Delay-Doppler Channel Estimation – Complexity Viewpoint

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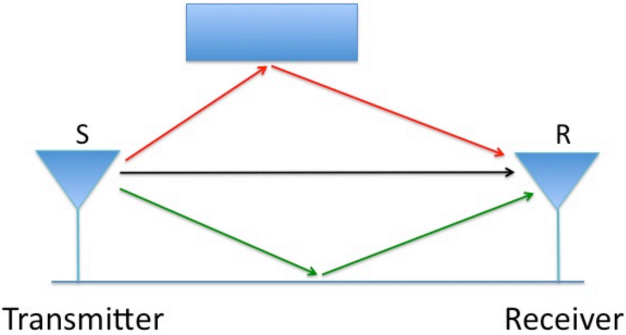
University of Sydney

Banff, 28 January 2015

Channel Model



Assumption 1: Physical channel is sparse, i.e., the number of "paths" is "small".

Channel Model

Assumption 2: carrier frequency $f_c \gg W$ – bandwidth.

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Conclusion: If $S_A(t), R_A(t) \in L^2(\mathbb{R})$ are sent and received signals in baseband $[\frac{-W}{2}, \frac{W}{2}]$, then

$$R_A(t) = \sum_{k=1}^{r} \alpha_k exp(2\pi i f_k t) S_A(t - t_k) + \mathcal{W}(t),$$

$f_k \in \mathbb{R}$ – Doppler shift along path k,

$t_k \in \mathbb{R}_+$ – delay along path k,

$\alpha_k \in \mathbb{C}$ – attenuation coefficient of path k,

$\mathcal{W}$ – additive (white) noise.

Channel Estimation Problem

Delay-Doppler Channel Estimation Problem: Given S_A (we can design our own signal!!!), and R_A satisfying

$$R_A(t) = \sum_{k=1}^{r} \alpha_k exp(2\pi i f_k t) S_A(t - t_k) + \mathcal{W}(t),$$

estimate (t_k, f_k), $k = 1, \ldots, r$.

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Applications: Radar, GPS, Wireless Communication.

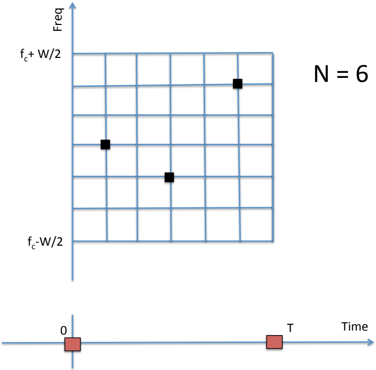
Discrete Channel Model

Discretization (Nyqvist-Shannon method): Given N, and $S \in \ell^2(\mathbb{Z}_N)$ at the transmitter, we obtain at the receiver $R \in \ell^2(\mathbb{Z}_N)$ satisfying

$$R(n) = \sum_{k=1}^{r} \beta_k e(\omega_k n) S(n - \tau_k) + \mathcal{W}(n), \; n \in \mathbb{Z}_N,$$

provided that all $t_k \in \frac{1}{W}\mathbb{Z}$, and $f_k \in \frac{W}{N}\mathbb{Z}$. Moreover, $|\beta_k| = |\alpha_k|$, and $\tau_k = t_k W$, and $\omega_k = f_k \frac{N}{W}$.

Discrete Channel Model



Assumption 3: All delay-Doppler shifts of the physical channel are on the lattice $\frac{1}{W}\mathbb{Z} \times \frac{W}{N}\mathbb{Z}$ in the time-frequency plane.

