Women in Geometry

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1 Overview

There is currently an underrepresentation of women in the mathematics faculty of Ph.D. granting universities, a relative underrepresentation of women in Geometry/Topology and low visibility of women at Geometry conferences. For example, in the last fifteen years roughly 30% of all mathematics Ph.D.s in the United States have been awarded to women, with only 20% of these degrees awarded in Geometry/Topology to women. Despite this, the percentage of tenure-track and tenured female math faculty members at Ph.D. granting institutions in the U.S. is only 11%. (Statistics from Proceedings of the National Academy of Sciences publications as well as the Annual Survey of the Mathematical Sciences.)

The overarching goal of the Women in Geometry (WIG) workshop was to increase the strength and visibility of the community of women geometers. The workshop hosted seven different research teams with anywhere from four to seven women each in the following areas: Calibrated geometry, Geometric aspects of soliton equations, Metric geometry, Minimal submanifolds, Ricci flow, Spectral geometry and Symplectic geometry. Led by women established in these areas of geometric research, each team worked together on an open problem in their respective area. Before the program began team leaders provided participants with a synopsis of the research problem and necessary background reading materials. Once at BIRS, the majority of the time was spent in working groups pursuing collaborative research. Further background was developed through team seminars held during the first day, as needed. Seven plenary talks, one for each group, were scheduled during the first four mornings to provide attendees with new insights on current trends in geometry, to build a feeling of community, to promote discussion and collaboration within and between groups, as well as to inspire participants to consider expanding and broadening their own research programs. On the last day of the program each team reported back to the full workshop on progress made and future goals.

Other highlights of the program included "Open Problem Sessions" scheduled on Monday evening and a "Group Discussion" on Tuesday evening. For the Open Problem Sessions, each team created a problem list, much as is done at AIM conferences (see the following link: http://aimpl.org). Participants were encouraged to think about open problems in their area, and try to arrive at the worksop with at least one problem to add to the team's problem list, or a place in their work where they would welcome ideas from others at WIG. Part of the motivation for doing this was to encourage teams to think about possible future collaborations, both with their own team members and as well as with members of other teams. In addition to the talks from each area, the exercise helped all participants gain a better understanding of what problems are open in each area as well as a better idea of what questions are of interest. The Group Discussion on Tuesday evening was an informal, but guided, conversation on the current status of women in mathematics with a view to finding ways to promote change. Topics included funding and job opportunities, improving the representation of women as speakers at conferences, how to lobby for funding for childcare to attend conferences, how to combat bias on award panels, and the mentoring of female graduate students as well as women in the early stages of their careers.

2 Presentation Highlights

There were seven plenary talks scheduled on the first four mornings of the workshop. A representative or representatives from each team gave a colloquium style lecture related to their team's research problem in one of these seven hour-long talks. The talks were accessible to all WIG participants for at least the first 30-40 minutes. A goal of WIG was for participants to broaden their horizons a bit and get to know what are important problems in each other's areas. WIG aimed to establish strong working relationships within teams. These lectures reached beyond this to lay the groundwork for possible cross-team collaboration.

Christine Guenther from Pacific University, Casey Kelleher from U.C. Irvine, and Xuan Hien Nguyen from Iowa State University, representing the Ricci Flow group, spoke on *An overview of geometric flows, Renormalization group flow, Self-similar surfaces under mean curvature flow, and Higher order Yang-Mills flow*.

Carolyn Gordon from Dartmouth College, representing the Spectral Geometry group, spoke on *The Steklov problem on orbifolds*.

Dusa McDuff from Barnard College and Columbia Unversity, representing the Symplectic Geometry group, spoke on *Symplectic and contact geometry*.

Gloria Mari-Beffa from the University of Wisconsin, representing the Geometric Aspects of Soliton Equations group, spoke on *Geometric realizations of completely integrable PDEs*

Chikako Mese from Johns Hopkins University, representing the Minimal Submanifolds group, spoke on *Harmonic maps and its generalization in singular geometry*.

Sema Salur from the University of Rochester, representing the Calibrated Geometries group, spoke on *Manifolds with special holonomy*.

Christina Sormani from CUNY Graduate Center, representing the Metric Geometry group, spoke on *Metric Geometry and Rectifiability*.

3 Scientific Progress Made

The majority of the time during the workshop was spent in working groups pursuing collaborative research. Here we summarize scientific progress made by each team. Each entry in the list contains: the team research area, the team members (with team leaders underlined), a description of the area, and a summary of the project and progress made during the workshop.

