

Operational Resource Theory of Coherence

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Outline

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“Operational”

Main results

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(Ir)reversibility and bound coherence

Summary

Resource theory

Ingredients

- ▶ Free states FS
- ▶ Resource states RS
- ▶ Free operations FO

Requirement

$FS \xrightarrow{FO} FS$: Resource states cannot be created from free states under free operations.

Task

$RS \xrightarrow{FO} RS$: The conversion between resource states under the restricted operations.

Coherence [Baumgratz/Cramer/Plenio, PRL(2014)]

Motivation: superposition principle + fixed basis $\{|i\rangle\}$

Ingredients

- ▶ Incoherent states: $\mathcal{I} = \{\rho = \sum_i p_i |i\rangle\langle i|\}$.
- ▶ Coherent states: $\rho \neq \sum_i p_i |i\rangle\langle i|$.
- ▶ Incoherent operations: $\mathcal{E} = \{K_\ell : K = \sum_i c(i) |j(i)\rangle\langle i|\}$.
 $K_\ell \mathcal{I} \subseteq \mathcal{I}$.

Composite system

Fixed basis = $\{|i\rangle \otimes |j\rangle\}$.

Example

- ▶ Phase unitary: $\sum e^{i\theta_i} |i\rangle\langle i|$.
- ▶ Permutation: $\sum |\pi(i)\rangle\langle i|$.
- ▶ Measurement: $\{|i\rangle\langle i|\}$.
- ▶ CNOT: $\sum |i \otimes (i \oplus j)\rangle\langle i \otimes j|$.

“Operational”

Usually resource monotone is employed to study the the conversion between resource state. Here by “operational”, we mean

- ▶ it is a resource monotone: $f(IC(\rho)) \leq f(\rho)$,
- ▶ it has a physical interpretation.

How to construct the operational coherence theory?

We will recall entanglement theory to show what questions should be asked in a resource theory?

Two facts

- ▶ General task: $\rho \xrightarrow{\text{LOCC}} \sigma$.
- ▶ Maximally entangled states play a special role because usually they are used to circumvent the restriction on operations. (teleportation)

Pure states transformations

- ▶ Entanglement Concentration: from partial entangled state to MES $|\Phi_2\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$,

$$\psi^{\otimes n} \xrightarrow{\text{LOCC}} \overset{1-\epsilon}{\approx} \Phi_2^{\otimes nR} \text{ as } n \rightarrow \infty, \epsilon \rightarrow 0.$$

$$\sup R = E(\psi) = -\text{Tr } \psi^A \log \psi^A.$$

- ▶ Entanglement Dilution: from Φ_2 to partial entangled state,

$$\Phi_2^{\otimes nR} \xrightarrow{\text{LOCC}} \overset{1-\epsilon}{\approx} \psi^{\otimes n} \text{ as } n \rightarrow \infty, \epsilon \rightarrow 0,$$

$$\inf R = E(\psi) = -\text{Tr } \psi^A \log \psi^A.$$

Remark

Important! Reversible \implies the standard unit entanglement.

Basic transformations and basic measures

- ▶ Entanglement Distillation: from mixed to pure

$$\rho^{\otimes n} \xrightarrow{\text{LOCC}} \overset{1-\epsilon}{\approx} \Phi_2^{\otimes nR} \text{ as } n \rightarrow \infty, \epsilon \rightarrow 0,$$

Distillable entanglement $E_d(\rho) := \sup R$.

- ▶ Entanglement formation: from pure to mixed

$$\Phi_2^{\otimes nR} \xrightarrow{\text{LOCC}} \overset{1-\epsilon}{\approx} \rho^{\otimes n} \text{ as } n \rightarrow \infty, \epsilon \rightarrow 0.$$

Entanglement cost $E_c(\rho) := \inf R$.

Task: to evaluate E_d and E_c .

Remark

- ▶ E_d and E_c have the operational interpretation.
- ▶ The evaluations are infeasible without additivity.

$$\lim_{n \rightarrow \infty} \frac{1}{n} E_f(\rho^{\otimes n}), \quad E_f(\rho) := \min_{\rho = \sum p_i |\psi_i\rangle\langle\psi_i|} E(\psi_i).$$

(Ir)reversibility and bound entanglement

Basic question: (Ir)reversible?

