

# On non-Archimedean curves omitting few components and their arithmetic analogues

Julie Tzu-Yueh Wang

Academia Sinica, Taiwan

Distribution of Rational and Holomorphic Curves  
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# A motivating example.

Let  $\mathbf{k}$  an algebraically closed field complete with respect to a non-Archimedean absolute value  $|\cdot|$  of arbitrary characteristic.

$f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathbf{k}[[z]]$  is analytic on  $z \in \mathbf{k}$  if  $|a_n z^n| \rightarrow 0$  as  $n \rightarrow \infty$ .

$f(z)$  is called a  $\mathbf{k}$  entire function if  $f$  is analytic on  $\mathbf{k}$ .

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# $\mathbf{k}$ hyperbolic

A variety  $X$  is said to be  $\mathbf{k}$  *hyperbolic* if any analytic map from  $\mathbf{k}$  to  $X$  is constant.

# Some examples of $\mathbf{k}$ hyperbolic varieties.

- ▶  $\mathbb{G}_m = \mathbb{A}^1 \setminus \{0\} = \mathbb{P}^1 \setminus \{0, \infty\}$ .
- ▶ curves of genus  $\geq 1$ . (Berkovich 1990)
- ▶ curve omitting two points
- ▶ semi-Abelian variety (Cherry 1994)
- ▶  $\mathbb{P}^n$  omitting  $n + 1$  hypersurfaces in g. p. (Ru 2001)
- ▶ projective variety omitting  $n + 1$  hypersurface divisors in g. p. (An 2007)
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## Further results.

Theorem (Lin-W.2010, An-Levin-W. 2011)

Let  $X$  be a nonsingular projective variety over  $\mathbf{k}$ . Let  $D_1, \dots, D_m$  be effective divisors on  $X$  with empty intersection. Let  $D = \sum_{i=1}^m D_i$ .

- 1 If  $\kappa(D_i) > 0$  for all  $i$ , then the image of an analytic map  $f : \mathbf{k} \rightarrow X \setminus D$  is contained in a proper subvariety of  $X$ .
- 2 If  $D_i$  is big for all  $i$ , then there exists a proper Zariski-closed subset  $Z \subset X$  such that the image of any non-constant analytic map  $f : \mathbf{k} \rightarrow X \setminus D$  is contained in  $Z$ .
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- ③ If  $D_i$  is ample for all  $i$ , then there is no non-constant analytic map from  $\mathbf{k}$  to  $X \setminus D$ .

# Question:

What if the intersection of the divisors  $D_i$ 's is not empty?

# A simple observation.

Let  $P_1, P_2$  be non-constant homogeneous polynomials in  $n + 1$  variables with degree  $d_1$  and  $d_2$  respectively. Assume that the divisors  $D_1 = \{P_1 = 0\}$  and  $D_2 = \{P_2 = 0\}$  in  $\mathbb{P}^n$  are distinct.

Let  $f = (f_0, \dots, f_n) : \mathbf{k} \rightarrow \mathbb{P}^n \setminus D_1 \cup D_2$  be an analytic map.

Then  $P_1^{d_2}(f)/P_2^{d_1}(f)$  is entire without zero and hence is constant.

Consequently, the image of  $f$  is contained in a subvariety of codimension one.

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### Theorem (An-W.-Wong 2008)

Let  $X$  be a nonsingular projective subvariety of  $\mathbb{P}^N$  of dimension  $n$ . Let  $P_1, \dots, P_q$  be non-constant homogeneous polynomials in  $N + 1$  variables. Let  $D_i = X \cap \{P_i = 0\}$ ,  $1 \leq i \leq q$ , be divisors of  $X$  in general position. Let  $f$  be an analytic map from  $\mathbf{k}$  to  $X \setminus \cup_{i=1}^q D_i$ . Then the image of  $f$  is contained in a subvariety of  $X$  of codimension  $\min\{n + 1, q\} - 1$  in  $X$ . In particular,  $f$  is algebraically degenerate if  $q \geq 2$ , and  $X \setminus \cup_{i=1}^q D_i$  is  $\mathbf{k}$ -hyperbolic if  $q \geq n + 1$ .

