Mass-Stationarity, Shift-Coupling, and Brownian Motion

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Based on

Invariant transports of stationary random measures and mass-stationarity. Annals of Probab 2009. Joint with Günter Last

and

What is typical? SØREN Festschrift 2011. Joint with Günter Last

and

Unbiased shifts of Brownian motion. Annals of Probab 2014. Joint with Günter Last and Peter Mörters

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Mandelbrot's Intuition

At the end of the 20th century, David Vere-Jones pointed out to me that Mandelbrot had in the early 80s suggested that the zero-set of two-sided standard Brownian motion $B = (B_s)_{s\in\mathbb{R}}$ should have a property similar to the one I was considering at that time for point processes on $\mathbb{R}^d$. Mandelbrot's intuitive idea was that B should look the same from all its zeros. Note that this idea has a well-known formalisation for a two-sided simple symmetric random walk on the integers.

In my case, the intuitive idea was that a Palm version of a stationary point process in $\mathbb{R}^d$ should look the same from all its points. This informal property has a well-known formalisation when $d = 1$. For $d > 1$, I had formalised this idea with an intuitively acceptable property that I named point-stationarity. In my 2000-book I suggested that the zero set of B might have that same property. It turns out my idea needed a modification. Also there is a simpler formalization that is basically obvious.

Mass-Stationarity

Setting: Let $(\Omega, \mathcal{F}, \mathbb{P})$ support the random elements below.

Let G be a locally compact second countable topological group with left-invariant Haar measure λ.

Let ξ be a random measure on G.

Let X be a random element in a space on which G acts.

Write θ_t for the shift map placing a new origin at $t \in G$.

E.g. $X = (X_s)_{s \in G}$ a shift-measurable r.f. and $\theta_t X = (X_{ts})_{s \in G}$.

Definition

The pair (X, ξ) is called mass-stationary if for all bounded λ-continuity sets $C \subseteq G$ of positive λ-measure

$$\theta_{V_C}(X, \xi, U_C^{-1}) \stackrel{D}{=} (X, \xi, U_C^{-1})$$

where U_C is such that $\mathbb{P}(U_C \in \cdot \mid X, \xi) = \lambda(\cdot \mid C)$

and V_C is such that $\mathbb{P}(V_C \in \cdot \mid X, \xi, U_C) = \xi(\cdot \mid \theta_{U_C} C)$.

Mass-Stationarity in the case when G is compact

Definition (from previous slide)

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and V_C is such that $\mathbb{P}(V_C \in \cdot \mid X, \xi, U_C) = \xi(\cdot \mid \theta_{U_C} C)$.

Note that when G be compact then $\mathbb{P}(V_G \in \cdot \mid X, \xi) = \xi(\cdot \mid G)$.

Theorem

Let G be compact and S be a random element in G such that

$$\mathbb{P}(S \in \cdot \mid X, \xi) = \xi(\cdot \mid G).$$

Then

$$(X, \xi) \text{ mass-stationary} \iff \theta_S(X, \xi) \stackrel{D}{=} (X, \xi)$$

Mass-stationarity and preserving shifts π

Let π be a measurable map taking ξ to a location $\pi(\xi)$ in G.
Define the induced allocation rule $\tau_\pi = \tau_\pi^\xi$ by

$$\tau_\pi(s) = \pi(\theta_s \xi)s, \qquad s \in G.$$

Call π preserving if τ_π preserves ξ, that is, if $\xi(\tau_\pi \in \cdot) = \xi$.

Theorem

(X, ξ) mass-stationary $\Rightarrow \forall$ preserving π: $\theta_{\pi(\xi)}(X, \xi) \overset{D}{=} (X, \xi)$

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Theorem: Let $G = \mathbb{R}$ and ξ be diffuse. Then

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Mass-stationarity when $G = \mathbb{R}$ and ξ is diffuse

Theorem (from previous slide): Let $G = \mathbb{R}$ and ξ diffuse. Then

(X, ξ) mass-stationary $\iff$ $\forall$ preserving π: $\theta_{\pi(\xi)}(X, \xi) \stackrel{D}{=} (X, \xi)$

The following shifts π_r move an amount r forward in the mass

$$\pi_r(\xi) = \sup\{t \in \mathbb{R} : \xi([0, t]) = r\}, \qquad r \in \mathbb{R}.$$

It is easy to show that these shifts are preserving.