Channel Model and Problem Formulation

Conclusion: Transmitting $S \in \ell^2(\mathbb{Z}_N)$ we obtain at the receiver $R \in \ell^2(\mathbb{Z}_N)$ satisfying:

$$R(n) = \sum_{k=1}^{r} \beta_k e(\omega_k n) S(n - \tau_k) + \mathcal{W}(n), \; n \in \mathbb{Z}_N,$$

$(\tau_k, \omega_k) \in \mathbb{Z}_N \times \mathbb{Z}_N$ - time-frequency shift of path k,

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r - channel sparsity,

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Channel Estimation Problem: Design S, and method to **extract** (τ_k, ω_k)'s, using S, and R.

NOISE FILTERING - Ambiguity Function

Definition

Ambiguity Function

$$\begin{cases} \mathcal{A}(R,S) : \overbrace{\mathbb{Z}_N \times \mathbb{Z}_N}^{\text{Delay-Doppler}} \to \mathbb{C}, \\ \mathcal{A}(R,S)[\tau,\omega] = \langle R(n)\ ,\ e(\omega n) \cdot S(n-\tau) \rangle\,. \end{cases}$$

- $\mathcal{A}(R,S) \approx \mathcal{A}(R_F,S)$ where $R_F = R - \mathcal{W}$.

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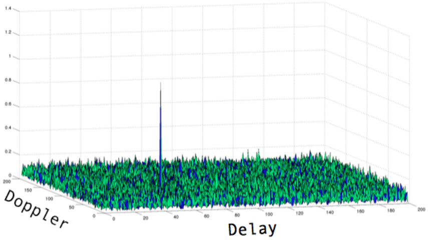
- $\mathcal{A}(R,S) \approx \mathcal{A}(R_F,S)$ where $R_F = R - \mathcal{W}$.
- $\Longrightarrow$ Reduction to R noiseless.

Pseudo Random Method

- S = pseudo-random.



$$|\mathcal{A}(R,S)|,\ r=1,\ (\tau_1,\omega_1)=(50,50).$$

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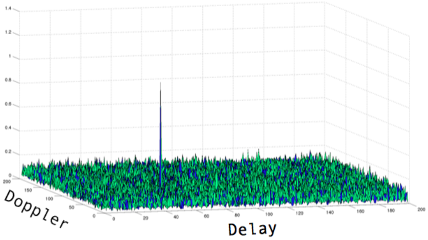

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- Using FFT compute $\mathcal{A}(R,S)$ in $O(N^2 \log N)$.

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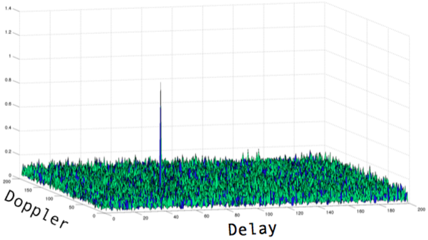

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- Using FFT compute $\mathcal{A}(R,S)$ in $O(N^2 \log N)$.

- **Question:** Can we design S and method which has lower complexity?

Sequence Design - Heisenberg (Chirp) Sequences

- Heisenberg operators

$$\pi(\tau,\omega)S(n) = e\left(\frac{-\tau\omega}{2}\right)e(\omega n)\cdot S(n-\tau), \qquad \tau,\omega \in \mathbb{Z}_N.$$

$\pi(\tau,\omega)$ — up to a scalar is time-frequency shift.

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- Take line $L \subset \mathbb{Z}_N \times \mathbb{Z}_N$, then

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- Basis of eigenfunctions $C_{L,b}$ of norm one, $b \in \mathbb{Z}_N$, satisfying

$$\pi(\ell)C_{L,b} = \underbrace{e_b(\ell)}_{\text{e.v.}}C_{L,b}, \quad \ell \in L.$$

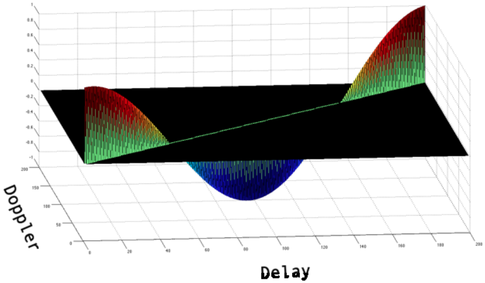
Sequence Design - Heisenberg (Chirp) Sequences

