Curves in Hilbert modular varieties

Erwan Rousseau (j.w.w. Frédéric Touzet)

Université d’Aix-Marseille

Banff, March 17th.
Conjecture

Let $\tilde{X}$ be a projective manifold defined over a number field $k$ and $D = \tilde{X} \setminus X$ a normal crossings divisor. If $(\tilde{X}, D)$ is of log-general type then for every ring of $S$-integers $\mathcal{O}_S$ the set of $S$-integral points $X(\mathcal{O}_S)$ is not Zariski-dense.
Conjecture

Let $\tilde{X}$ be a projective manifold defined over a number field $k$ and $D = \tilde{X} \setminus X$ a normal crossings divisor. If $(\tilde{X}, D)$ is of log-general type then for every ring of $S$-integers $\mathcal{O}_S$ the set of $S$-integral points $X(\mathcal{O}_S)$ is not Zariski-dense.

An algebraic function field analogue predicts

Conjecture

Let $\tilde{X}$ be a complex projective manifold, $D = \tilde{X} \setminus X$ a normal crossings divisor, $\tilde{C}$ a smooth projective curve and $S \subset \tilde{C}$ a finite subset. If $(\tilde{X}, D)$ is of log-general type then there exists a bound for the degree of the images of non-constant morphisms $\tilde{C} \setminus S \to X$. 
Theorem (Bogomolov, Miyaoka)

Let $X$ be a minimal complex projective surface of general type and $C \subset X$ an irreducible curve of geometric genus $g$. If $c_1^2 > c_2$ then

$$C.K_X \leq a(2g - 2) + b$$

where $a$ and $b$ are effective functions of $c_1^2$ and $c_2$. 

Erwan Rousseau (j.w.w. Frédéric Touzet) 
Curves in Hilbert modular varieties
### Known results

**Theorem (Bogomolov, Miyaoka)**

Let $X$ be a minimal complex projective surface of general type and $C \subset X$ an irreducible curve of geometric genus $g$. If $c_1^2 > c_2$ then

$$C.K_X \leq a(2g - 2) + b$$

where $a$ and $b$ are effective functions of $c_1^2$ and $c_2$.

**Theorem (Chen, Pacienza-Rousseau)**

Let $D \subset \mathbb{P}^n$ be a very general hypersurface of degree $d \geq 2n + 1$ and $f : \tilde{C} \to \mathbb{P}^n$ a finite morphism from a smooth projective curve such that $f(\tilde{C}) \not\subset D$. Then

$$(d - 2n) \deg_C(f^*(\mathcal{O}(1))) \leq 2g(\tilde{C}) - 2 + N_1(f^*D).$$
Known results

Remark

\[ \tilde{X} = \mathbb{P}^2, \text{ } D \text{ a quartic, union of a smooth conic and two lines in general position} \text{ (Corvaja-Zannier)} \]
Known results

Remark

- $\tilde{X} = \mathbb{P}^2$, $D$ a quartic, union of a smooth conic and two lines in general position (Corvaja-Zannier)
- $\tilde{X} = \mathbb{P}^2$, $D$ a very general quartic (Turchet)
Known results

Remark
- \( \tilde{X} = \mathbb{P}^2 \), \( D \) a quartic, union of a smooth conic and two lines in general position (Corvaja-Zannier)
- \( \tilde{X} = \mathbb{P}^2 \), \( D \) a very general quartic (Turchet)

Theorem (Demailly)

Let \( X \) be complex projective Kobayashi hyperbolic manifold. Then there exists a constant \( a > 0 \) such that for every projective curve \( C \) and any finite morphism \( f : C \to X \)

\[
\deg_C(f^* K_X) \leq a(2g - 2).
\]
Theorem (Autissier - Chambert-Loir - Gasbarri)

Let $X$ be a compact quotient of a bounded symmetric domain. Then for any projective curve $C$ and any finite morphism $g : C \to X$,

$$\deg_C(f^*K_X) \leq \dim(X)(2g - 2).$$
Hilbert modular varieties

- $K$ a totally real number field of degree $n$. 
Hilbert modular varieties

- $K$ a totally real number field of degree $n$.
- $\mathcal{O}$ the ring of algebraic integers of $K$. 
Hilbert modular varieties

- $K$ a totally real number field of degree $n$.
- $\mathcal{O}$ the ring of algebraic integers of $K$.
- The Hilbert modular group is $\Gamma_K := SL_2(\mathcal{O})$. 
Hilbert modular varieties

- $K$ a totally real number field of degree $n$.
- $\mathcal{O}$ the ring of algebraic integers of $K$.
- The Hilbert modular group is $\Gamma_K := SL_2(\mathcal{O})$.
- $\Gamma_K$ acts on the product of $n$ upper half-planes $\mathbb{H}^n$. 

\[ \text{Curves in Hilbert modular varieties} \]
Hilbert modular varieties

- $K$ a totally real number field of degree $n$.
- $\mathcal{O}$ the ring of algebraic integers of $K$.
- The Hilbert modular group is $\Gamma_K := SL_2(\mathcal{O})$.
- $\Gamma_K$ acts on the product of $n$ upper half-planes $\mathbb{H}^n$.
- There is a natural compactification of the quotient $\mathbb{H}^n/\Gamma_K$ adding finitely many cusps.

Definition

A non-singular model $X_K$ is called a Hilbert modular variety.

Theorem (Rousseau-Touzet)

There is a projective resolution $\pi : X \to \mathbb{H}^n/\Gamma_K$ with $E$ the exceptional divisor, $K_X$ the canonical line bundle of $X$ such that if $C$ is a smooth projective curve and $f : C \to X$ a finite morphism such that $f(C) \not\subset E$.

Then $\deg C(f^*(K_X+E)) \leq n(2g(C) - 2 + N_1(f^*E))$.
Hilbert modular varieties

- $K$ a totally real number field of degree $n$.
- $\mathcal{O}$ the ring of algebraic integers of $K$.
- The Hilbert modular group is $\Gamma_K := SL_2(\mathcal{O})$.
- $\Gamma_K$ acts on the product of $n$ upper half-planes $\mathbb{H}^n$.
- There is a natural compactification of the quotient $\mathbb{H}^n/\Gamma_K$ adding finitely many cusps.