## Theorem (An-Cherry-W. 2008 & 2015 )

*Let  $Y$  be a closed positive dimensional subvariety of a non-singular projective variety  $X$ . Let  $\{D_i\}_{i=1}^{\ell}$  be  $\ell$  irreducible, effective, ample divisors in general position on  $X$ . Let  $r$  be the rank of the subgroup of  $\text{NS}(X)$  generated by  $\{c_1(D_i)\}_{i=1}^{\ell}$ . If there exists an algebraically non-degenerate analytic map from  $\mathbf{k}$  to  $Y$  omitting each of the  $D_i$  that does not contain all of  $Y$ , then*

$$\ell \leq \min\{r + \text{codim } Y, \dim X\}.$$



## $n$ -component

### Theorem (An-W.-Wong 2008)

Let  $D_1, \dots, D_n$  be nonsingular hypersurfaces in  $\mathbb{P}^n$  intersecting transversally. Then  $\mathbb{P}^n \setminus \cup_{i=1}^n D_i$  is  $\mathbf{k}$  hyperbolic if  $\deg D_i \geq 2$  for each  $1 \leq i \leq n$ .

### Theorem (An-W.-Wong 2008)

Let  $D_1$  and  $D_2$  be nonsingular projective curves in  $\mathbb{P}^2$ . Assume that  $D_1$  and  $D_2$  intersect transversally and  $\deg D_1 \leq \deg D_2$ . Then  $\mathbb{P}^2 \setminus \{D_1 \cup D_2\}$  is  $\mathbf{k}$ -hyperbolic if and only if either  $\deg D_1, \deg D_2 \geq 2$  or  $\deg D_1 = 1, \deg D_2 \geq 3$  and  $D_1$  does not intersect  $D_2$  at any maximal inflexion point.

### Corollary

If  $D_1$  and  $D_2$  are two generic curves in  $\mathbb{P}^2(\mathbf{k})$  with  $\deg D_1 + \deg D_2 \geq 4$ , then  $\mathbb{P}^2 \setminus \{D_1 \cup D_2\}$  is  $\mathbf{k}$  hyperbolic.

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# Main Theorem 1.

## Theorem (Levin-W.)

Let  $D_1, \dots, D_n$  be effective nef divisors intersecting transversally in an  $n$ -dimensional nonsingular projective variety  $X$  over  $\mathbf{k}$ . Let  $K_X$  denote the canonical divisor on  $X$ .

- 1 Assume that either  $D_i^n > 1$  or that  $D_i^n = 1$  and  $K_X \cdot D_i^{n-1} < 1 - n$  for each  $1 \leq i \leq n$ . Then the image of an analytic map  $f : \mathbf{k} \rightarrow X \setminus \cup_{i=1}^n D_i$  is contained in a proper subvariety of  $X$ .
- 2 If  $D_i^n > 1$  for all  $i$ , then there exists a proper Zariski-closed subset  $Z \subset X$  such that the image of any non-constant analytic map  $f : \mathbf{k} \rightarrow X \setminus \cup_{i=1}^n D_i$  is contained in  $Z$ .

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## Corollary

Let  $X$  be an  $n$ -dimensional nonsingular projective variety over  $\mathbf{k}$ . Suppose that  $-K_X$  is nef and

$$(-K_X)^n > (n-1)^n.$$

Let  $D_1, \dots, D_n$  be effective nef and big divisors on  $X$  intersecting transversally. Then the image of an analytic map  $f : \mathbf{k} \rightarrow X \setminus \bigcup_{i=1}^n D_i$  is contained in a proper subvariety of  $X$ .