3.1 Calibrated geometries

Members: Jeanne Clelland (University of Colorado, Boulder), Rebecca Glover (University of Rochester), Eleonora Di Nezza (Imperial College), Kimberly Moore (University of Cambridge), Colleen Robles (Duke University), and Sema Salur (University of Rochester)

Area: A calibrated manifold is Riemannina manifold (M, g) of dimension n equiped with a p-form ϕ whihc is closed and such that $\phi|_\xi = \lambda \mathrm{vol}_\xi$ with $\lambda \leq 1$ for any oriented p-dimensional subspace ξ of the tangent space to M, where vol_ξ is the volume form of ξ with respect to the metric g. The theory of calibrations is due to Harvey and Lawson and others.

Summary: The team made progress on two problems. Let (M, ϕ) be a G_2–manifold.

(1) Sema Salur has conjectured the existence of a natural contact structure on M which is compatible with ϕ. There are at least two different possible constructions/definitions of the desired contact structure. The team made partial progress toward demonstrating the viability of the first construction in the flat case $M = \mathbb{R}^7$.

(2) The associative 3–folds $X \subset M$ are characterized by an exterior differential system (EDS) $\mathcal{I} \subset \Omega(M)$. Associated to that system is a characteristic cohomology (CC). The CC gives obstructions to the following problem. Given a closed surface $S \subset M$, when can the surface be realized as the boundary of an associative submanifold X? The CC yields "moment conditions" on S which obstruct the existence of X. The team studied the flat case $\mathbb{R}^7$ and showed that there exist both cohomology obstructed and unobstructed surfaces.

3.2 Geometric aspects of soliton equations

Members: Annalisa Callini (College of Charleston), Katrin Leschke (University of Leicester), Gloria Mari-Beffa (University of Wisconsin), and Jing Ping Wang (University of Kent)

Area: Soliton equations are non-linear evolution PDEs that admit many remarkable properties. Many soliton equations can be constructed from Lie algebra splittings. Terng and Uhlenbeck showed that the properties of soliton equations can be obtained in a unified way from Lie algebra splittings. Soliton equations occur naturally in submanifold geometry and in geometric curve flows.

Summary: There were two problems the group discussed, with some progress. The first was the geometric interpretation of the Miura transformation at the curve evolution level. The KdV equation has a geometric realization as a projective flow of curves, meaning that as the curves flow following the geometric realization, their projective invariant flows following KdV. This can be generalized to Adler-Gel'fand-Dikii flows in $\mathbb{RP}^n$. The so-called Miura transformation $u \to v' - v^2$ takes solutions of KdV to solutions of mKdV. On the other hand, mKdV does not have geometric realizations in projective space, only in Euclidean spaces. The question was: what is the interpretation of the Miura transformation at the curve level? Early on, the group noticed that if u is a projective invariant of a curve, then v given as $u = v' - v^2$ is invariant only under an affine subgroup of the projective group PSL(2). Since then we have generalized it to $\mathbb{RP}^n$ and we have identified the Miura map at the level of curve evolutions. We are now trying to find a measure that will tie the new moving frame to a parallel frame usually associated to mKdV realizations.

Since Chuu-Lian Terng did not attend, Katrin Leshke was left a little isolated (she works in problems closer to Chuu-Lian's), so instead of working on the other problems we had proposed, we looked for one closer to Katrin's area of specialization. She has been working on Darboux transforms of surfaces, while JP Wang has worked on Darboux transforms of integrable PDEs. We had long discussions about how to bridge the two approaches, and we saw some similarities that we plan to pursue. We did not have any breakthroughs, but we learned a lot from Katrin.

3.3 Metric geometry

Members: Maree Jaramillo (University of Cennecticut), Raquel Perales (SUNY at Stonybrook), Rajan Priyanka (U.C. Riverside), Catherine Searle (Wichita State University), Anna Siffert (Max-Planck-Institut), and Christina Sormani (Lehman College and CUNY Graduate Center)

Area: Metric geometry has blossomed from its origins when Gauss first defined curvature, to the study of Alexandrov spaces of curvature bounded from below, and most recently to the application of Gromov-Hausdorff convergence in Perelman's proof of the Poincare Conjecture. There are many open questions in this active field and new synthetic notions of convergence are being developed.

Summary: The Metric Geometry group considered questions of intrinsic flat convergence and Alexandrov spaces during the workshop. The intrinsic flat convergence, first defined by Sormani and Wenger for oriented countably H^m rectifiable metric spaces has more flexibility than Gromov-Hausdorff convergence.