Proposition

We have

$$\mathcal{A}(C_{L,b}, C_{L,b})[v] = \begin{cases} e_b(v), & v \in L; \\ 0, & v \notin L. \end{cases}$$

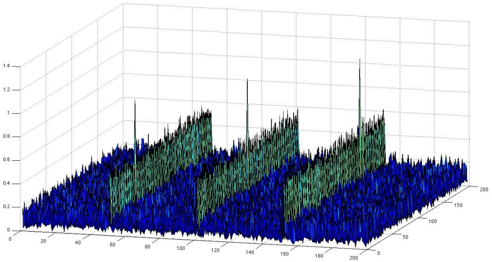
and for $L \neq M$ we have $|\mathcal{A}(C_L, C_M)| = \frac{1}{\sqrt{N}}$.



$$\mathcal{A}(C_D, C_D), \quad D = \{(\tau, \tau)\}.$$

Flag Algorithm - F., Gurevich, Hadani, Sayeed, Schwartz

- For every line $L \subset \mathbb{Z}_N \times \mathbb{Z}_N$ we construct flag sequence $S = \psi + S_L$, where ψ is a pseudo-random sequence, and S_L is the chirp sequence associated with L.



$|\mathcal{A}(R, S)|$, $L = \{(0, \omega)\}$, and three delay-Doppler shifts.

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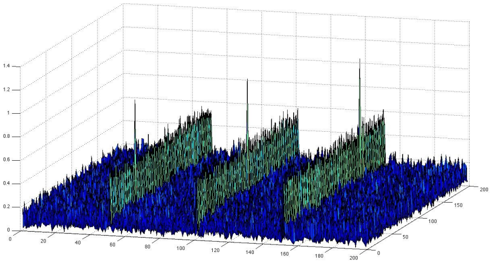

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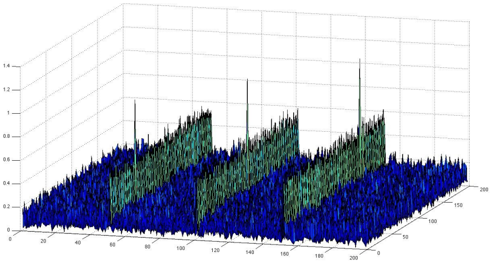
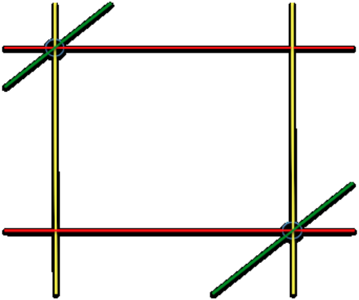


$|\mathcal{A}(R,S)|$, $L = \{(0,\omega)\}$, and three delay-Doppler shifts.

- **Complexity:** $O(rN \cdot \log(N))$.
- **Question:** Can we design algorithm with almost linear complexity for $r = o(N)$?

Incidence Algorithm – Idea

Idea: 3 generic slops and r points configuration



3 slops and $r = 2$ points.

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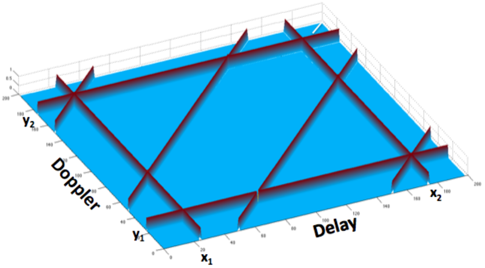

