Geometric Flows: Recent Developments and Applications

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1 Overview of the Field

Geometric analysis emerged as a field of its own about 40 years ago, having formed at the nexus of PDEs, Riemannian geometry, and related fields such as Kähler geometry, general relativity, and applied mathematics. Few fields of mathematics can claim to have had such success over that period of time. Much of this success has arisen out of the study of geometric flow equations. The study of geometric flows has led to proof of the Poincaré conjecture [12, 13, 10, 11], the Thurston geometrization conjecture [12, 13, 10], the Riemannian Penrose conjecture [8, 9], the differentiable sphere theorem [2, 1], and the n-dimensional Rauch-Hamilton spherical space forms conjecture.

Geometric analysis has also benefited greatly from its interaction with physics. The positive energy conjecture of general relativity motivated the work of Schoen and Yau, who proved the conjecture [15, 16] as part of a much wider programme of using analytical techniques to study the geometry of manifolds. But this example shows that physics has also gained much from geometric analysis. Another example is provided by Yau's proof of the Calabi conjecture [18, 19], which made it possible to use Calabi-Yau manifolds to construct models of string theory phenomenology [3]. A third example is Geroch's fundamental observation that inverse mean curvature flow has a monotonic quantity [5], which could be used as the basis of a proof of a conjecture of Penrose related to cosmic censorship. The eventual proof was given by Huisken and Ilmanen 25 years later [8, 9], after much progress in mean curvature flow (which, in turn, is important in engineering and materials science).

It is therefore natural for physicists, engineers, and applied mathematicians to search for ways to exploit the mathematical progress made in geometric flow equations, and natural as well for mathematicians to use these fields, and physics in particular, as a source for new geometric flow problems. There have been a number of meetings in recent years, typically consisting mostly of geometric analysts together with a smaller number of physicists and applied mathematicians, who gather together to discuss progress in the field and possible new applications. The last such meeting was a workshop held at BIRS in 2011. There has been tremendous progress since then, and it had become time to meet again.

2 Recent Developments and Open Problems

In the last few years, new themes have emerged in the subject. Traditionally, Ricci flow has been applied to Riemannian metrics on closed manifolds. The non-compact case has been less well studied and the case of manifolds-with-boundaries has been studied least of all. Pulemotov has, however, advanced the latter subject [14] and recent work of Gianniotis has produced the current state of the art for Ricci flow with boundary [6, 7]. Meanwhile, physics has motivated the study of asymptotically hyperbolic Ricci flow, and also numerical Ricci flow. Wiseman, Figueras, and their collaborators have used a numerical Ricci flow with boundary to produce compelling numerical evidence for certain black holes (for an overview see [17]), including one which resolves an important question in the Randall-Sundrum "braneworld" scenario and another which may shed light on conformal field theory in Schwarschild spacetime.

There has been progress on questions arising from Hamilton's original programme to prove the Poincaré conjecture and Perelman's remarkable completion of it. Two such questions are whether there exists a (weak) Ricci flow that can be continued through singularities and whether scalar curvature always blows up when a singularity forms. Among the highlights of this workshop were the new results of Haslhofer and the remarkable work of Lott (joint with Kleiner) on the former question, and the new developments presented by Bamler on the latter question.

The Kähler-Ricci flow has also been a focus of recent developments. The objective here has been to obtain a classification of low-dimensional compact Kähler manifolds. Several speakers at the workshop presented recent developments. Most notably, Prof Gang Tian of Princeton University gave an excellent overview of progress within the last year.

3 Presentation Highlights

The organizing committee decided to emphasize early-career researchers. We decreased the length of most talks to 30 to 40 minutes, allowing more of the younger people to speak while still preserving a large portion of free time between lunch and afternoon coffee to encourage collaboration and informal discussions. The strategy was very successful, both in providing opportunities for younger researchers to present work and in encouraging collaboration. After lunch most days, the small break-out rooms in the basement of BIRS were full with small groups discussing work. And the quality of the work presented by so many relatively young researchers in the field was remarkable, as can be perceived from the contents of the next section.

Sigurd Angenent: Rigorous Asymptotics for Ancient Solutions of Curve Shortening and Mean Curvature Flow

Angenant discussed the construction and asymptotic description of ancient solutions in two settings: first ancient solutions with finite total curvature to curve shortening in the plane, then, a proof of the precise asymptotics of the White-Haslhofer-Kleiner ancient convex solution to mean curvature flow. He also commented on the possible existence of other compact ancient solutions to mean curvature flow.

Eric Bahuaud: The Ricci flow of asymptotically hyperbolic metrics

Bahuaud discussed two projects related to behavior of the normalized Ricci flow evolving from a conformally compact asymptotically hyperbolic metric. In the first part, he discussed his joint work with Mazzeo and Woolgar on the behavior of the renormalized volume along the flow of asymptotically Poincaré-Einstein metrics (APEs). In the second part, he discussed his joint work with Woolgar on the long-time existence of the flow for rotationally symmetric asymptotically hyperbolic initial data.