## Remark

Let  $X$  be a smooth projective surface over an algebraically closed field. Then  $-K_X$  is nef and  $(-K_X)^2 > 1$  if and only if  $X$  is one of the following:

- 1  $X \cong \mathbf{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$  over  $\mathbb{P}^1$ ;
- 2  $X \cong \mathbb{P}^2$  or  $X \cong \mathbb{P}^1 \times \mathbb{P}^1$ ;
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## Question.

Is it possible to impose conditions on the self-intersection numbers of the divisors or the degrees of the divisors (in some fixed projective embedding) to ensure that  $X \setminus (D_1 \cup \cdots \cup D_n)$  is  $\mathbf{k}$  hyperbolic?

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## Example.

Let  $m$  and  $n$  be positive integers. Let  $X = \mathbb{P}^1 \times \mathbb{P}^1$  and let  $D_1$  and  $D_2$  be effective divisors of type  $(1, m)$  and  $(1, n)$ , respectively. Let  $D = D_1 + D_2$ . Let  $P$  be a point in the intersection of the supports of  $D_1$  and  $D_2$  and let  $L$  be the line on  $X$  of type  $(0, 1)$  through  $P$ . Then since  $L \setminus D = L \setminus \{P\} \cong \mathbb{A}^1$ , there exists a non-constant analytic map  $f : \mathbf{k} \rightarrow X \setminus D$ . Note that  $D_1$  and  $D_2$  are very ample and  $D_1^2 = 2m$ ,  $D_2^2 = 2n$ .

## Main Theorem 2.

### Theorem (Levin-W.)

Let  $X$  be a rational ruled surface over  $\mathbf{k}$ . Let  $D_1$  and  $D_2$  be effective divisors intersecting transversally in  $X$ .

- ① If  $D_1$  and  $D_2$  are big, then the image of an analytic map  $f : \mathbf{k} \rightarrow X \setminus D_1 \cup D_2$  is contained in a proper subvariety of  $X$ .
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# Nevanlinna theory v.s. Diophantine approximation

Let  $X$  be a variety over a number field  $K$ .

existence of a nonconstant analytic curve in  $X$



existence of an infinite set of integral points on  $X$   
(over a finite extension of  $K$ )

## Definition.

Let  $X$  be an affine variety over a number field  $k$ . Let  $S$  be a finite set of places of  $k$  containing the archimedean places and let  $\mathcal{O}_{k,S}$  denote the ring of  $S$ -integers of  $k$ . We define a set  $R \subset X(k)$  to be a set of  $\mathcal{O}_{k,S}$ -integral points on  $X$  if there exists an affine embedding  $\phi : X \hookrightarrow \mathbb{A}^n$  such that  $\phi(R) \subset X \cap \mathbb{A}^n(\mathcal{O}_{k,S})$ .

## non-Archimedean Nevanlinna theory $\leftrightarrow$ ?

Note that  $\mathbf{k}$  entire functions without zeros are constant.

It suggests that we should look at ring of integers of a number field  $k$  with finite unit groups.

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$\mathbb{G}_m$  is  $\mathbf{k}$  hyperbolic.

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# An Example

$C : x^2 - 2y^2 = 1$ . Over  $\overline{\mathbb{Q}}$ , we have  $C \cong \mathbb{G}_m$ .  $C$  does admit infinitely many  $\mathbb{Z}$ -integral points, while  $C$  is **k** hyperbolic.

Let  $\tilde{C}$  be the projective closure of  $C$  defined in the projective plane by  $x^2 - 2y^2 = z^2$ . Then  $C$  has two points at infinity  $P_{\pm} = (\pm\sqrt{2}, 1, 0)$ .

Every regular function on  $C$  over  $\mathbb{Q}$  has a pole at both  $P_+$  and  $P_-$  on  $\tilde{C}$ . Over  $\mathbb{Q}(\sqrt{2})$ , however, there are regular functions on  $C$  with a pole only at, say,  $P_+$  on  $\tilde{C}$ . One might view the problem here as being that  $C$  doesn't have enough regular functions over  $\mathbb{Q}$  to have a good notion of  $\mathbb{Z}$ -integral points.