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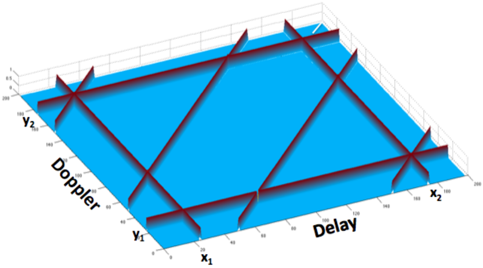

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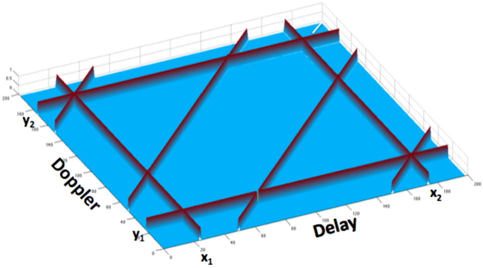

Profile of $\mathcal{A}(R, S)$ for $r = 2$

- Find delay-Doppler points with the incidence $= 3$.
- **Complexity:**
  $O(N \log N + r^2 + r \log r + r^2 \log r) = O(N \log N + r^2 \log r)$

Sublinear complexity?

Sparse Fourier Transform (Hassanieh, Indyk, Katabi, Price 2012): Given $R \in \mathbb{C}(\mathbb{Z}_N)$ having sparsity r in frequency, i.e., $|supp\left(\hat{R}\right)| = r$, there exists a randomised algorithm of complexity $O(r \log N)$ which finds all frequencies of R.

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If we are interested in r highest frequencies for not necessarily sparse function, then there exists a randomised algorithm of complexity $O(r \log(N) \log\left(\frac{N}{r}\right))$.

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Conclusion: The incidence algorithm can be done in complexity $O(r \log(N) \log\left(\frac{N}{r}\right) + r^2 \log r)$.

Performance – PR method

Assumption 1 (uniformity of attenuation coefficients):
The attenuation coefficients of the channel operator $\sum_{k=1}^{r} \beta_k \pi(\tau_k, \omega_k)$ are chosen uniformly on the complex unit sphere

$$S_{\mathbb{C}}^{r-1} = \{(\beta_1, \ldots, \beta_r) \mid \sum_{k=1}^{r} |\beta_k|^2 = 1\}.$$

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Assumption 2: (square root cancellation of the noise):
For any $\varepsilon > 0$ $\exists C > 0$ such that for any N^2 vectors $u_1, \ldots, u_{N^2} \in S_{\mathbb{C}}^{N-1}$ we have

$$P(|\langle \mathcal{W}, u_j \rangle| \leq N^{-\frac{1}{2}+\varepsilon}, j = 1, \ldots, N^2) \geq 1 - e^{-CN}.$$

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Assumption 3 (independence):
The attenuation coefficients and the noise are independent.

Performance – PR method

Probability of detection:

$$P_D = Prob(\text{a true time-frequency shift is detected}) = \frac{1}{r}\int_{\Sigma} \mathcal{N}_t(\sigma)dP(\sigma),$$

where $\mathcal{N}_t$ – number of true time-frequency shifts detected.

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Theorem (F., Gurevich)

For any $\delta > 0$, if $r \leq N^{1-\delta}$ then $P_D \to 1$, and $E_{FT} \to 0$ as $N \to \infty$, for PR Method with threshold equal to $N^{-\frac{1}{2}+\frac{\delta}{4}}$.

Further research — Open Questions

Question (complexity): Is there a method for channel estimation of the almost linear complexity in r? The best known method (this lecture) has complexity $O(r^2 \log r + r \log(N) \log\left(\frac{N}{r}\right))$.

Further research — Open Questions

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Question (performance): Analyse the performance of the Incidence Algorithm.

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Question (performance): Analyse the performance of the Incidence Algorithm.

Question (sparse discrete channel model): How to estimate the real channel parameters (t_k, f_k) in the case where they are **not** on the grid $\frac{1}{W}\mathbb{Z}_N \times \frac{W}{N}\mathbb{Z}_N$?