- ▶ State reversible: $E_d(\rho) = E_c(\rho)$.
Theory reversible: all states reversible.
- ▶ State irreversible: $E_d(\rho) < E_c(\rho)$.
Theory irreversible: not all states reversible.

Bound entanglement?

Bound entanglement: $E_d = 0$ but $E_c > 0$.

What do we study in coherence theory?

- ▶ coherence concentration
- ▶ coherence dilution
- ▶ coherence distillation
- ▶ coherence formation
- ▶ (ir)reversible
- ▶ bound coherence?

Pure states transformations [Yuan et al arXiv:1505.04032]

Denote $|\Phi_2\rangle := \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $\Delta(\rho) := \sum_i \langle i|\rho|i\rangle |i\rangle\langle i|$.

- ▶ Coherence concentration: from partial coherent state to $|\Phi_2\rangle$,

$$\psi^{\otimes n} \xrightarrow{\text{IC}} \approx^{1-\epsilon} \Phi_2^{\otimes nR} \text{ as } n \rightarrow \infty, \epsilon \rightarrow 0.$$

- ▶ Coherence dilution: from Φ_2 to partial coherent state,

$$\Phi_2^{\otimes nR} \xrightarrow{\text{IC}} \approx^{1-\epsilon} \psi^{\otimes n} \text{ as } n \rightarrow \infty, \epsilon \rightarrow 0.$$

Answer

$$C(\psi) = S(\Delta(\psi))$$

Remark

Reversible! $\implies |\Phi\rangle_2$ is the unit coherence state.

two basic transformations and two basic measures

- ▶ Coherence distillation: from mixed to pure

$$\rho^{\otimes n} \xrightarrow{\text{IC}} \overset{1-\epsilon}{\approx} \Phi_2^{\otimes nR} \text{ as } n \rightarrow \infty, \epsilon \rightarrow 0.$$

Distillable coherence

$$C_d(\rho) := \sup R.$$

- ▶ Coherence formation: from pure to mixed

$$\Phi_2^{\otimes nR} \xrightarrow{\text{IC}} \overset{1-\epsilon}{\approx} \rho^{\otimes n} \text{ as } n \rightarrow \infty, \epsilon \rightarrow 0.$$

Coherence cost

$$C_c(\rho) := \inf R.$$

Answers

- ▶ Distillable coherence

$$C_d(\rho) = C_r(\rho) := \min_{\sigma \in \mathcal{I}} S(\rho \| \sigma) = S(\Delta(\rho)) - S(\rho),$$

- ▶ Coherence cost

$$C_c(\rho) = C_f(\rho) := \min_{\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|} \sum_i p_i C(\psi_i)$$

Remark

- ▶ Operational interpretation.
- ▶ Computable single-letter formulas.
- ▶ Additivity

$$C_r(\rho \otimes \sigma) = C_r(\rho) + C_r(\sigma),$$

$$C_f(\rho \otimes \sigma) = C_f(\rho) + C_f(\sigma).$$

(Ir)reversible

- ▶ State reversible: $C_d(\rho) = C_c(\rho)$.
Theory reversible: all states reversible.
- ▶ State irreversible: $C_d(\rho) < C_c(\rho)$.
Theory irreversible: not all states reversible.

Criterion of reversibility

$$\rho = \bigoplus_j p_j |\phi_j\rangle\langle\phi_j|,$$

$|\phi_j\rangle \in \mathcal{H}_j = \text{span}\{|i_j\rangle\}$, and $\mathcal{H}_j \perp \mathcal{H}_k$ for $j \neq k$.

Remark

- (1) It is simple to check.
- (2) It completely characterizes **all** the reversible states.

Bound coherence?

Question

Does there exist bound coherent state s.t $C_d(\rho) = 0$ but $C_c(\rho) > 0$?

Answer: NO!

Reason

$C_d = C_r$ is faithful in the sense that $C_r = 0$ implies $C_f = 0$.

$C(\psi) = S(\Delta(\psi))$: sketched proof

$$\psi \xrightarrow{\text{IC}} \varphi \iff \Delta(\psi) \prec \Delta(\varphi). [\text{Du et al, arXiv:1503.09176}]$$
$$\implies \Phi_d \xrightarrow{\text{IC}} \rho, \text{ where } \text{rank}(\rho) \leq d.$$

Suppose $|\psi\rangle = \sum_{i=1}^d \sqrt{q_i} |i\rangle$, then $|\psi\rangle^{\otimes n} = \sum_{i^n} \sqrt{q_{i^n}} |i^n\rangle$.