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# A necessary condition

Let  $X$  be an affine variety over  $k$  where  $k = \mathbb{Q}$  or an imaginary quadratic field.

Condition (\*): There exists a projective closure  $\tilde{X}$  of  $X$  nonsingular at every point in  $\tilde{X} \setminus X$  and such that every (geometric) irreducible component of  $\tilde{X} \setminus X$  is defined over  $k$ .

## Theorem

*Let  $k = \mathbb{Q}$  or an imaginary quadratic field and suppose that  $\text{char } \mathbf{k} = 0$ . If  $X$  is an affine curve over  $k$  satisfying  $(*)$  then  $X$  contains an infinite set of  $\mathcal{O}_k$ -integral points if and only if there exists a non-constant analytic map  $f : \mathbf{k} \rightarrow X$  if and only if  $X$  is rational with a single point at infinity.*

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Let  $k = \mathbb{Q}$  or an imaginary quadratic field. Let  $X$  be an  $n$ -dimensional nonsingular projective variety over  $k$ . Let  $D_1, \dots, D_n$  be effective nef divisors on  $X$ , all defined over  $k$ , intersecting transversally in  $X$ . Let  $K_X$  denote the canonical divisor on  $X$ . **Suppose that every point in the intersection  $\cap_{i=1}^n D_i$  is  $k$ -rational.**

- 1 Assume that either  $D_i^n > 1$  or that  $D_i^n = 1$  and  $K_X \cdot D_i^{n-1} < 1 - n$  for each  $1 \leq i \leq n$ . Then any set  $R$  of  $\mathcal{O}_k$ -integral points on  $X \setminus \cup_{i=1}^n D_i$  is contained in a proper Zariski-closed subset of  $X$ .
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## Corollary

Let  $k = \mathbb{Q}$  or an imaginary quadratic field. Let  $X$  be an  $n$ -dimensional nonsingular projective variety over  $k$ . Suppose that  $-K_X$  is nef and

$$(-K_X)^n > (n-1)^n.$$

Let  $D_1, \dots, D_n$  be effective nef and big divisors on  $X$ , all defined over  $k$ , intersecting transversally in  $X$ . **Suppose that every point in the intersection  $\cap_{i=1}^n D_i$  is  $k$ -rational.** Then any set  $R$  of  $\mathcal{O}_k$ -integral points on  $X \setminus \cup_{i=1}^n D_i$  is contained in a proper Zariski-closed subset of  $X$ .

## Theorem

Let  $k = \mathbb{Q}$  or an imaginary quadratic field. Let  $\pi : X \rightarrow \mathbb{P}^1$  be a rational ruled surface over  $k$ . Let  $D_1$  and  $D_2$  be effective divisors intersecting transversally in  $X$ . Suppose that all the irreducible components of  $D_1$  and  $D_2$  are defined over  $k$  and that **all of the points in the intersection  $D_1 \cap D_2$  are  $k$ -rational**.

- 1 If  $D_1$  and  $D_2$  are big, then any set  $R$  of  $\mathcal{O}_k$ -integral points on  $X \setminus D_1 \cup D_2$  is contained in a proper Zariski-closed subset of  $X$ .
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# Basic ingredient for the non-Archimedean case

Theorem (Lin-W.2010, An-Levin-W.2011)

Let  $X$  be a nonsingular projective variety over  $\mathbf{k}$ . Let  $D_1, \dots, D_m$  be effective divisors on  $X$  with empty intersection. Let  $D = \sum_{i=1}^m D_i$ .

- 1 If  $\kappa(D_i) > 0$  for all  $i$ , then the image of an analytic map  $f : \mathbf{k} \rightarrow X \setminus D$  is contained in a proper subvariety of  $X$ .
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## Theorem (Levin, 2008)

Let  $k = \mathbb{Q}$  or an imaginary quadratic field. Let  $X$  be a nonsingular projective variety over  $k$ . Let  $D_1, \dots, D_m$  be effective divisors on  $X$ , defined over  $k$ , with empty intersection. Let  $D = \sum_{i=1}^m D_i$ .