**Definition**

A non-singular model $X_K$ is called a *Hilbert modular variety*. 
Hilbert modular varieties

- $K$ a totally real number field of degree $n$.
- $\mathcal{O}$ the ring of algebraic integers of $K$.
- The Hilbert modular group is $\Gamma_K := SL_2(\mathcal{O})$.
- $\Gamma_K$ acts on the product of $n$ upper half-planes $\mathbb{H}^n$.
- There is a natural compactification of the quotient $\mathbb{H}^n/\Gamma_K$ adding finitely many cusps.

**Definition**

A non-singular model $X_K$ is called a *Hilbert modular variety*.

**Theorem (Rousseau-Touzet)**

There is a projective resolution $\pi : X \to \overline{\mathbb{H}^n/\Gamma_K}$ with $E$ the exceptional divisor, $K_X$ the canonical line bundle of $X$ such that if $C$ is a smooth projective curve and $f : C \to X$ a finite morphism such that $f(C) \not\subset E$. Then

$$\deg_C(f^*(K_X + E)) \leq n(2g(C) - 2 + N_1(f^*E)).$$
Hilbert modular foliations

- $dz_i = 0$ defines a holomorphic codimension-one foliation on $\mathcal{H}^n$. 
Hilbert modular foliations

- $dz_i = 0$ defines a holomorphic codimension-one foliation on $\mathbb{H}^n$.
- Projecting gives a codimension-one foliation $\mathcal{G}_i$ on $\overline{\mathbb{H}}^n/\Gamma$. 

---

Erwan Rousseau (j.w.w. Frédéric Touzet)  
Curves in Hilbert modular varieties
$dz_i = 0$ defines a holomorphic codimension-one foliation on $\mathfrak{H}^n$.
projecting gives a codimension-one foliation $\mathcal{G}_i$ on $\overline{\mathfrak{H}^n/\Gamma}$.
pulling back to the resolution $\pi : X \to \overline{\mathfrak{H}^n/\Gamma}$ gives the $i^{th}$
codimension-one holomorphic foliation $\mathcal{F}_i = \pi^* \mathcal{G}_i$ on $X$. 

---

Erwan Rousseau (j.w.w. Frédéric Touzet)  
Curves in Hilbert modular varieties
Hilbert modular foliations

- $dz_i = 0$ defines a holomorphic codimension-one foliation on $\mathcal{H}^n$.
- Projecting gives a codimension-one foliation $\mathcal{G}_i$ on $\mathcal{H}^n/\Gamma$.
- Pulling back to the resolution $\pi : X \to \mathcal{H}^n/\Gamma$ gives the $i^{th}$ codimension-one holomorphic foliation $\mathcal{F}_i = \pi^* \mathcal{G}_i$ on $X$.

We have the following tangency formula

**Theorem**

There exists a projective resolution $\pi : X \to \mathcal{H}^n/\Gamma$ such that

$$N_{\mathcal{F}_1}^* \otimes \ldots \otimes N_{\mathcal{F}_n}^* = K_X \otimes \mathcal{O}(- (n-1)E)$$
Hilbert modular foliations

- $dz_i = 0$ defines a holomorphic codimension-one foliation on $\mathcal{H}^n$.
- projecting gives a codimension-one foliation $\mathcal{G}_i$ on $\overline{\mathcal{H}^n}/\Gamma$.
- pulling back to the resolution $\pi : X \to \overline{\mathcal{H}^n}/\Gamma$ gives the $i^{th}$ codimension-one holomorphic foliation $\mathcal{F}_i = \pi^* \mathcal{G}_i$ on $X$.

We have the following tangency formula

**Theorem**

There exists a projective resolution $\pi : X \to \overline{\mathcal{H}^n}/\Gamma$ such that

$$N_{\mathcal{F}_1}^* \otimes \ldots \otimes N_{\mathcal{F}_n}^* = K_X \otimes \mathcal{O}(- (n-1)E)$$

**Proposition**

Let $E_c \subset E$ be the hypersurface corresponding to the cusps resolution. Let $C$ be a germ of analytic curve tangent to $\mathcal{F}_i$ passing through $p \in E_c$. Then $C$ is contained in $E_c$. 
Proof.
The change of coordinates in the resolution of cusps is (locally) given by

\[ z_i = \sum_j a_{ij} \log u_j. \]

where the coefficients \( a_{ij} \) are positive real numbers.
Proof.

The change of coordinates in the resolution of cusps is (locally) given by

\[ z_i = \sum_j a_{ji} \log u_j. \]

where the coefficients \( a_{ij} \) are positive real numbers.

In particular the (locally defined) real valued function

\[ g = \prod_j |u_j|^{a_{ij}} \]

is continuous, constant along the leaves of \( \mathcal{F}_i \) and vanishes precisely on \( E_c \). Therefore \( g \) necessarily vanishes along \( C \), hence \( C \) is contained in \( E_c \).
Proposition

Leaves of $\mathcal{G}_i$ are (Brody) hyperbolic.
Proposition

Leaves of $G_i$ are (Brody) hyperbolic.

Proof.

- If the leaf avoids orbifold points: trivial.
Proposition

Leaves of $G_i$ are (Brody) hyperbolic.

Proof.

- If the leaf avoids orbifold points: trivial.
- If the leaf passes through an orbifold point: next lemma.
Proposition

Leaves of $G_i$ are (Brody) hyperbolic.

Proof.

- If the leaf avoids orbifold points: trivial.
- If the leaf passes through an orbifold point: next lemma.

Lemma

Leaves passing through an orbifold point are quotients of polydisk by finite groups, in particular they are Stein and hyperbolic.
Proof.

Let us prove that the stabilizer of a leaf $\mathcal{H}^{n-1}$ in $\mathcal{H}^n$ passing through an orbifold point is finite.
Proof.

Let us prove that the stabilizer of a leaf $\mathcal{H}^{n-1}$ in $\mathcal{H}^n$ passing through an orbifold point is finite. The stabilizer is a commutative group.
Proof.

Let us prove that the stabilizer of a leaf $\mathcal{H}^{n-1}$ in $\mathcal{H}^n$ passing through an orbifold point is finite.