All Alexandrov spaces are countably H^m rectifiable, but not necessarily orientable, however using the existence of ramified orientable double covers, which themselves are Alexandrov spaces with the same lower curvature bound defined in Harvey and Searle, one can pass to such spaces to consider the intrinsic flat convergence.

In particular, they studied the convergence of Alexandrov spaces with curvature bounded below with respect to intrinsic flat convergence, to determine whether the Gromov-Hausdorff limits and intrinsic flat limits agree when they exist, and to determine what properties are conserved under this limit.

The problem breaks up naturally into two problems.

Problem 1. *Prove that every oriented Alexandrov space is an integral current space.*

The notion of integral current space as in Sormani and Wenger extends the notion of a Riemannian manifold with boundary: it involves finding a bi-Lipschitz collection of charts almost everywhere which preserve the orientation and then proving the boundary has finite mass, where the mass is defined to be the mass of the current structure.

Problem 2. *Prove that the Gromov-Hausdorff and Intrinsic flat limits of oriented Alexandrov spaces without boundary that are non-collapsing and have uniformly non-negative curvature agree.*

This has been proven for Riemannian manifolds with non-negative Ricci curvature in Sormani and Wenger. Major progress on Problem 2 is found in Li and Perales, where they consider integral current spaces with non-negative Alexandrov curvature. Problem 2 follows once they have proven Problem 1.

The time spent at the workshop was used to consider the case where the Alexandrov space X has no boundary. The group made significant progress on this question and hopes to complete this case soon, which they will then publish in a peer-reviewed journal. A large number of the participants plan to then collaborate to investigate the case where the Alexandrov space has boundary. We note that this will also require generalizing the result in Li and Perales to include the case where X may have boundary.

3.4 Minimal submanifolds

Members: Christine Breiner (Fordham Unviersity), Ailana Fraser (University of British Columbia), Lan-Hsuan Huang (University of Connecticut), Chikako Mese (Johns Hopkins University), Pam Sargent (University of British Columbia), Karen Uhlenbeck (University of Texas Austin and Institute for Advanced Study), and Yingying Zhang (Johns Hopkins University)

Area: Solutions to variational problems are critical points of an energy functional and as such are deeply connected with physical phenomenon. Understanding these solutions is a foundational aspect of geometric analysis and has classical roots dating back to the ancient Greeks. As a particular example, minimal surfaces are critical points for area and on sufficiently small scales are area minimizers.

Summary: Our goal is to prove the following conjecture:

Conjecture: *Let Σ be a compact Riemannian surface, X a compact CAT(1) space and $\varphi : \Sigma \to X \in C^0 \cap W^{1,2}$. Then there exists a harmonic map $u : \Sigma \to X$ homotopic to φ or a conformal harmonic map $v : \mathbb{S}^2 \to X$.*

CAT(1) spaces are metric spaces with an upper curvature bound determined via comparison triangles on the unit sphere.

In the smooth setting, the conjecture was first shown by Sacks-Uhlenbeck in 1981. They used a perturbed energy functional that implied convenient Sobolev embedding theorems. They then had to establish that the solutions to the perturbed energy problems converged to a harmonic map as the perturbed energy converged to the Dirichlet energy. Among other things, their work relied on the Palais-Smale theory for the perturbed functional. Jost demonstrated that one may instead use the idea of *harmonic replacement*. Jost's argument also used the Palais-Smale condition but we noticed that this is unnecessary. Our argument demonstrates that the main tools needed for harmonic replacement are: existence of a minimizer for the Dirichlet problem on sufficiently small balls, convexity of the energy, and a compactness theory for uniform limits of harmonic maps on small balls.

During this week, we worked out the necessary steps to prove each of the previously mentioned elements. We also noted that we should carefully prove the regularity theory for minimizers. We expect to produce at least two papers based on this project. The first paper will address the existence and regularity of Dirichlet solutions. The second paper will address the previously mentioned conjecture. We might also attempt to answer questions of energy quantization in a third paper. Our time together was incredibly productive and fruitful and we are grateful to have had the opportunity to spend a week together at BIRS.

3.5 Ricci flow

Members: Christine Guenther (Pacific Unviersity), Casey Kelleher (U.C. Irvine), Xuan Hien Nguyen (Iowa State University), and Guofang Wei (U.C. Santa Barbara)

Area: The Ricci flow is informally the process of stretching the metric in directions of negative Ricci curvature, and contracting the metric in directions of positive Ricci curvature, as a way of smoothing out irregulairties in the metric. It is the primary tool used in Hamilton-Perelman's solution of the Poincar conjecture, as well as in the proof of the differentiable sphere theorem by Brendle-Schoen.

