Linear Index Codes are Optimal Up To Five Nodes

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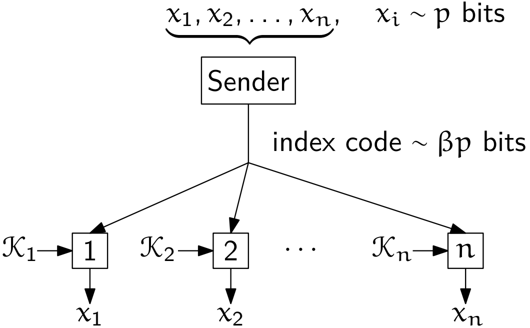
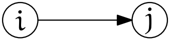
BIRS Workshop: Between Shannon and Hamming: Network Information Theory and Combinatorics

Unicast Index Coding



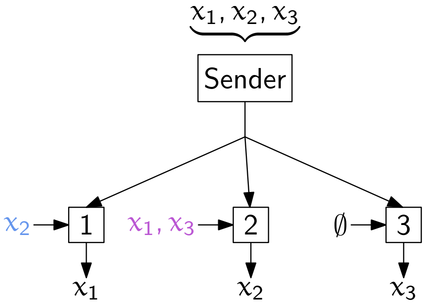
What is the minimum β, denoted as β^*?
(broadcast rate / coded packets)

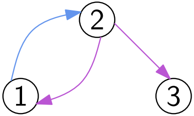
Directed Side-Information Graph



≡ Node i knows x_j

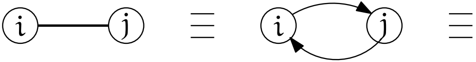
Example:





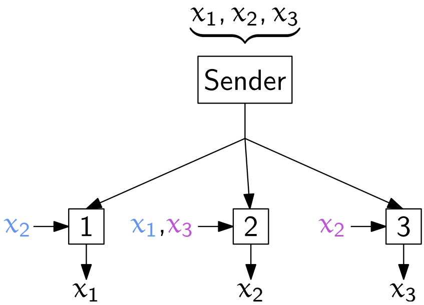
Undirected Side-Information Graph

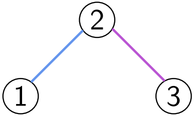
For symmetrical side information



Node i knows x_j
&
Node j knows x_i

Example:





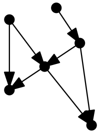
Problem Formulation Using Graphs

Index-Coding Problem

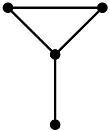
Given $\mathbb{G}$ and p, find $\beta^*(\mathbb{G}, p)$.

Problem open to date, except for special cases, e.g.,
Bar-Yossef et al. (2011):

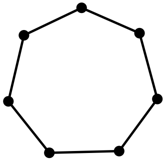
Directed
acyclic



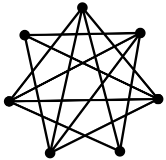
Undirected
perfect



Undirected
odd hole



Undirected
odd anti-hole



Unicast: Results for Up To Five Nodes

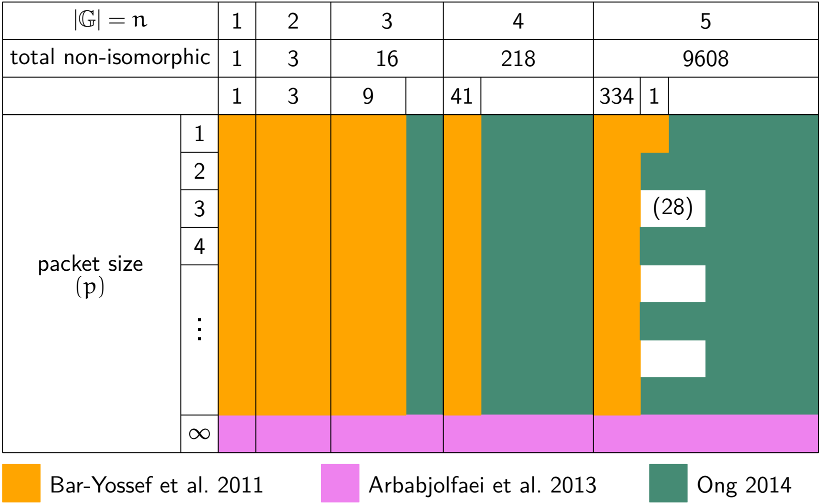
$\beta^*(\mathbb{G}, p)$ found for the following:

$\vert\mathbb{G}\vert = n$		1	2	3		4		5		
total non-isomorphic		1	3	16		218		9608		
		1	3	9		41		334	1	
packet size (p)	1									
	2									
	3									
	4									
	$\vdots$									
	∞									