The stabilizer is a commutative group. Indeed, $\Gamma_K = SL(2, \mathcal{O})$ acts on $\mathcal{H}^n$ via the embedding of groups $SL(2, K) \hookrightarrow SL(2, \mathbb{R})^n$ and the projections $p_i : SL(2, \mathbb{R})^n \to SL(2, \mathbb{R})$ have restrictions to $\Gamma_K$ which are injective.
Hilbert modular foliations

Proof.

Let us prove that the stabilizer of a leaf $\mathcal{H}^{n-1}$ in $\mathcal{H}^n$ passing through an orbifold point is finite.

The stabilizer is a commutative group. Indeed, $\Gamma_K = \text{SL}(2, \mathcal{O})$ acts on $\mathcal{H}^n$ via the embedding of groups $\text{SL}(2, K) \hookrightarrow \text{SL}(2, \mathbb{R})^n$ and the projections $p_i : \text{SL}(2, \mathbb{R})^n \to \text{SL}(2, \mathbb{R})$ have restrictions to $\Gamma_K$ which are injective.

If two elements $g$ and $h$ of $\Gamma_K$ are in the stabilizer of a leaf, it means that the projections $g_i := p_i(g)$ and $h_i := p_i(h)$ of $g$ and $h$ on the corresponding factor of $\text{SL}(2, \mathbb{R})^n$ have the same fixed point and so commute.
Hilbert modular foliations

Proof.

Let us prove that the stabilizer of a leaf $\mathcal{H}^{n-1}$ in $\mathcal{H}^n$ passing through an orbifold point is finite.

The stabilizer is a commutative group. Indeed, $\Gamma_K = SL(2, \mathcal{O})$ acts on $\mathcal{H}^n$ via the embedding of groups $SL(2, K) \hookrightarrow SL(2, \mathbb{R})^n$ and the projections $p_i : SL(2, \mathbb{R})^n \to SL(2, \mathbb{R})$ have restrictions to $\Gamma_K$ which are injective.

If two elements $g$ and $h$ of $\Gamma_K$ are in the stabilizer of a leaf, it means that the projections $g_i := p_i(g)$ and $h_i := p_i(h)$ of $g$ and $h$ on the corresponding factor of $SL(2, \mathbb{R})^n$ have the same fixed point and so commute.

This implies that $g$ and $h$ must commute.
Proof.

Let us prove that the stabilizer of a leaf $\mathcal{H}^{n-1}$ in $\mathcal{H}^n$ passing through an orbifold point is finite.

The stabilizer is a commutative group. Indeed, $\Gamma_K = SL(2, \mathcal{O})$ acts on $\mathcal{H}^n$ via the embedding of groups $SL(2, K) \hookrightarrow SL(2, \mathbb{R})^n$ and the projections $p_i : SL(2, \mathbb{R})^n \to SL(2, \mathbb{R})$ have restrictions to $\Gamma_K$ which are injective.

If two elements $g$ and $h$ of $\Gamma_K$ are in the stabilizer of a leaf, it means that the projections $g_i := p_i(g)$ and $h_i := p_i(h)$ of $g$ and $h$ on the corresponding factor of $SL(2, \mathbb{R})^n$ have the same fixed point and so commute.

This implies that $g$ and $h$ must commute. Now suppose that $g$ is in the stabilizer of the orbifold point, which is assumed to be non trivial. Then any other element $h$ of the stabilizer of the leaf commutes with $g$ which means that $h$ is in fact in the stabilizer of the orbifold point:
Proof.

Let us prove that the stabilizer of a leaf $\mathcal{H}^{n-1}$ in $\mathcal{H}^n$ passing through an orbifold point is finite.

The stabilizer is a commutative group. Indeed, $\Gamma_K = SL(2, \mathcal{O})$ acts on $\mathcal{H}^n$ via the embedding of groups $SL(2, K) \hookrightarrow SL(2, \mathbb{R})^n$ and the projections $p_i : SL(2, \mathbb{R})^n \to SL(2, \mathbb{R})$ have restrictions to $\Gamma_K$ which are injective.

If two elements $g$ and $h$ of $\Gamma_K$ are in the stabilizer of a leaf, it means that the projections $g_i := p_i(g)$ and $h_i := p_i(h)$ of $g$ and $h$ on the corresponding factor of $SL(2, \mathbb{R})^n$ have the same fixed point and so commute.

This implies that $g$ and $h$ must commute.

Now suppose that $g$ is in the stabilizer of the orbifold point, which is assumed to be non trivial. Then any other element $h$ of the stabilizer of the leaf commutes with $g$ which means that $h$ is in fact in the stabilizer of the orbifold point: the stabilizer of the leaf and of the orbifold point coincide.
Theorem (Rousseau-Touzet)

There is a projective resolution $\pi : X \to \mathfrak{H}^n/\Gamma_K$ with $E$ the exceptional divisor, $K_X$ the canonical line bundle of $X$ such that if $C$ is a smooth projective curve and $f : C \to X$ a finite morphism such that $f(C) \not\subset E$. Then

$$\deg_C(f^*(K_X + E)) \leq n(2g(C) - 2 + N_1(f^*E)).$$
Theorem (Rousseau-Touzet)

There is a projective resolution \( \pi : X \to \tilde{\mathcal{H}}^n/\Gamma_K \) with \( E \) the exceptional divisor, \( K_X \) the canonical line bundle of \( X \) such that if \( C \) is a smooth projective curve and \( f : C \to X \) a finite morphism such that \( f(C) \not\subset E \). Then

\[
\deg_C(f^*(K_X + E)) \leq n(2g(C) - 2 + N_1(f^*E)).
\]

- The morphism \( f : C \to X \) induces a morphism \( f' : C \to \mathbb{P}(T_X(-\log E)) \)
Proof of geometric Lang-Vojta for Hilbert modular varieties

Theorem (Rousseau-Touzet)