Summary: In the field of geometric flows, one studies the evolution of geometric objects in time. The flows have extensive applications to physical problems such as crystal growth and eroding stones, as well as geometric problems such as the Thurston Geometrization conjecture. Important examples include the curve shortening flow, the mean curvature flow, and the Ricci flow. A prototype for many analytical techniques is the heat equation.

Our Geometric Flow group was unique in that, due to cancellations before the BIRS workshop, each of the four members of the group brought ideas for problems on which to work (instead of working on a problem that was posed beforehand). Throughout the week we discussed the feasibility of these projects, and decided to work on four of them. To this end we have set up a Dropbox in order to easily share information at a distance. The problems are as follows:

Problem A: Given an entire graph u over $\mathbb{R}^2$ that satisfies the equation for self-translating surfaces under mean curvature flow:

$$\sqrt{1+|Du|^2}\mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 1,$$

must the graph be the rotationally symmetric "bowl soliton"?

This Bernstein-type problem for self-translating hypersurfaces in $\mathbb{R}^{n+1}$ is solved in dimension $n \geq 3$ (the answer is no). In dimension $n = 2$, the conclusion holds under the additional assumption that the initial surface $u(\mathbb{R}^2)$ is non-collapsed (Haslhofer) or convex (Wang).

Progress: We wrote the equation in polar coordinates and decided to study possible energy functionals and monotonicity formulas in order to control the area growth.

Problem B: Ricci Flow for Smooth Metric Measure Space

One may consider the flow

$$\frac{\partial g}{\partial t} = -2Rc - 2\nabla\nabla f$$
$$\frac{\partial f}{\partial t} = \Delta f - |\nabla f|^2.$$

Is positivity of the Bakry-Emery Ricci tensor $Rc + \nabla\nabla f$ preserved in dim $n = 3$?

In Hamiltons original paper in 1982, he showed that the positivity of the Ricci tensor is preserved under the Ricci flow, by applying the tensor maximum principle to the evolution equation of the Ricci tensor.

Progress: During the BIRS workshop we considered the evolution equations of the curvatures, and decided to work with weighted differential operators and try to apply the maximum principle.

Problem C: The second order Renormalization Group flow (RG-2) equation is a nonlinear perturbation of the Ricci flow that arises in quantum field theory, and is given by

$$\frac{\partial}{\partial t}g = -2Rc - \frac{\alpha}{2}Rm^2.$$

Here $\alpha > 0$, Rc is the Ricci tensor, and $Rm^2_{ij} := R_{iklm}R_j^{klm}$. The fixed points of the RG-2 flow in dimension $n = 3$ include the space forms, as well as the product spaces $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$ (allowing $\alpha < 0$).

In the cases of Nil, Sol, and $\mathrm{SL}(2, \mathbb{R})$, the flow exhibits dichotomous behavior depending on the initial conditions, with a curve separating the "Ricci flow" type behavior from asymptotic solutions where all directions collapse to zero. Are the dividing curves for Nil, Sol, and $\mathrm{SL}(2, \mathbb{R})$ self-similar solutions? Can one use this flow to construct a flow whose fixed points are locally homogeneous spaces?

Progress: During the Workshop we decided to write out the soliton equation for the geometries in question, and see if the dividing curves could be shown to satisfy them, as well as writing out the 3-loop flow in dimensions $n = 4$.

Problem D: We decided to investigate the functional

$$\int |\nabla Rm|^2,$$

both varying the connection and the metric, to see if the resulting flow has interesting properties; in particular, it may be useful for finding geometric structures such as symmetric spaces.

Progress: We will proceed with calculations guided by previous work on the variation of $\int |Rm|^2$ and the Yang Mills functional.

3.6 Spectral geometry

Members: Teresa Arias-Marco (Universidad de Extremadura), Emily Dryden (Bucknell University), Carolyn Gordon (Dartmouth College), Asma Hassannezhad (Max-Planck Institute for Mathematics), Allie Ray (Trinity College), Liz Stanhope (Lewis and Clark College)

Area: A central question in Spectral Geometry is: "What information about a geometric object is encoded in the eigenvalue spectrum of the Laplace operator acting on differentiable functions on that object?" This compelling question, often phrased as "Can you hear the shape of a manifold?", has yielded a rich area of research.

Summary: Our project was to consider the Steklov problem on orbifolds.