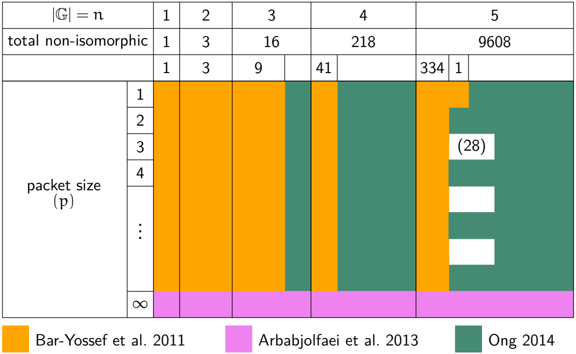
Bar-Yossef et al. 2011   Arbabjolfaei et al. 2013

Our Recent Results

We found $\beta^*(\mathbb{G}, p)$ for the following:



Up to Five Users (Ong, 2014)



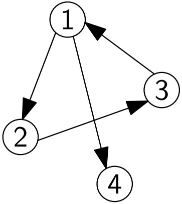
- For 9818 graphs, **bit-wise linear** encoding is optimal
- For the rest (28 graphs), double-bit linear encoding is optimal
- Results on achieving $\beta^*(\mathbb{G}, \infty)$ using sufficiently large p omitted here (Yu & Neely 2013, Arbabjolfaei & Kim 2014)

Maximum Acyclic Induced Subgraph (MAIS)

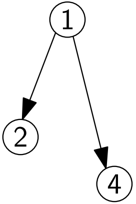
A Lower Bound to β^* (Bar-Yossef et al. 2011)

$$\mathsf{MAIS}(\mathbb{G}) \leqslant \beta^*(\mathbb{G}, p), \qquad \text{for any } p \geqslant 1$$

Example:



Graph $\mathbb{G}$



$\mathsf{MAIS}(\mathbb{G}) = 3 \leqslant \beta^*(\mathbb{G}, p)$

A Key Result (Ong, 2014)

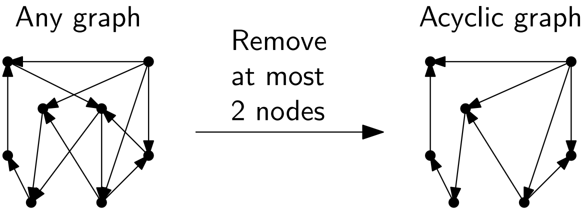
For any $\mathbb{G}$ and any p, if

$$\mathsf{MAIS}(\mathbb{G}) \geqslant n - 2,$$

then

$$\mathsf{MAIS}(\mathbb{G}) = \beta^*(\mathbb{G}, p),$$

which is achievable by bit-wise linear codes.



A brief proof for the following:

1. If $\mathsf{MAIS}(\mathbb{G}) \geqslant n - 2$, then $\mathsf{MAIS}(\mathbb{G}) = \beta^*(\mathbb{G}, p)$.

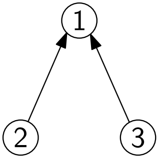
2. Linear codes are optimal up to five nodes.

Proof: If $\mathsf{MAIS}(\mathbb{G}) \geqslant n - 2$ **then** $\mathsf{MAIS}(\mathbb{G}) = \beta^*(\mathbb{G}, p)$

Case 1: $\mathsf{MAIS}(\mathbb{G}) = n$

- $\mathbb{G}$ is acyclic
- Send n symbols uncoded

Example:



(x_1, x_2, x_3)

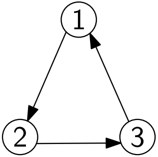
$\beta^*(\mathbb{G}, p) = 3$

Proof: If $\mathsf{MAIS}(\mathbb{G}) \geqslant n - 2$ **then** $\mathsf{MAIS}(\mathbb{G}) = \beta^*(\mathbb{G}, p)$

Case 2: $\mathsf{MAIS}(\mathbb{G}) = n - 1$

- $\mathbb{G}$ contains a cycle
- Use cyclic code on one cycle to save one symbol

Example:



$(x_1 \oplus x_2, x_2 \oplus x_3)$

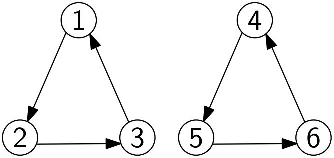
$\beta^*(\mathbb{G}, p) = 2$

Proof: If MAIS($\mathbb{G}$) $\geqslant n - 2$ **then** MAIS($\mathbb{G}$) $= \beta^*(\mathbb{G}, p)$