There is a projective resolution $\pi : X \to \mathcal{H}^n/\Gamma_K$ with $E$ the exceptional divisor, $K_X$ the canonical line bundle of $X$ such that if $C$ is a smooth projective curve and $f : C \to X$ a finite morphism such that $f(C) \not\subset E$. Then

$$\deg_C(f^*(K_X + E)) \leq n(2g(C) - 2 + N_1(f^*E)).$$

- The morphism $f : C \to X$ induces a morphism $f' : C \to \mathbb{P}(T_X(-\log E))$
- We have an inclusion $f'^*(\mathcal{O}(1)) \hookrightarrow K_C(f^*(E)_{\text{red}})$
Theorem (Rousseau-Touzet)

There is a projective resolution $\pi : X \to \bar{\mathcal{H}}^n/\Gamma_K$ with $E$ the exceptional divisor, $K_X$ the canonical line bundle of $X$ such that if $C$ is a smooth projective curve and $f : C \to X$ a finite morphism such that $f(C) \not\subset E$. Then

$$\deg_C(f^*(K_X + E)) \leq n(2g(C) - 2 + N_1(f^*E)).$$

- The morphism $f : C \to X$ induces a morphism $f' : C \to \mathbb{P}(T_X(-\log E))$
- We have an inclusion $f'^*(\mathcal{O}(1)) \hookrightarrow K_C(f^*(E)_{red})$
- This gives the algebraic tautological inequality $\deg_C(f'^*(\mathcal{O}(1)) \leq 2g(C) - 2 + N_1(f^*E)$. 
Theorem (Rousseau-Touzet)

There is a projective resolution $\pi : X \to \bar{\mathcal{H}}^n/\Gamma_K$ with $E$ the exceptional divisor, $K_X$ the canonical line bundle of $X$ such that if $C$ is a smooth projective curve and $f : C \to X$ a finite morphism such that $f(C) \not\subset E$. Then

$$\deg_C(f^*(K_X + E)) \leq n(2g(C) - 2 + N_1(f^*E)).$$

- The morphism $f : C \to X$ induces a morphism $f' : C \to \mathbb{P}(T_X(-\log E))$
- We have an inclusion $f'^*(\mathcal{O}(1)) \hookrightarrow K_C(f^*(E)_{\text{red}})$
- This gives the algebraic tautological inequality $\deg_C(f'^*(\mathcal{O}(1)) \leq 2g(C) - 2 + N_1(f^*E)$.

We distinguish two cases: either $f(C)$ is contained in a leaf of a Hilbert modular foliation or not.
Proof of geometric Lang-Vojta for Hilbert modular varieties

If $f(C)$ is not tangent to a Hilbert modular foliation.
If $f(C)$ is not tangent to a Hilbert modular foliation.

- Let $\mathcal{F}$ be one of the canonical Hilbert modular foliation on $X$. 
If $f(C)$ is not tangent to a Hilbert modular foliation.

- Let $\mathcal{F}$ be one of the canonical Hilbert modular foliation on $X$.
- To the foliation $\mathcal{F}$ is associated a divisor $Z \subset \mathbb{P}(T_X(-\log E))$, linearly equivalent to $\mathcal{O}(1) + N_\mathcal{F}(-E)$. 
If $f(C)$ is not tangent to a Hilbert modular foliation.

- Let $\mathcal{F}$ be one of the canonical Hilbert modular foliation on $X$.
- To the foliation $\mathcal{F}$ is associated a divisor $Z \subset \mathbb{P}(T_X(-\log E))$, linearly equivalent to $\mathcal{O}(1) + N_\mathcal{F}(-E)$.
- The algebraic tautological inequality gives: $\deg_C(f^*(N^*_\mathcal{F}(E))) \leq \deg_C(f^*(Z)) + \deg_C(f'^*(N^*_\mathcal{F}(E))) \leq 2g(C) - 2 + N_1(f^*E)$. 

Erwan Rousseau (j.w.w. Frédéric Touzet)

Curves in Hilbert modular varieties
If \( f(C) \) is not tangent to a Hilbert modular foliation.

- Let \( \mathcal{F} \) be one of the canonical Hilbert modular foliation on \( X \).
- To the foliation \( \mathcal{F} \) is associated a divisor \( Z \subset \mathbb{P}(T_X(-\log E)) \), linearly equivalent to \( \mathcal{O}(1) + N_{\mathcal{F}}(-E) \).
- The algebraic tautological inequality gives: \( \deg_C(f^*(N_{\mathcal{F}}^*(E))) \leq \deg_C(f^*(Z)) + \deg_C(f'^*(N_{\mathcal{F}}^*(E))) \leq 2g(C) - 2 + N_1(f^*E) \).

Now we use the tangency formula

\[
N_{\mathcal{F}_1}^* \otimes \ldots \otimes N_{\mathcal{F}_n}^* = K_X \otimes \mathcal{O}(- (n-1)E).
\]
If $f(C)$ is not tangent to a Hilbert modular foliation.

- Let $\mathcal{F}$ be one of the canonical Hilbert modular foliation on $X$.
- To the foliation $\mathcal{F}$ is associated a divisor $Z \subset \mathbb{P}(T_X(-\log E))$, linearly equivalent to $\mathcal{O}(1) + N_{\mathcal{F}}(-E)$.
- The algebraic tautological inequality gives: \(\deg_C(f^*(N_{\mathcal{F}}^*(E))) \leq \deg_C(f^*(Z)) + \deg_C(f'^*(N_{\mathcal{F}}^*(E))) \leq 2g(C) - 2 + N_1(f^*E)\).

Now we use the tangency formula

\[ N_{\mathcal{F}_1}^* \otimes \cdots \otimes N_{\mathcal{F}_n}^* = K_X \otimes \mathcal{O}(- (n - 1)E). \]

We obtain

\[ \deg_C(f^*(K_X + E)) = \sum_i \deg_C(f^*(N_{\mathcal{F}_i}^*(E))) \leq n(2g(C) - 2 + N_1(f^*E)). \]
If $f(C)$ is contained in a leaf of a Hilbert modular foliation:
Proof of geometric Lang-Vojta for Hilbert modular varieties

If $f(C)$ is contained in a leaf of a Hilbert modular foliation: it has to avoid cusps and orbifold points.
If $f(C)$ is contained in a leaf of a Hilbert modular foliation: it has to avoid cusps and orbifold points. Therefore, $f(C)$ is contained in the smooth part of $\mathfrak{H}^n/\Gamma$. 