Our first goal was to obtain an upper bound on Steklov eigenvalues in the orbifold setting. We considered previous results by Hassannezhad in the Riemannian setting, and showed that the main arguments in the proof could be adapted to the orbifold setting for dimension two. One tool in this proof is the Ricci flow. Consultation with the Women in Geometry Geometric Flow team greatly expedited our solution of an important case. This gives us the main strategy for generalizing this result, although we still need to consider more minor details. In the future, we intend to publish this result as well as considering the higher dimensional case.

Our second goal was to consider whether the Steklov spectrum on 2-orbifolds determines the number of boundary components as well as their lengths. We looked at a similar result by Girouard-Parnovski-Polterovich-Sher in the Riemannian setting, and are currently working through their proof, adapting steps as needed to the orbifold setting. Once obtained, this result will distinguish 2-orbifolds with an odd number of singular boundary components from Riemannian surfaces.

The group has applied for support to the Max Planck Institute in Bonn, Germany, and received an informal acceptance, in order to meet together again in the summer of 2016 for future collaboration.

3.7 Symplectic geometry

Members: Maia Fraser (University of Ottawa), Ailsa Keating (Columbia Unversity), Joan Licata (Australian National University), Dusa McDuff (Barnard College, Columbia University), Sheila Sandon (University of Strasbourg), and Lisa Traynor (Bryn Mawr College)

Area: Symplectic and contact structures are closely related geometric structures that can be put on even (resp. odd) dimensional smooth manifolds. By Darboux's theorem these structures are locally unique so that all invariants are global in nature. Moreover in each case the group of structure-preserving transformations has a rich structure.

Summary: The symplectic geometry group discussed a variety of topics that eventually converged to three questions. Overlapping teams from our group are interested in pursuing each of these.

1. Symplectic Embeddings

   One of the classical results in symplectic geometry is Gromov's Non-Squeezing Theorem for symplectic balls. In dimension 4, interesting embedding results due to McDuff exist for ellipses, and we began by reviewing a recent result addressing the "stabilized" problem: for what A can $E(1, S, T)$, embed symplectically into $B^4(A) \times \mathbb{R}^2$? Hind gives an explicit construction which is close to optimal, and we hope to generalize his technique to high dimensions in which essentially nothing is currently known.

2. Legendrian norms

   Two of the members of the group (Fraser & Sandon) have studied metrics on the group of contactomorphisms of a fixed contact manifold. In-progress work of Traynor and a collaborator associates to a pair of cobordant Legendrian submanifolds the minimal length of a Lagrangian cobordism between them. We have developed a potential connection between these two topics: to a contactomorphism, we may associate a pair of cobordant Legendrian submanifolds of a higher-dimensional contact manifold. We conjecture that the length associated to these submanifolds is bounded from below by a function of the norm of the original contactomorphism.

3. (Non-)squeezing and metric properties of the group of contactomorphisms

   Prior work of McDuff and others in the symplectic setting can be interpreted as addressing this relationship, and in the contact setting, fundamental work of Eliashberg-Kim-Polterovich establishes a connection between null homotopies of certain loops of contactomorphisms and squeezing of domains in a related contact manifold. We are looking at shorter homotopies that reduce one of the existing norms (or related real-valued length) of loops and whether this corresponds to new squeezing phenomena.

4 Outcome of the Meeting

Each of the research teams made significant progress on their research problems during the workshop. The teams plan to continue their collaborations, and resulting publications are expected.

Participants benefited on the individual level by building background knowledge on a new problem, by strengthening and broadening their research programs, and, in some cases, by being provided with a re-entry point after being sidetracked by any or all of family duties, high service loads or high teaching loads. By building teams that included women at all career stages, from advanced graduate students and recent Ph.D.s to associate professors seeking to invigorate their research programs to senior researchers, the workshop formed mentoring and collaborative networks that will strengthen the careers of all participants. All attending gained an overview of seven exciting areas of current research in geometry, and all contributed to progress in their own area.

The community of women geometers was strengthened by WIG's supportive research community, mentorship of women just beginning or at the middle of their research careers, and the new collaborative links forged between women geometers working within and between their respective areas of specialization. It is important to note that the areas of geometry featured in the WIG program are strongly interrelated, so the potential for cross-area collaboration is high and at least one group of women from three different teams have told us of their intention of forging a future collaboration. The visibility of the community of women geometers was increased by highlighting the work of established female leaders in geometry, by bringing attention to the work of outstanding new women geometers, and, very simply, by having this many women together to do geometry research.