Concentration: Perform type measurement $\{M_P\}$

$$M_P = \sum_{i_1 \dots i_n \in T(P)} |i_1 \dots i_n\rangle \langle i_1 \dots i_n|,$$
$$2^{nS(P)} \geq |T(P)| \geq (n+1)^{-d} 2^{nS(P)},$$
$$P_{\text{typ}} \approx Q,$$

Dilution: Prepare the typical part,

$$|\psi\rangle^{\otimes n} = \sqrt{\Pr(T_{Q,\delta}^n)} |\text{typ}\rangle + \sqrt{1 - \Pr(T_{Q,\delta}^n)} |\text{atyp}\rangle.$$
$$|T_{Q,\delta}^n| \leq 2^{n(S(Q)+\delta)}.$$

$C_d = C_r$: sketched roof

Achievability: type measurement + operator covering lemma + proper IC + Uhlmann's theorem

The purification $|\phi\rangle^{AE} = \sum \sqrt{q_i} |i\rangle^A |\phi_i\rangle^E$

Step 1. Perform type measurement $\{M_P\}$, $P_{\text{typ}} \approx Q$,

$$\begin{aligned} |\phi(P)\rangle^{AE} &= \frac{1}{\sqrt{|T(P)|}} \sum_{i^n \in T(P)} |i^n\rangle^A |\phi_{i^n}\rangle^E, \\ &= \frac{1}{\sqrt{M}} \sum_m |m\rangle^{A_1} \frac{1}{\sqrt{S}} \sum_s |s\rangle^{A_2} |\phi_{ms}\rangle = \frac{1}{\sqrt{M}} \sum_m |m\rangle^{A_1} |\phi(P)\rangle_m^{A_2 E}, \\ &= \frac{1}{\sqrt{(1-\epsilon)M}} \sum_{\text{good } m} |m\rangle^{A_1} |\phi(P)\rangle_m^{A_2 E} + \frac{1}{\sqrt{\epsilon M}} \sum_{\text{bad } m} |m\rangle^{A_1} |\phi(P)\rangle_m^{A_2 E}, \end{aligned}$$

“good m ”: $|\phi(P)\rangle_m^E \approx |\phi(P)\rangle^E$,

“bad m ”: $|\phi(P)\rangle_m^E \not\approx |\phi(P)\rangle^E$.

Step 2. Perform measurement $\{P_{\text{good } m}^{A_1} \otimes |A_2\rangle, P_{\text{bad } m}^{A_1} \otimes |A_2\rangle\}$,

$$\frac{1}{\sqrt{(1-\epsilon)M}} \sum_{\text{good } m} |m\rangle |\phi(P)\rangle_m^{A_2 E}.$$

For good m , by Uhlmann's theorem: $\exists U_m^{A_2}$,

$$U_m^{A_2} \otimes I^E |\phi(P)\rangle_m^{A_2 E} \approx |\phi(P)\rangle_0^{A_2 E}.$$

Step 3. Apply IC operation $\{K_S\}$

$$K_S = \sum_{\text{good } m} |m\rangle\langle m|^{A_1} \otimes |0\rangle\langle s| U_m^{A_2}.$$

$$\approx \frac{1}{\sqrt{(1-\epsilon)M}} \sum_{\text{good } m} |m\rangle\langle\phi(P)|_0^{A_2 E}.$$

Step 4. Count $|T(P)|$ and $S = S(\rho)$ [operator covering lemma].

Optimality: monotonicity + additivity + asymptotical continuity.