Let $g$ denote the Bergmann metric with Kähler form $\omega$ on $\mathfrak{H}^n$ such that $\text{Ricci}(g) = -g$ and having holomorphic sectional curvature $\leq -\frac{1}{n}$. It descends to a metric on the regular part of $\mathfrak{H}^n/\Gamma$. 

$$\int_C f^*(\omega) = 2\pi \deg C(f^*(K_X))$$

$f^*(\omega)$ induces a hermitian metric on the canonical line bundle $K_C(-Rf)$ where $Rf$ is the branching divisor of $f$. If $\Theta$ denotes its curvature, one has

$$\int_C \Theta = \deg(K_C(-Rf)) = 2g(C) - 2 - \deg(Rf).$$

From the definition of the holomorphic sectional curvature one has,

$$\Theta \geq \omega^2 \pi^n.$$ 

Therefore \(\deg C(f^*(K_X)) \leq n(2g(C) - 2)\).
If \( f(C) \) is contained in a leaf of a Hilbert modular foliation: it has to avoid cusps and orbifold points. Therefore, \( f(C) \) is contained in the smooth part of \( \mathcal{H}^n/\Gamma \).

Let \( g \) denote the Bergmann metric with Kähler form \( \omega \) on \( \mathcal{H}^n \) such that \( \text{Ricci}(g) = -g \) and having holomorphic sectional curvature \( \leq -1/n \).
If $f(C)$ is contained in a leaf of a Hilbert modular foliation: it has to avoid cusps and orbifold points. Therefore, $f(C)$ is contained in the smooth part of $\mathbb{H}^n/\Gamma$.

Let $g$ denote the Bergmann metric with Kähler form $\omega$ on $\mathbb{H}^n$ such that $\text{Ricci}(g) = -g$ and having holomorphic sectional curvature $\leq -1/n$. It descends to a metric on the regular part of $\mathbb{H}^n/\Gamma$. 

$$\int_C f^* (\omega) = 2\pi \deg_C (f^* (K_X))$$

$f^* (\omega)$ induces a hermitian metric on the canonical line bundle $K_C (-R_f)$ where $R_f$ is the branching divisor of $f$. If $\Theta$ denotes its curvature, one has

$$\int_C \Theta = \deg (K_C (-R_f)) = 2g(C) - 2 - \deg (R_f).$$

From the definition of the holomorphic sectional curvature one has, $\Theta \geq \omega$.

Therefore $\deg_C (f^* (K_X)) \leq n(2g(C) - 2)$. 
If $f(C)$ is contained in a leaf of a Hilbert modular foliation: it has to avoid cusps and orbifold points. Therefore, $f(C)$ is contained in the smooth part of $\mathcal{H}^n/\Gamma$. Let $g$ denote the Bergmann metric with Kähler form $\omega$ on $\mathcal{H}^n$ such that $\text{Ricci}(g) = -g$ and having holomorphic sectional curvature $\leq -1/n$. It descends to a metric on the regular part of $\mathcal{H}^n/\Gamma$.

- $\int_C f^*(\omega) = 2\pi \deg_C(f^*(K_X))$. 

Erwan Rousseau (j.w.w. Frédéric Touzet)
Proof of geometric Lang-Vojta for Hilbert modular varieties

If $f(C)$ is contained in a leaf of a Hilbert modular foliation: it has to avoid cusps and orbifold points. Therefore, $f(C)$ is contained in the smooth part of $\mathcal{H}^n/\Gamma$.

Let $g$ denote the Bergmann metric with Kähler form $\omega$ on $\mathcal{H}^n$ such that $Ricci(g) = -g$ and having holomorphic sectional curvature $\leq -1/n$. It descends to a metric on the regular part of $\mathcal{H}^n/\Gamma$.

- $\int_C f^*(\omega) = 2\pi \deg_C(f^*(K_X))$.
- $f^*(\omega)$ induces a hermitian metric on the canonical line bundle $K_C(-R_f)$ where $R_f$ is the branching divisor of $f$. 
Proof of geometric Lang-Vojta for Hilbert modular varieties

If \( f(C) \) is contained in a leaf of a Hilbert modular foliation: it has to avoid cusps and orbifold points. Therefore, \( f(C) \) is contained in the smooth part of \( \mathcal{H}^n/\Gamma \).

Let \( g \) denote the Bergmann metric with Kähler form \( \omega \) on \( \mathcal{H}^n \) such that \( \text{Ricci}(g) = -g \) and having holomorphic sectional curvature \( \leq -1/n \). It descends to a metric on the regular part of \( \mathcal{H}^n/\Gamma \).

- \( \int_C f^*(\omega) = 2\pi \deg_C(f^*(K_X)) \).
- \( f^*(\omega) \) induces a hermitian metric on the canonical line bundle \( K_C(-R_f) \) where \( R_f \) is the branching divisor of \( f \).
- If \( \Theta \) denotes its curvature, one has \( \int_C \Theta = \deg(K_C(-R_f)) = 2g(C) - 2 - \deg(R_f) \).
Proof of geometric Lang-Vojta for Hilbert modular varieties

If $f(C)$ is contained in a leaf of a Hilbert modular foliation: it has to avoid cusps and orbifold points. Therefore, $f(C)$ is contained in the smooth part of $\mathcal{H}^n/\Gamma$.

Let $g$ denote the Bergmann metric with Kähler form $\omega$ on $\mathcal{H}^n$ such that $Ricci(g) = -g$ and having holomorphic sectional curvature $\leq -1/n$. It descends to a metric on the regular part of $\mathcal{H}^n/\Gamma$.

- $\int_C f^*(\omega) = 2\pi \deg_C(f^*(K_X))$.
- $f^*(\omega)$ induces a hermitian metric on the canonical line bundle $K_C(-R_f)$ where $R_f$ is the branching divisor of $f$.
- If $\Theta$ denotes its curvature, one has $\int_C \Theta = \deg(K_C(-R_f)) = 2g(C) - 2 - \deg(R_f)$.
- From the definition of the holomorphic sectional curvature one has, $\Theta \geq \frac{\omega}{2\pi.n}$.
Proof of geometric Lang-Vojta for Hilbert modular varieties

If $f(C)$ is contained in a leaf of a Hilbert modular foliation: it has to avoid cusps and orbifold points. Therefore, $f(C)$ is contained in the smooth part of $\mathbb{H}^n/\Gamma$.