Richard Bamler: On the scalar curvature blow up conjecture in Ricci flow

It is a basic fact that the Riemannian curvature becomes unbounded at every finite-time singularity of the Ricci flow. Sesum showed that, more precisely, even the Ricci curvature becomes unbounded at every such singularity. Whether the same can be said about the scalar curvature has since remained a conjecture, which has resisted several attempts of resolution.

Bamler presented a new result that partially confirms this conjecture in dimension 4 and motivates some interesting questions in 4 dimensional Ricci flow. Its proof relies on a combination of multi-scale arguments and Perelman's Harnack inequality on the conjugate heat equation. As a byproduct, he obtains an unconventional backwards pseudolocality theorem, which holds in any dimension. This project is joint work with Qi Zhang.

Paul Bryan: A viscosity equation and applications of the maximum principle for the isoperimetric profile

The distance function on a Riemannian manifold holds much (all?) geometric information. As such, studying how it evolves under geometric flows can prove a very useful, and powerful technique. The chord-arc profile of a curve in the plane is defined to be the least planar distance between two points a given length apart along the curve. Such a geometric functional, as an extremum, is amenable to study via variational techniques, leading to a weak differential inequality. The profile is strongly related to curvature and so may be used to control the curvature of a family of curves of evolving by a geometric evolution equation. A comparison result yields an explicit curvature bound for curves evolving by curve shortening, which may be applied to deduce long time convergence results in a very direct manner.

Mauro Carfora: Heat kernel embedding, dilatonic sigma models and Ricci flow extensions

By generalizing the heat kernel injection of a Riemannian manifold into Wasserstein spaces of probability measures introduced by N Gigli and C Mantegazza, Carfora presented a non-perturbative (toy) model for the renormalization group flow for the dilatonic non-linear sigma model. The beta functions of this flow characterize a non-perturbative extension of the Hamilton-Perelman version of the Ricci flow.

Jingyi Chen: Compact branched shrinkers to Lagrangian MCF in the complex plane: Rigidity, compactness, F-stability

Chen discussed properties of the space of compact self-shrinking solutions to Lagrangian mean curvature flow in the complex plane. He is able to show that there is no compact branched Lagrangian shrinker of genus zero, and that the space of compact Lagrangian immersed shrinkers can be compactified by the branched ones, under the assumption that the areas and the conformal structures are bounded. He also explained how to prove that compact branched Lagrangian shrinkers are all F-unstable.

Alix Deruelle: Conical expanding gradient Ricci solitons

Deruelle discussed various questions about Ricci gradient expanders coming out of smooth metric cones, focusing on the positively curved case and the asymptotically Ricci flat case.

Pau Figueras: Numerical Ricci flows and black holes

Figueras described a novel use of Ricci flows that has attracted a lot of interest in recent years in the theoretical physics community. Black hole spacetimes are (Lorentzian) Einstein manifolds that play a central role in our understanding of general relativity, Einsteins theory of gravity. In his talk, Figueras explained how one can use Ricci flows to find, numerically, equilibrium black hole spacetimes. He provided some simple examples, emphasising their physical relevance. Note that the flows that often arise in black hole physics are Ricci flows on Lorentzian non-compact manifolds, with various asymptotic boundary conditions, including (but not restricted to) asymptotic flatness.

Panagiotis Gianniotis: The Ricci flow on manifolds with boundary

The behaviour of the Ricci flow on manifolds with boundary seems to be a hard problem and little is known. Gianniotis described progress towards developing a theory for the Ricci flow on such spaces. In particular, he addressed the issues of local existence and uniqueness, Shi-type a priori estimates, break down criteria, and compactness of flows.

Christine Guenther: Second order renormalization group flow

The Ricci flow arises in physics as the first order approximation of the renormalization group flow for the nonlinear sigma model of quantum field theory. The *second* order approximation is given by the system $\frac{\partial}{\partial t} g = -2\mathrm{Rc} - \frac{\alpha}{2}\mathrm{Rm}^2$, and can be considered as a natural nonlinear perturbation of the Ricci flow (here g is a Riemannian metric, Rc is Ricci curvature, $\alpha > 0$ is a small parameter and $\mathrm{Rm}^2_{ij} := R_{iklm}R_j^{klm}$). In this talk, Guenther surveyed what is known mathematically about this system, including a recent proof of short term existence in n-dimensions from joint work she did with James Isenberg and Karsten Gimre. She concluded with a list of open problems.