For any protocol \mathcal{L}_n such that $\mathcal{L}_n(\rho^{\otimes n}) \approx |\Phi_2\rangle\langle\Phi_2|^{\otimes nR}$,

$$nC_r(\rho) = C_r(\rho^{\otimes n}) \tag{1}$$

$$\geq C_r(\mathcal{L}_n(\rho^{\otimes n})) \tag{2}$$

$$\geq C_r(\Phi_2^{\otimes nR}) - 2n\epsilon \log d - 2\eta(\epsilon) \tag{3}$$

$$= nR - 2n\epsilon \log d - 2\eta(\epsilon), \tag{4}$$

$C_c = C_f$: sketched proof

Achievability: typicality + Caratheodory's Theorem

For an optimal decomposition, $\rho = \sum_i^M p_i \psi_i$, by Caratheodory's Theorem, $M \leq d^2$.

$$\rho^{\otimes n} = \sum_{i^n} p_{i^n} \psi_{i^n} = (1 - \epsilon_1) \rho(\mathcal{T}) + \epsilon_1 \rho_0,$$

\mathcal{T} is the set of typical sequences.

Prepare each state whose index sequence in the \mathcal{T} with coherence concentration.

Error: $(1 - \epsilon_1)(1 - \epsilon_2)^M$.

Consumption of $|\Phi_2\rangle$: $n(p_j + \delta_1)(S(\Delta(\psi_j)) + \delta_2)$.

Optimality: monotonicity + additivity + asymptotical continuity

For any protocol \mathcal{L}_n such that $\mathcal{L}(\Phi_2^{\otimes nR}) = \rho^{(n)} \approx \rho^{\otimes n}$.

$$nR = C_f(\Phi_2^{\otimes nR}) \tag{5}$$

$$\geq C_f(\rho^{(n)}) \tag{6}$$

$$\geq C_f(\rho^{\otimes n}) - 10n\epsilon \log d - 2\eta(\epsilon) \tag{7}$$

$$= nC_f(\rho) - 10n\epsilon \log d - 2\eta(\epsilon), \tag{8}$$

Additivity

Additivity of C_r : $C_r(\rho) = S(\Delta\rho) - S(\rho)$.

Additivity of C_f :

$$C_f(\rho) = E_f(\tilde{\rho}),$$
$$\rho = \sum_{ij} \rho_{ij} |i\rangle\langle j| \longleftrightarrow \tilde{\rho} = \sum_{ij} \rho_{ij} |ii\rangle\langle jj|,$$

$\tilde{\rho}$ is a maximally correlated state (MCS).

$$E_f(\tilde{\rho} \otimes \tilde{\sigma}) = E_f(\tilde{\rho}) + E_f(\tilde{\sigma}). [Vidal et al, PRL(2002)]$$

Criterion: sketched proof

$$\rho = \sum_{ij} \rho_{ij} |i\rangle\langle j| \longleftrightarrow \tilde{\rho} = \sum_{ij} \rho_{ij} |ii\rangle\langle jj|$$

$$C_r(\rho) = S(\tilde{\rho}^B) - S(\tilde{\rho}^{AB})$$

$$C_f(\rho) = E_f(\tilde{\rho}^{AB}).$$

Apply a lemma [Winter and Yang, arXiv:1505.00907]:

If $S(\sigma^B) - S(\sigma^{AB}) = E_f(\sigma^{AB})$, then

$$\sigma^{AB} = \bigoplus_i \rho_i \rho_i^{B_i^L} \otimes |\phi_j\rangle\langle\phi_j|^{AB_i^R},$$

$$\mathcal{H}_B = \bigoplus_i \mathcal{H}_{B_i^L} \otimes \mathcal{H}_{B_i^R}.$$

Summary

- ▶ Pure states transformation: $C(\psi) = S(\Delta(\psi))$
- ▶ Distillation: $C_d(\rho) = C_r(\rho) = S(\Delta(\rho)) - S(\rho)$.
- ▶ Cost: $C_c(\rho) = C_f(\rho)$.
- ▶ Reversible: $\rho = \bigoplus_j p_j |\phi_j\rangle\langle\phi_j|$, $\Delta(\phi_i) \perp \Delta(\phi_j)$, $i \neq j$.
- ▶ No bound coherence.

Open questions

Q1.

$$\rho = \sum_{ij} \rho_{ij} |i\rangle\langle j| \longleftrightarrow \tilde{\rho} = \sum_{ij} \rho_{ij} |ii\rangle\langle jj|,$$

Coherence theory \longleftrightarrow Entanglement theory of MCS.

Incoherent operation \longleftrightarrow LOCC ?

Q2. How to expand the restricted operations to recover reversible resource theory? [Brandao and Gour, arXiv:1502.03149.]

**Thank you
for your attention !**