Let $g$ denote the Bergmann metric with Kähler form $\omega$ on $\mathbb{H}^n$ such that $\text{Ricci}(g) = -g$ and having holomorphic sectional curvature $\leq -1/n$. It descends to a metric on the regular part of $\mathbb{H}^n/\Gamma$.

- $\int_C f^*(\omega) = 2\pi \deg_C(f^*(K_X))$.
- $f^*(\omega)$ induces a hermitian metric on the canonical line bundle $K_C(-R_f)$ where $R_f$ is the branching divisor of $f$.
- If $\Theta$ denotes its curvature, one has $\int_C \Theta = \deg(K_C(-R_f)) = 2g(C) - 2 - \deg(R_f)$.
- From the definition of the holomorphic sectional curvature one has, $\Theta \geq \frac{\omega}{2\pi.n}$.
- Therefore $\deg_C(f^*(K_X)) \leq n(2g(C) - 2)$.
The same proof gives the following Second Main Theorem:

**Theorem**

Consider a projective resolution \( \pi : X \rightarrow \mathcal{H}^n/\Gamma \) as above, \( E \) the exceptional divisor, \( K_X \) the canonical line bundle of \( X \). Let \( f : \mathbb{C} \rightarrow X \) be a non-constant entire curve such that \( f(\mathbb{C}) \) is not contained in \( E \). Then

\[
T_f(r, K_X) + T_f(r, E) \leq nN_1(r, f^*E) + S_f(r)\|,
\]

where \( S_f(r) = O(\log^+ T_f(r)) + o(\log r) \), and \( \| \) means that the estimate holds outside some exceptional set of finite measure.
The same proof gives the following Second Main Theorem:

**Theorem**

Consider a projective resolution \( \pi : X \to \mathfrak{H}^n/\Gamma \) as above, \( E \) the exceptional divisor, \( K_X \) the canonical line bundle of \( X \). Let \( f : \mathbb{C} \to X \) be a non-constant entire curve such that \( f(\mathbb{C}) \) is not contained in \( E \). Then

\[
T_f(r, K_X) + T_f(r, E) \leq nN_1(r, f^* E) + S_f(r),
\]

where \( S_f(r) = O(\log^+ T_f(r)) + o(\log r) \), and \( \| \) means that the estimate holds outside some exceptional set of finite measure.

**Remark**

This generalizes results of Tiba on Hilbert modular surfaces.
Proof.

- Replace the algebraic tautological inequality with the analytic tautological inequality of McQuillan.
Proof.

- Replace the algebraic tautological inequality with the analytic tautological inequality of McQuillan.
- Use hyperbolicity of leaves to exclude the case $f : \mathbb{C} \to X$ tangent to a Hilbert modular foliation $\mathcal{F}$. 

**Corollary**
Let $X$ as above be a Hilbert modular variety of general type. Let $f : \mathbb{C} \to X$ be a non-constant entire curve which ramifies over $E$ with order at least $n$, i.e. $f^* E \geq n \text{ supp } f^* E$. Then $f(\mathbb{C})$ is contained in $E$. 

Erwan Rousseau (j.w.w. Frédéric Touzet)  
Curves in Hilbert modular varieties
Proof.

- Replace the algebraic tautological inequality with the analytic tautological inequality of McQuillan.
- Use hyperbolicity of leaves to exclude the case $f : \mathbb{C} \to X$ tangent to a Hilbert modular foliation $\mathcal{F}$.

Corollary

Let $X$ as above be a Hilbert modular variety of general type. Let $f : \mathbb{C} \to X$ be a non-constant entire curve which ramifies over $E$ with order at least $n$, i.e. $f^*E \geq n \text{supp } f^*E$. Then $f(\mathbb{C})$ is contained in $E$. 
Proof.

If $f(\mathbb{C})$ is not contained in $E$ then

$$nN_1(r, f^*E) \leq N(r, f^*E) \leq T_f(r, E) + O(1).$$
Proof.
If $f(\mathbb{C})$ is not contained in $E$ then

$$nN_1(r, f^*E) \leq N(r, f^*E) \leq T_f(r, E) + O(1).$$

The Second Main Theorem then gives

$$T_f(r, K_X) \leq S_f(r)\|.$$

Since $K_X$ is supposed to be a big line bundle, this gives a contradiction.
Proof.

If $f(\mathbb{C})$ is not contained in $E$ then

$$nN_1(r, f^*E) \leq N(r, f^*E) \leq T_f(r, E) + O(1).$$

The Second Main Theorem then gives

$$T_f(r, K_X) \leq S_f(r).$$

Since $K_X$ is supposed to be a big line bundle, this gives a contradiction.

Remark

For all $K$ except a finite number, $X_K$ is of general type (Tsuyumine).
The Green-Griffiths-Lang conjecture

Conjecture (Green-Griffiths-Lang)

Let $X$ be a complex projective variety of general type. Then there exists a proper algebraic subvariety $Z \subseteq X$ such that every (non-constant) entire curve $f : \mathbb{C} \to X$ satisfies $f(\mathbb{C}) \subseteq Z$.

Theorem (Rousseau-Touzet)

Let $n \geq 2$. Then, except finitely many possible exceptions, Hilbert modular varieties of dimension $n$ satisfy the Green-Griffiths-Lang conjecture.