Case 3a: MAIS($\mathbb{G}$) $= n - 2$ & there are two disjoint cycles

- Use cyclic code on each disjoint cycle to save two symbols

Example:



$$(x_1 \oplus x_2, x_2 \oplus x_3,\ x_4 \oplus x_5, x_5 \oplus x_6)$$

$$\beta^*(\mathbb{G}, p) = 4$$

Proof: If $\mathsf{MAIS}(\mathbb{G}) \geqslant n-2$ then $\mathsf{MAIS}(\mathbb{G}) = \beta^*(\mathbb{G}, p)$

Case 3b: $\mathsf{MAIS}(\mathbb{G}) = n-2$ & there are NO two disjoint cycles

Problem: Cannot use cyclic codes to save two symbols

A Solution for Case 3b

Lemma

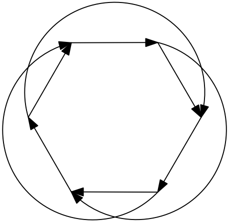
If

$$\text{MAIS}(\mathbb{G}) = n - 2$$

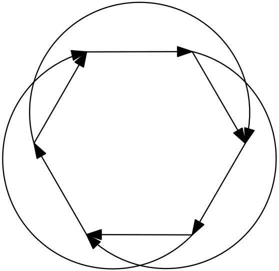
and

there are NO two disjoint cycles,

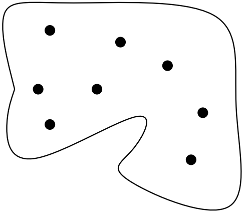
then $\mathbb{G}$ contains a subgraph of the following form:



- Suppose Lemma is true.
- Let the subgraph $\mathbb{G}'$ contain $n' \leqslant n$ nodes.



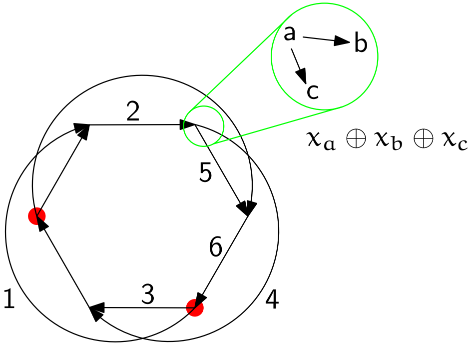
Code with $n' - 2$ symbols



Send $n - n'$ uncoded symbols

- Total code length $= (n' - 2) + (n - n') = n - 2$.
- Save two symbols.

A Coding Scheme on the Subgraph $\mathbb{G}'$



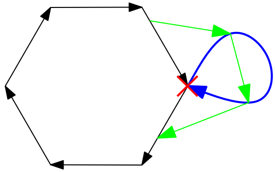
Send $x_i \quad \bigoplus_{j \in \mathcal{N}^+_{\mathbb{G}'}(i)} x_j$, for each vertex i except the two red nodes,

where $\mathcal{N}^+_{\mathbb{G}'}(i)$ = out-neighbourhood.

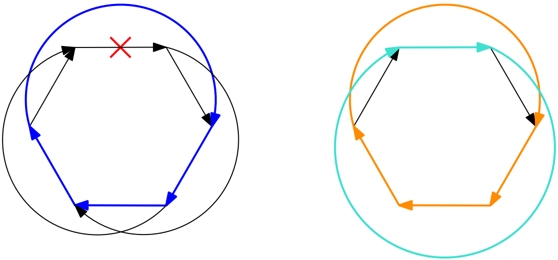
Sketch of Proof for Lemma (Existence of Subgraph)

- $\mathsf{MAIS}(\mathbb{G}) = n - 2 \quad \Rightarrow \quad \geqslant 2$ cycles
- Pick a “centre” cycle.
- No two disjoint cycles $\quad \Rightarrow \quad$ All other cycles share nodes with “centre” cycle.

$\exists$ one cycle sharing 1 node with “centre” cycle



all other cycles share $\geqslant 2$ nodes with “centre” cycle



Theorem

For any $\mathbb{G}$ and any p, if

$$\mathsf{MAIS}(\mathbb{G}) \geqslant n - 2,$$

then

$$\mathsf{MAIS}(\mathbb{G}) = \beta^*(\mathbb{G}, p) \quad \& \quad \text{bit-wise linear codes are optimal.}$$

Proof: Linear Codes Are Optimal for $n \leqslant 5$

For any $\mathbb{G}$, $1 \leqslant \text{MAIS}(\mathbb{G}) \leqslant n$

- $n = 1, 2, 3$
  $\Rightarrow \text{MAIS}(\mathbb{G}) \geqslant n - 2 \Rightarrow \beta^*(\mathbb{G}, p)$ found