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Moreover, the following holds:

Theorem: Let $G = \mathbb{R}$ and ξ be diffuse. Then

(X, ξ) mass-stationary $\iff \forall\, r \in \mathbb{R}:\ \theta_{\pi_r(\xi)}(X, \xi) \overset{D}{=} (X, \xi)$

Mandelbrot was right about the zeros of $B = (B_s)_{s\in\mathbb{R}}$

Recall the shifts $\pi_r(\xi) = \sup\{t \in \mathbb{R} :\ \xi([0,t]) = r\},\ r \in \mathbb{R},$ and

Theorem : Let $G = \mathbb{R}$ and ξ be diffuse. Then

$$(X,\xi) \text{ mass-stationary} \iff \forall\, r \in \mathbb{R}:\ \theta_{\pi_r(\xi)}(X,\xi) \overset{D}{=} (X,\xi)$$

Now let ℓ^0 be local time at zero.
This random measure represents the zeros of B and is diffuse.
Moreover, with $T_r = \pi_r(\xi)$ the following holds:

Theorem: (B,ℓ^0) is mass-stationary, that is,

$\theta_{T_r}B = (B_{T_r+t})_{t\in\mathbb{R}}$ is a two-sided Brownian motion for all $r \in \mathbb{R}$.

Thus: when traveling in time accoring to the clock of local time at zero you always see globally a two-sided Brownian motion.

Mass-Stationarity and Shift-Coupling when $G = \mathbb{R}$

Now let ξ and η be random measures such that $\theta_t\xi = f(\theta_t X)$ and $\theta_t\eta = g(\theta_t X)$ for some measurable f and g and all $t \in \mathbb{R}$.

Let π be a measurable map taking X to a location $\pi(X)$ in $\mathbb{R}$.
Say that τ_π balances ξ and η if $\xi(\tau_\pi \in \cdot) = \eta$.

Let X' be a random element in the same space as X.
Put $\xi' = f(X')$ and $\eta' = g(X')$.

Theorem: Let $G = \mathbb{R}$ (for simplicity).

Let (X, ξ) and (X', η') both be mass-stationary. Let

$$0 < \mathbb{E}\Big[\int_0^{\pi_1(X)} \xi([t, t+1])dt\Big] = \mathbb{E}\Big[\int_0^{\pi_1(X')} \eta'([t, t+1])dt\Big] < \infty.$$

Let X and X' have the same trivial distribution on invariant sets.
Then

$$\theta_{\pi(X)}X \overset{D}{=} X' \iff \tau_\pi \text{ balances } \xi \text{ and } \eta$$

Remark

$\theta_{\pi(X)}X \overset{D}{=} X'$ means $T = \pi(X)$ is shift-coupling time for X and X'.

Unbiased shifts of two-sided Brownian motion B

Definition: Let $B = (B_t)_{t\in\mathbb{R}}$ be a standard Brownian motion.

An unbiased shift of B is a random time T in $\mathbb{R}$ such that:

- $T = \pi(B)$ for some measurable map π,
- $(B_{T+t} - B_T)_{t\in\mathbb{R}}$ is a standard Brownian motion,
- $(B_{T+t} - B_T)_{t\in\mathbb{R}}$ is independent of B_T.

Remark

Thus, $T = \pi(B)$ is an unbiased shift if and only if $\theta_T B = (B_{T+t})_{t\in\mathbb{R}}$ is a two-sided standard Brownian motion not necessarily taking the value 0 at time 0.

That is, $\theta_T B \stackrel{D}{=} B'$ where $B' = B'_0 + B$ with B'_0 distributed as B_T and independent of B.

Thus T is a shift-coupling time for B and B'.

Note that $T = \pi_r(X)$ is an unbiased shift with $B_T = 0$.

Examples of times T that are NOT unbiased

Example

If $T \geq 0$ is a stopping time, then $(B_{T+t} - B_T)_{t \geq 0}$ is a one-sided Brownian motion independent of B_T. However, the example

$$T := \inf\{t \geq 0 \colon B_t = y\} = \text{hitting time of a non-zero state } y$$

shows $(B_{T+t} - B_T)_{t \in \mathbb{R}}$ need not be two-sided Brownian motion.