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# Ideas of the proof of Main Theorem 1.

Since  $D_1, \dots, D_n$  intersect transversally, their intersection contains only points. Let  $\cap_{i=1}^n D_i = \{p_1, \dots, p_m\}$ .

Let  $\pi : \tilde{X} \rightarrow X$  be the blow-up along  $p_1, \dots, p_m$ , successively, and let  $E_j := \pi^{-1}(p_j)$  be the exceptional divisor for  $1 \leq j \leq m$ .

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## Theorem

Let  $X$  be a nonsingular projective variety of dimension  $n$  over a field  $k$  and let  $D$  be a nef divisor on  $X$ . Let  $K_X$  denote the canonical divisor on  $X$ . Let  $\pi : \tilde{X} \rightarrow X$  be the blow-up along  $m$  distinct points  $P_1, \dots, P_m$  of  $X$ , successively, and let  $E_i := \pi^{-1}(P_i)$  be the exceptional divisor for  $1 \leq i \leq m$ .

- 1 If  $D^n > 1$ , then  $\pi^*D - E_i$  is big for each  $1 \leq i \leq m$ .
- 2 If  $D^n = 1$  and  $K_X \cdot D^{n-1} < 1 - n$ , then  $\kappa(\pi^*D - E_i) \geq n - 1$  for each  $1 \leq i \leq m$ .

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# Sketch of Proof

The case of blowing-up one point:  $\pi : \tilde{X} \rightarrow X$ . Let  $i : E \rightarrow \tilde{X}$  be the inclusion map. We have an exact sequence

$$\begin{aligned} 0 \rightarrow H^0(\tilde{X}, \mathcal{O}(m\pi^*D - (j+1)E)) &\rightarrow H^0(\tilde{X}, \mathcal{O}(m\pi^*D - jE)) \\ &\rightarrow H^0(E, i^*\mathcal{O}(m\pi^*D - jE)). \end{aligned}$$

A standard computation shows

$$h^0(m\pi^*D) - h^0(m\pi^*D - mE) \leq \frac{m^n}{n!} + \frac{1}{2} \frac{m^{n-1}}{(n-2)!} + O(m^{n-2}).$$

Apply the following asymptotic Riemann-Roch formula.

### Theorem (Matsusaka)

*Let  $X$  be a nonsingular projective variety of dimension  $n$  and let  $D$  be a nef and big divisor on  $X$ . Then*

$$h^0(mD) = \frac{D^n}{n!} m^n - \frac{K_X \cdot D^{n-1}}{2(n-1)!} m^{n-1} + O(m^{n-2}).$$

Then

$$h^0(m\pi^*D - mE) \geq \frac{D^n - 1}{n!} m^n - \frac{K_X \cdot D^{n-1} + (n-1)}{2(n-1)!} m^{n-1} + O(m^{n-2}).$$

# Basics of rational ruled surfaces

## Proposition

A rational ruled surface  $X$  (over  $\bar{k}$ ) is isomorphic to  $X_e := \mathbf{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$  over  $\mathbb{P}^1$  for some nonnegative integer  $e$ . Let  $F$  denote a fiber on  $X$  and let  $C_0$  denote a section of  $X$  such that  $\mathcal{O}(C_0) \cong \mathcal{O}_{X_e}(1)$ . Then

- (a)  $\text{Pic}X \cong \mathbb{Z} \oplus \mathbb{Z}$  generated by  $C_0 \subset X$  and  $F$  with  $C_0^2 = -e$ ,  $F^2 = 0$ , and  $C_0 \cdot F = 1$ .
- (b) Let  $K_X$  be the canonical divisor on  $X$ . Then  $K_X \sim -2C_0 - (2 + e)F$ . In particular,  $K_X^2 = 8$ .
- (c) Let  $D$  be a divisor on  $X$  equivalent to  $aC_0 + bF$  in  $\text{Pic}X$ . Then
  - (i) If  $D$  is an irreducible curve  $\neq C_0, F$ , then  $a, b > 0$ ,  $b \geq ae$ , and  $D^2 > 0$ .
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Since  $D_2$  is linearly equivalent to a positive integral linear combination of  $C$  and  $F'$ , there exists a non-constant function  $\phi \in \mathbf{k}(X)^*$  with poles and zeros only in the support of  $D_1$  and  $D_2$ .