- $n = 4$
  - If $\text{MAIS}(\mathbb{G}) = 2, 3, 4$
    $\Rightarrow \text{MAIS}(\mathbb{G}) \geqslant n - 2 \Rightarrow \beta^*(\mathbb{G}, p)$ found
  - Else ($\text{MAIS}(\mathbb{G}) = 1$)
    $\Rightarrow$ Any two nodes contains a cycle
    $\Rightarrow$ Each node i knows all $x_j, j \neq i$
    $\Rightarrow \text{MAIS}(\mathbb{G}) = 1$ achievable using $x_1 \oplus x_2 \oplus \cdots \oplus x_n$
    $\Rightarrow \beta^*(\mathbb{G}, p)$ found

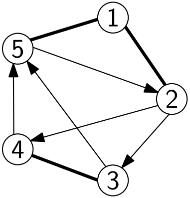
Proof: Linear Codes Are Optimal for $n \leqslant 5$

For $n = 5$

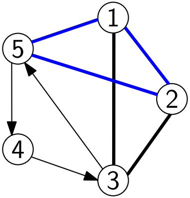
- If $\text{MAIS}(\mathbb{G}) = 3, 4, 5$
  $\Rightarrow \text{MAIS}(\mathbb{G}) \geqslant n - 2 \Rightarrow \beta^*(\mathbb{G})$ found
- If $\text{MAIS}(\mathbb{G}) = 1$
  $\Rightarrow$ Any two nodes contains a cycle
  $\Rightarrow x_1 \oplus x_2 \oplus \cdots \oplus x_n$ optimal
- Need to solve $(n = 5)$ & $(\text{MAIS} = 2)$

$(n = 5)$ & $(\mathsf{MAIS} = 2)$: **Classification**

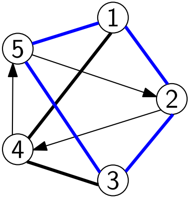
Case 1: No undirected cycle



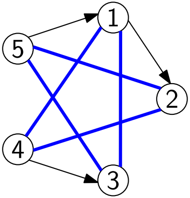
Case 2: ∃ undirected 3-cycle



Case 3: No undirected 3-cycle
∃ 4-cycle



Case 4: No undirected 3-, 4-cycle
∃ 5-cycle



$(n = 5)$ & $(\mathsf{MAIS} = 2)$: **Lemmas**

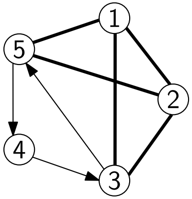
Lemma 1

Any induced subgraph with 3 nodes must contain a cycle.

Lemma 2

Any induced subgraph with 4 nodes must contain an (undirected) edge.

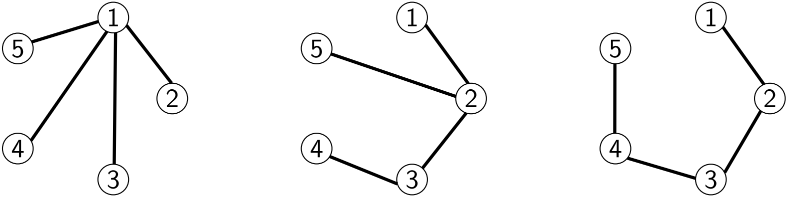
Example:



$(n = 5)$ & $(\mathsf{MAIS} = 2)$: Necessary Arcs and Edges

Example: No undirected cycle and 4 edges

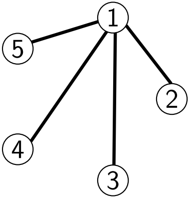
Only 3 non-isomorphic configurations:



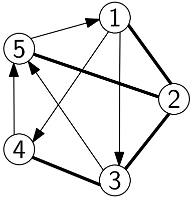
$(n = 5)$ & $(\mathsf{MAIS} = 2)$: Necessary Arcs and Edges

Example: No undirected cycle and 4 edges

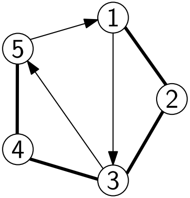
Applying Lemmas 1 and 2:



$\{2, 3, 4, 5\}$ has no edge
not possible



$x_3 \oplus x_4 \oplus x_5$
$x_1 \oplus x_2 \oplus x_5$



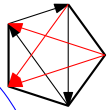
four sub-cases

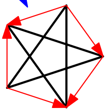
$(n = 5)$ & $(\text{MAIS} = 2)$:

- Classified graphs according to undirected cycles
- Need to consider only 13 configurations

number of nodes (n)		1	2	3	4	5		
total non-isomorphic		1	3	16	218	9608		
							8	20
packet size (p)	1						$\beta^* = 3$	
	2							
	$\vdots$							
	∞							

MAIS lower bound is tight





$\beta^* = 2.5$

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Thank you!