Example

Consider a deterministic $T = t_0$.
Then $\tilde{B} := (B_{t_0+t} - B_{t_0})_{t \in \mathbb{R}}$ is a two-sided Brownian motion.
However, it is not independent of B_{t_0} since $B_{t_0} = -\tilde{B}_{-t_0}$.

Remark

We might see later that an unbiased shift need not be a stopping time, even when it is nonnegative.

Mass-Stat. and Shift-Coupl. when $G = \mathbb{R}$ and ξ diffuse

Recall: $\theta_t\xi = f(\theta_t X)$, $\theta_t\eta = g(\theta_t X)$, $\xi' = f(X')$, $\eta' = g(X')$.

Theorem (from some slides ago): Let $G = \mathbb{R}$.

Let (X, ξ) and (X', η') both be mass-stationary. Let
$0 < \mathbb{E}\Big[\int_0^{\pi_1(X)} \xi([t, t+1])dt\Big] = \mathbb{E}\Big[\int_0^{\pi_1(X')} \eta'([t, t+1])dt\Big] < \infty.$
Let X and X' have the same trivial distribution on invariant sets.
Then $\theta_{\pi(X)}X \overset{D}{=} X' \iff \tau_\pi$ balances ξ and η

Theorem

In addition to the conditions in the above theorem, let ξ and η be diffuse and orthogonal. Then the map π defined by

$$\pi(X) := \inf\{t > 0 \colon \xi([0, t]) = \eta([0, t])\}$$

is such that the induced allocation rule

$$\tau_\pi(s) := \inf\{t > s \colon \xi([s, t]) = \eta([s, t])\}, \quad s \in \mathbb{R},$$

balances ξ and η.

Existence of unbiased shifts of B

Let ν be a probability measure on $\mathbb{R}$. Let B_0^ν be a random variable with distribution ν and independent of B. Define a standard Brownian motion with distribution ν at 0 by

$$B^\nu = B_0^\nu + B.$$

Let ℓ^x be local time of B at $x \in \mathbb{R}$ and set $\ell^\nu = \int \ell^x \nu(dx)$. These random measures are diffuse.

Theorem

The pair (B^ν, ℓ^ν) is mass-stationary and has the same trivial distribution as (B, ℓ^0) on invariant sets. Further,

$$0 < \mathbb{E}\Big[\int_0^{\pi_1(B^\nu)} \ell^\nu([t, t+1])dt\Big] = \mathbb{E}\Big[\int_0^{\pi_1(B)} \ell^0([t, t+1])dt\Big] < \infty.$$

Due to this and the previous slide we now obtain the following.

Theorem

If $\nu\{0\} = 0$ then $T^\nu := \inf\{t > 0 \colon \ell^0([0, t]) = \ell^\nu([0, t])\}$ is an unbiased shift and B_T has distribution ν (that is, T imbeds ν).

The Brownian Bridge

Jim Pitman and Wenpin Tang have just shown in their paper

> The Slepian zero set, and Brownian bridge
> embedded in Brownian motion by a spacetime shift,
> http://arxiv.org/abs/1411.0040

that the Slepian process $(B_{t+1} - B_t)_{t \in \mathbb{R}}$
has its own 'local time at zero' γ.

Note that $(B_{t+1} - B_t)_{t \in \mathbb{R}}$ is stationary.
This implies that $((B_{t+1} - B_t)_{t \in \mathbb{R}}, \lambda)$ is mass-stationary,
here λ is Lebesgue measure.

This also implies that γ does not have where 0 in its support.

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Theorem

Set

$$T = \inf\{t > 0 \colon \gamma([0, t]) = t\}.$$

Then $(B_{T+u} - B_T)_{0 \le u \le 1}$ is a standard Brownian Bridge.

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Theorem

Set
$$T = \inf\{t > 0 \colon \gamma([0, t]) = t\}.$$

Then $(B_{T+u} - B_T)_{0 \leq u \leq 1}$ is a standard Brownian Bridge.

Outline of proof: Set $X = (X_t)_{t \in \mathbb{R}}$ where $X_t = (B_{t+u} - B_t)_{0 \leq u \leq 1}$.
Note that X is stationary so (X, λ) is mass-stationary.
The Palm version (X', γ') of X w.r.t. γ is mass-stationary.
Moreover, X'_0 is a standard Brownian Bridge.
The conditions of the shift-coupling theorem are satisfied.
Thus $(B_{T+u} - B_T)_{0 \leq u \leq 1} \overset{D}{=} X'_0$. □

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