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Since  $D_2$  is linearly equivalent to a positive integral linear combination of  $C$  and  $F'$ , there exists a non-constant function  $\phi \in \mathbf{k}(X)^*$  with poles and zeros only in the support of  $D_1$  and  $D_2$ .

Consider an analytic map  $f : \mathbf{k} \rightarrow X \setminus (D_1 \cup D_2)$ . Then

$\phi \circ f : \mathbf{k} \rightarrow \mathbb{A}^1 \setminus \{0\}$  is analytic, and hence constant. It follows that the image of  $f$  is contained in a proper subvariety of  $X$ .

Assume furthermore that  $D_1$  and  $D_2$  are ample.

By (a), the image of a non-constant analytic map  $f : \mathbf{k} \rightarrow X \setminus (D_1 \cup D_2)$  is contained in a curve in  $X$ .

Let  $C$  be the Zariski closure of the image of  $f$  in  $X$ .

If  $C \cap (D_1 \cup D_2)$  contains more than one point, then  $f$  must be constant.

On the other hand,  $D_i \cap C \neq \emptyset$  for  $i = 1, 2$ , since  $D_1$  and  $D_2$  are ample.

Therefore, we only need to consider when  $C \cap D_1 = C \cap D_2 = \{x\}$  for some  $x \in X$ .

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Let  $\pi : \tilde{X} \rightarrow X$  be the blow-up of  $X$  at  $x$  with exceptional divisor  $E$ .

Let  $\tilde{C}$  be the strict transform of  $C$  and  $\tilde{D}_i$  the strict transform of  $D_i$ ,  $i = 1, 2$ .

Then  $f : \mathbf{k} \rightarrow X \setminus D_1 \cup D_2$  lifts to  $\tilde{f} : \mathbf{k} \rightarrow \tilde{X} \setminus \tilde{D}_1 \cup \tilde{D}_2$  with  $f = \pi \circ \tilde{f}$  and the image of  $\tilde{f}$  is contained in  $\tilde{C}$ .

Denote by  $m = m_x(C)$  the multiplicity of  $C$  at  $x$ .

If  $(C \cdot D_i)_x > m = m_x(C) \cdot m_x(D_i)$  for  $i = 1, 2$ , then each  $\tilde{D}_i$  must intersect  $\tilde{C}$  at some point on  $\tilde{X}$  lying above  $x$ .

Since  $D_1$  and  $D_2$  intersect transversally,  $\bigcap_{i=1}^2 \tilde{D}_i \cap E = \emptyset$ . Thus, there must be at least two points on  $\tilde{C}$  lying above  $x$ .

Consequently,  $\tilde{f} : \mathbf{k} \rightarrow \tilde{X} \setminus \tilde{D}_1 \cup \tilde{D}_2$  is constant and hence  $f$  is also constant.

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Let  $(C.D_1)_x = m$ .

Suppose that  $C \equiv cC_0 + dF$  and  $D_1 \equiv aC_0 + bF$ .

Then  $(C.D_1)_x = C.D_1 = ad + c(b - ae)$ .

Assume that  $C$  is not linearly equivalent to  $C_0$  or  $F$ .

From Proposition, we have that  $a, b - ae, c, d > 0$ .

By taking  $F$  to be the fiber passing through  $x$ , we see that

$$c = C.F \geq m.$$

Then  $m = C.D_1 = ad + c(b - ae) > c \geq m$ , a contradiction.

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