Robert Haslhofer: Weak solutions for the Ricci flow

Haslhofer characterized solutions of the Ricci flow in terms of various infinite dimensional estimates. Namely, given an evolving family of Riemannian manifolds, he considers the path space of its space-time. His first characterization says that the family evolves by Ricci flow if and only if a certain infinite dimensional gradient estimate holds for all L^2 functions on its path space. He proved further characterizations in terms of the regularity of martingales, a log-Sobolev inequality, and a spectral gap. Based on these characterizations he can define a notion of weak solutions for the Ricci flow. This was joint work with Aaron Naber.

James Isenberg: Asymptotic Behavior of Non-Round Neckpinches in Ricci Flow

Neckpinch singularities are a prevalent feature of Ricci flow, and recent work has given us a good picture of their asymptotic behavior, so long as the geometries are rotationally symmetric. Isenberg discussed this asymptotic behavior, both for degenerate and non-degenerate neckpinches. It has been conjectured that neckpinch singularities which develop in non-rotationally symmetric Ricci flows do asymptotically approach roundness, and consequently have very similar asymptotic behavior to those which are rotationally symmetric. He discussed recent work which supports this conjecture.

Brett Kotschwar: An energy approach to uniqueness for geometric flows

Kotschwar described a simple alternative method by which the uniqueness of solutions to a variety of curvature flows of all orders, including the Ricci flow, the cross-curvature flow, and the L^2-curvature flow, can be established without recourse to DeTurck's trick.

John Lott: Ricci flow through singularities

Perelman's Ricci flow-with-surgery involves a surgery parameter δ, which describes the scale at which surgery is performed. Lott is able to show that there is a subsequential limit as δ goes to zero, thereby partially answering a question of Perelman. The limiting object is called a singular Ricci flow. Such objects can considered to be flows through singularities, and studied in their own right. Lott proved some geometric and analytical properties of such singular Ricci flows. This was joint work with Bruce Kleiner.

Warner Miller: Discrete Hamiltons Ricci Flow in Higher Dimensions

Recently Miller and coworkers defined a discrete form of Hamiltons Ricci flow equations for an n-dimensional piecewise flat simplicial geometry S, where $n \geq 2$. These algebraic equations are derived using a discrete formulation of Einsteins theory of general relativity known as Regge calculus, or equivalently discrete exterior calculus. An algebraic Regge-Ricci flow equation is naturally associated with an edge l in S and is construction using the circumcentric dual lattice S^*. The inherent orthogonality between elements of S and their duals in S^* provide a simple geometric representation of Hamilton's Ricci flow equations. In this talk, Miller outlined the construction of these equations in 3-dimensions and discussed their solutions for a few illustrative examples including neck-pinch singularities.

Andrea Mondino: Properties of non-smooth spaces with Ricci curvature lower bounds

The idea of compactifying the space of Riemannian manifolds satisfying Ricci curvature lower bounds goes back to Gromov in the 1980s and was pushed by Cheeger and Colding in the 1990s who investigated the structure of the spaces arising as Gromov-Hausdorff limits of smooth Riemannian manifolds satisfying Ricci curvature lower bounds. A completely new approach via optimal transportation was proposed by Lott-Villani and Sturm almost ten years ago; with this approach one can a give a precise meaning of what it means for a non-smooth space to have Ricci curvature bounded from below by a constant. This approach has been refined in the last years by a number of authors (see the fundamental work of Ambrosio-Gigli-Savaré, among others) and a number of fundamental tools have now been established (for instance the Bochner inequality, the splitting theorem, etc), permitting further insight into the theory. In his seminar, Mondino gave an overview of the topic.

Reto Mueller: Harmonic Ricci Flow on Surfaces

The Harmonic Ricci Flow is a coupling of the Harmonic Map Flow and the Ricci Flow (on the domain manifold of the map). While it is known that for this flow the energy density cannot blow up without the curvature of the domain manifold also becoming unbounded, Mueller showed that if the domain manifold is of dimension two then also the converse result holds. From this, he can immediately obtain smooth long-time existence for the flow for large enough coupling constants. This was joint work of Mueller with Melanie Rupflin.

Eleonora di Nezza: Smoothing properties of the Kähler-Ricci flow

In connection with the "analytic analogue" of the Minimal Model Program, it is important to analyse the long-term behaviour of the Kaehler-Ricci flow. This motivated attempts to run the flow on a compact compact Kähler manifold X from degenerate initial data. Di Nezza showed that the Kähler-Ricci flow can be run from any arbitrary positive closed current, and that it is immediately smooth in a Zariski open subset of X. This was joint work with Chinh Lu, Chalmers University of Technology.

Huy Nguyen: Mean Curvature Flow of Surfaces of Codimension Two

Nguyen described joint work with Charles Baker, considering surfaces of co-dimension two in Euclidean space moving by the mean curvature flow. He showed that if the initial surface lies satisfies a nonlinear curvature condition depending on the normal curvature tensor then the mean curvature flow deforms the surface to a round point.