Remark

Hilbert modular varieties provide counter-examples to the so-called “jet differentials” strategy developed (by Bloch, Green-Griffiths, Demailly, Siu...) to attack the Green-Griffiths-Lang conjecture (Diverio-Rousseau).
The Green-Griffiths-Lang conjecture

<table>
<thead>
<tr>
<th>Conjecture (Green-Griffiths-Lang)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Let $X$ be a complex projective variety of general type. Then there exists a proper algebraic subvariety $Z \subseteq X$ such that every (non-constant) entire curve $f : \mathbb{C} \to X$ satisfies $f(\mathbb{C}) \subset Z$.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Theorem (Rousseau-Touzet)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Let $n \geq 2$. Then, except finitely many possible exceptions, Hilbert modular varieties of dimension $n$ satisfy the Green-Griffiths-Lang conjecture.</td>
</tr>
</tbody>
</table>
The Green-Griffiths-Lang conjecture

**Conjecture (Green-Griffiths-Lang)**

Let $X$ be a complex projective variety of general type. Then there exists a proper algebraic subvariety $Z \subsetneq X$ such that every (non-constant) entire curve $f : \mathbb{C} \to X$ satisfies $f(\mathbb{C}) \subset Z$.

**Theorem (Rousseau-Touzet)**

Let $n \geq 2$. Then, except finitely many possible exceptions, Hilbert modular varieties of dimension $n$ satisfy the Green-Griffiths-Lang conjecture.

**Remark**

Hilbert modular varieties provide counter-examples to the so-called “jet differentials” strategy developed (by Bloch, Green-Griffiths, Demailly, Siu...) to attack the Green-Griffiths-Lang conjecture (Diverio-Rousseau).
The Green-Griffiths-Lang conjecture

Let $g$ denote the Bergmann metric with Kähler form $\omega$ on $\mathcal{H}^n$ such that $\text{Ricci}(g) = -g$ and having holomorphic sectional curvature $\leq -1/n$. It descends to a (singular) metric on $\mathcal{H}^n/\Gamma$. 
The Green-Griffiths-Lang conjecture

Let $g$ denote the Bergmann metric with Kähler form $\omega$ on $\mathcal{H}^n$ such that $Ricci(g) = -g$ and having holomorphic sectional curvature $\leq -1/n$. It descends to a (singular) metric on $\mathcal{H}^n/\Gamma$.

The main point of the proof is to extend it to $\pi : X \to \mathcal{H}^n/\Gamma$. The exceptional divisor $E$ splits as $E = E_c + E_e$. The metric $g$ has Poincaré growth near $E$.

Let $F$ be a Hilbert modular form of weight 2 and $\omega = dz_1 \wedge \cdots \wedge dz_n$.

$s := F \omega \otimes l$ provides a section $s \in H^0(X \setminus E, K \otimes l_X)$. Consider its norm $||s||_h^2$ with respect to the metric induced by $h = (\det g)^{-1}$.

Problem: Find conditions on $F$ under which $||s||_b / \ln g$ will extend as a pseudo-metric on $X$ for some $b > 0$ suitably chosen.
Let $g$ denote the Bergmann metric with Kähler form $\omega$ on $\mathcal{H}^n$ such that $\text{Ricci}(g) = -g$ and having holomorphic sectional curvature $\leq -1/n$. It descends to a (singular) metric on $\mathcal{H}^n/\Gamma$.

The main point of the proof is to extend it to $\pi : X \to \mathcal{H}^n/\Gamma$.

The exceptional divisor $E$ splits as $E = E_c + E_e$. 
The Green-Griffiths-Lang conjecture

Let $g$ denote the Bergmann metric with Kähler form $\omega$ on $\mathcal{H}^n$ such that $Ricci(g) = -g$ and having holomorphic sectional curvature $\leq -1/n$. It descends to a (singular) metric on $\mathcal{H}^n/\Gamma$. The main point of the proof is to extend it to $\pi : X \rightarrow \mathcal{H}^n/\Gamma$. The exceptional divisor $E$ splits as $E = E_c + E_e$. The metric $g$ has Poincaré growth near $E$. 
The Green-Griffiths-Lang conjecture

Let $g$ denote the Bergmann metric with Kähler form $\omega$ on $\mathcal{H}^n$ such that $Ricci(g) = -g$ and having holomorphic sectional curvature $\leq -1/n$. It descends to a (singular) metric on $\mathcal{H}^n/\Gamma$. The main point of the proof is to extend it to $\pi: X \to \mathcal{H}^n/\Gamma$. The exceptional divisor $E$ splits as $E = E_c + E_e$. The metric $g$ has Poincaré growth near $E$.

- Let $F$ be a Hilbert modular form of weight $2l$ and $\omega = dz_1 \wedge \cdots \wedge dz_n$. 

$s := F \omega \otimes l$ provides a section $s \in H^0(X \setminus E, K \otimes l_X)$. Consider its norm $\|s\|^2_h$ with respect to the metric induced by $h = (\det g)^{-1}$. The problem is to find conditions on $F$ under which $\|s\|^2_b/\ln b$ will extend as a pseudo-metric on $X$ for some $b > 0$ suitably chosen.
The Green-Griffiths-Lang conjecture

Let $g$ denote the Bergmann metric with Kähler form $\omega$ on $\mathfrak{H}^n$ such that $\text{Ricci}(g) = -g$ and having holomorphic sectional curvature $\leq -1/n$. It descends to a (singular) metric on $\mathfrak{H}^n/\Gamma$.

The main point of the proof is to extend it to $\pi : X \to \mathfrak{H}^n/\Gamma$.

The exceptional divisor $E$ splits as $E = E_c + E_e$.

The metric $g$ has Poincaré growth near $E$.

- Let $F$ be a Hilbert modular form of weight $2l$ and $\omega = dz_1 \wedge \cdots \wedge dz_n$.
- $s := F \omega^\otimes l$ provides a section $s \in H^0(X \setminus E, K_X^\otimes l)$. 

Problem

Find conditions on $F$ under which $||s||^2_b/\ln$ will extend as a pseudo-metric on $X$ for some $b > 0$ suitably chosen.
The Green-Griffiths-Lang conjecture

Let $g$ denote the Bergmann metric with Kähler form $\omega$ on $\mathbb{H}^n$ such that $\text{Ricci}(g) = -g$ and having holomorphic sectional curvature $\leq -1/n$. It descends to a (singular) metric on $\mathbb{H}^n/\Gamma$.

The main point of the proof is to extend it to $\pi : X \to \mathbb{H}^n/\Gamma$.

The exceptional divisor $E$ splits as $E = E_c + E_e$.

The metric $g$ has Poincaré growth near $E$.

