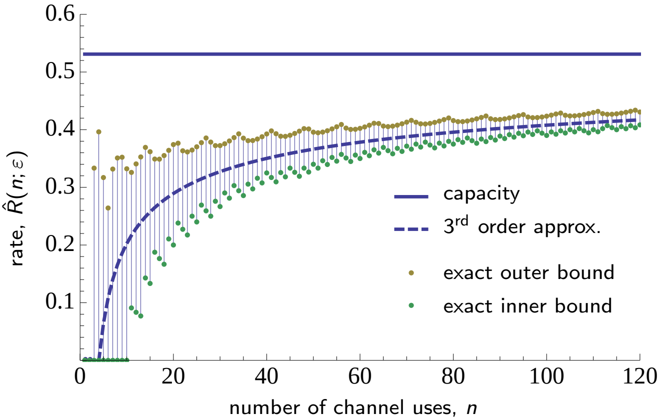
Moreover, equality holds if $\mathcal{N}$ is also dephasing.

- $V(\mathcal{N})$ is (Rains) quantum channel dispersion. Here,

$$R(\mathcal{N}) = \min_{\sigma_{AB}\in \mathrm{Rains}(A:B)} D\big(\mathcal{N}_{A'\to B}(\phi^{\rho}_{A'A}) \,\big\|\, \sigma_{AB}\big),$$
$$V(\mathcal{N}) = V\big(\mathcal{N}_{A'\to B}(\phi^{\rho}_{A'A}) \,\big\|\, \sigma^{*}_{AB}\big)\,.$$

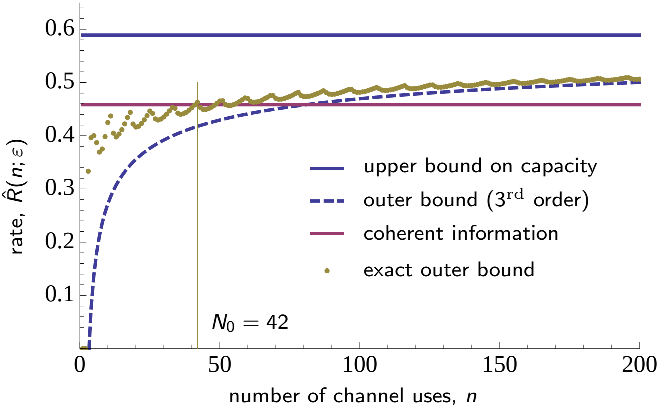
  with $V(\rho\|\sigma) = \mathrm{tr}\left(\rho(\log\rho - \log\sigma)^2\right) - D(\rho\|\sigma)^2$.
- Φ is cumulative (normal) Gaussian distribution function.

Non-Asymptotical Achievable Region for Qubit Dephasing



- Dephasing channel: $\gamma = 0.1$ and fixed fidelity $1 - \varepsilon = 95\%$.
- Corresponds to classical binary symmetric channel.

Qubit Depolarizing Channel



- Depolarizing channel: $\rho \mapsto (1-\alpha)\rho + \frac{\alpha}{3}\big(X\rho X + Y\rho Y + Z\rho Z\big)$.
- Exact outer bound for $\alpha = 0.0825$ and $\varepsilon = 5.5\%$.