Artem Pulemotov: The prescribed Ricci curvature problem on homogeneous spaces.

Pulemotov discussed the problem of finding a Riemannian metric whose Ricci curvature coincides with a given invariant $(0, 2)$-tensor on a homogeneous space.

Frédéric Rochon: Uniform construction of the heat kernel under a neck pinching

Rochon presented a uniform construction of the heat kernel under a neck pinching leading to the formation of cusps, for instance when a geodesic is pinched on a hyperbolic surface. This allows one to study the limit of spectral invariants like the determinant of the Laplacian or analytic torsion under such a neck pinching. In particular, this leads to a Cheeger-Müller theorem on manifolds with cusps. This was joint work with Pierre Albin and David Sher.

Felix Schulze: A local regularity theorem for mean curvature flow with triple edges

Schulze considers the evolution by mean curvature flow of surface clusters, where along triple edges three surfaces are allowed to meet under an equal angle condition. He shows that any such smooth flow, which is weakly close to the static flow consisting of three half-planes meeting along the common boundary, is smoothly close with estimates. Furthermore, he shows how this can be used to prove a smooth short-time existence result. This was joint work with B White.

Benjamin Sharp: Compactness theorems for minimal surfaces with bounded index

Sharp presented a new compactness theorem for minimal hypersurfaces embedded in a closed Riemannian manifold N^{n+1} with $n < 7$. When $n = 2$ and N has positive Ricci curvature, Choi and Schoen proved that a sequence of minimal hypersurfaces with bounded genus converges smoothly and graphically to some minimal limit. A corollary of Sharp's main theorem recovers the result of Choi-Schoen and extends this appropriately for all $n < 7$.

Miles Simon: Some integral curvature estimates for four dimensional Ricci flows and consequences thereof

Simon presented some integral curvature estimates for four dimensional solutions to Ricci flow. These estimates hold for any compact, connected, smooth, four dimensional solution. In the special case that the existence time $T > 0$ is finite and the scalar curvature is uniformly bounded on $[0, T)$, he presented various consequences thereof.

Gang Tian: Some progress on Káhler-Ricci flow

Tian started with a brief tour onf the Analytic Minimal Model Programme through Ricci flow. Then he discussed several new results on Káhler-Ricci flow.

Bing Wang: Regularity scales and convergence of the Calabi flow

Wang defined regularity scales to study the behavior of the Calabi flow. Based on estimates of the regularity scales, we obtain convergence theorems of the Calabi flow on extremal Kähler surfaces, under the assumption of global existence of the Calabi flow solutions. His results partially confirm Donaldson's conjectural picture for the Calabi flow in complex dimension 2. Similar results hold in high dimension with an extra assumption that the scalar curvature is uniformly bounded. This was joint work with HZ Li and K Zheng.

Claude Warnick: Stability problems in Anti-de Sitter

The anti-de Sitter spacetime is the simply connected spacetime of constant sectional curvature -1. Einsteins equations in this spacetime have the character of an initial-boundary value problem, with boundary data specified on the timelike conformal infinity. Warnick discussed the possible boundary conditions, and presented some recent results with Holzegel, Luk, and Smulevici which indicate that for a particular, dissipative, choice of boundary condition the spacetime is stable against small perturbations to the initial data.

Burkhard Wilking: The $\hat{A}$ genus of almost non-negatively curved manifolds vanishes

Wilking investigated the sequences of spin manifolds with lower sectional curvature bound upper diameter bound and the property that the Dirac operator is not invertible. By comparing averaged curvature with averaged holonomy he is able to establish the topological conclusion indicated in the title.

4 Scientific Progress Made and Outcome of the Meeting

It is of course not possible to judge the outcome of such a meeting until long after it finishes. Nearly all of the work that was presented was very recent and it will take time for it to be disseminated and its effects to be felt. Amongst the recent scientific breakthroughs presented at the meeting, we have already mentioned the work of Haslhofer and of Kleiner and Lott on the flow through singularities, the work of Bamler on scalar curvature blow-up at singular times, and the recent progress in Káhler-Ricci flow and the minimal model programme described by Tian.

Many participants took away new ideas arising from interaction with others who brought their own perspectives. The Regge-calculus inspired discrete Ricci flow of Miller may help to catapult work on numerical geometric flows. The work of Schulze on mean curvature flow of networks of surfaces may have applications in cosmology, as observed by Wiseman. The new Ricci flow estimates of Simon may simplify proofs of long-time exitence and convergence of Ricci flow in certain cases.

A number of new collaborations appear to have arisen from the meeting, and the work of other colaborations was advanced. The physical structure of the small meeting rooms in the basement of the lecture facility helped quite a bit with this, as did the organizational decision to have a long lunch break during which people could meet in small groups. This will likely be an important part of the legacy of the meeting.

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