- Let $F$ be a Hilbert modular form of weight $2l$ and $\omega = dz_1 \wedge \cdots \wedge dz_n$.
- $s := F \omega^{\otimes l}$ provides a section $s \in H^0(X \setminus E, K_X^{\otimes l})$.
- Consider its norm $||s||^2_{h'}$ with respect to the metric induced by $h = (\det g)^{-1}$.
The Green-Griffiths-Lang conjecture

Let $g$ denote the Bergmann metric with Kähler form $\omega$ on $\mathbb{H}^n$ such that $\text{Ricci}(g) = -g$ and having holomorphic sectional curvature $\leq -1/n$. It descends to a (singular) metric on $\mathbb{H}^n/\Gamma$.

The main point of the proof is to extend it to $\pi : X \to \mathbb{H}^n/\Gamma$.

The exceptional divisor $E$ splits as $E = E_c + E_e$.

The metric $g$ has Poincaré growth near $E$.

- Let $F$ be a Hilbert modular form of weight $2l$ and $\omega = dz_1 \wedge \cdots \wedge dz_n$.
- $s := F\omega^\otimes l$ provides a section $s \in H^0(X \setminus E, K_X^\otimes l)$.
- Consider its norm $\|s\|_{h}^2$ with respect to the metric induced by $h = (\det g)^{-1}$.

Problem

*Find conditions on $F$ under which $\|s\|^{2b/\ln g}$ will extend as a pseudo-metric on $X$ for some $b > 0$ suitably chosen.*
Denote $S^m_k$ the space of Hilbert modular form of weight $k$ and order at least $m$, where the order is the vanishing order at the cusps.
The Green-Griffiths-Lang conjecture

Denote $S^m_k$ the space of Hilbert modular form of weight $k$ and order at least $m$, where the order is the vanishing order at the cusps.

**Proposition**

Let $F \in S^\nu_{2l}$ and $b > 0$ then $||s||^{2b}/\ln g$ extends as a pseudo-metric over cusps vanishing on $E_c$ if $\nu > \frac{n}{b}$.
The Green-Griffiths-Lang conjecture

Denote $S^m_k$ the space of Hilbert modular form of weight $k$ and order at least $m$, where the order is the vanishing order at the cusps.

Proposition

Let $F \in S_{2l}^\nu$ and $b > 0$ then $\|s\|^{2b/\ln}g$ extends as a pseudo-metric over cusps vanishing on $E_c$ if $\nu > \frac{n}{b}$.

Proposition

Let $b > 0$. There is a constant $c$ depending only on the order of the stabilizer of the elliptic fixed point such that if $F$ is a Hilbert modular form of weight $2l$ vanishing with order $c\ln$ at elliptic fixed points then $\|s\|^{2b/\ln}g$ extends as a pseudo-metric over elliptic singularities vanishing on $E_e$. 
The Green-Griffiths-Lang conjecture

Let $F$ be a Hilbert modular form and $0 < \epsilon < 1/n$ such that $|s|^{2(1-n\epsilon)/\ln g}$ extends as a pseudo-metric on $X$ vanishing on $E$. 

Let $F$ be a Hilbert modular form and $0 < \epsilon < 1/n$ such that $||s||^{2(1-n\epsilon)/\ln g}$ extends as a pseudo-metric on $X$ vanishing on $E$.

**Proposition**

There exists a constant $\beta > 0$ such that

$$\tilde{g} := \beta.||s||^{2(1-n\epsilon)/\ln g}$$

satisfies the following property: for any holomorphic map $f : \Delta \to X$ from the unit disc equipped with the Poincaré metric $g_P$, we have

$$f^* \tilde{g} \leq g_P.$$
The Green-Griffiths-Lang conjecture

Corollary

Let $d_X$ be the Kobayashi pseudo-distance, $\beta$ and $F$ a Hilbert modular form as above. Then $\tilde{g} \leq d_X$. In particular, the degeneracy locus of $d_X$ is contained in the base locus of these Hilbert modular forms.
Corollary

Let $d_X$ be the Kobayashi pseudo-distance, $\beta$ and $F$ a Hilbert modular form as above. Then $\tilde{g} \leq d_X$. In particular, the degeneracy locus of $d_X$ is contained in the base locus of these Hilbert modular forms.

Corollary

Let $X$ be a Hilbert modular variety such that there exists a Hilbert modular form as above. Then $X$ satisfies the strong Green-Griffiths-Lang conjecture.
Existence of Hilbert modular forms

We use the following formula due to Tsuyumine

$$\dim S^\nu_k (\Gamma_K) \geq (2^{-2n+1} \pi^{-2n} d_K^{3/2} \zeta_K(2) - 2^{n-1} \nu^n n^{-n} d_K^{1/2} hR) k^n + O(k^{n-1})$$

for even $k \geq 0$, where $h, d_K, R, \zeta_K$ denote the class number of $K$, the absolute value of the discriminant, the positive regulator and the zeta function of $K$. 

Corollary

For $n$ fixed, except for a finite number of $K$, there is a Hilbert modular form $F$ such that

$$||s||^{2(1-n \varepsilon)/\ln g}$$

extends as a pseudo-metric over cusps. Moreover as $d_K$ tends to infinity, the number of such forms grows at least with order $O(d_K^{3/2})$. 

Erwan Rousseau (j.w.w. Frédéric Touzet) Curves in Hilbert modular varieties
Existence of Hilbert modular forms

We use the following formula due to Tsuyumine

$$\dim S_k^{\nu_k}(\Gamma_K) \geq (2^{-2n+1} \pi^{-2n} d_K^{3/2} \zeta_K(2) - 2^{n-1} \nu^n n^{-n} d_K^{1/2} hR) k^n + O(k^{n-1})$$

for even $k \geq 0$, where $h, d_K, R, \zeta_K$ denote the class number of $K$, the absolute value of the discriminant, the positive regulator and the zeta function of $K$.

In particular, there is a modular form $F$ with $\text{ord}(f)/\text{weight}(f) \geq \nu$, if

$$\nu < 2^{-3} \pi^{-2} n \left( \frac{4d_K \zeta_K(2)}{hR} \right)^{1/n}.$$
Existence of Hilbert modular forms

We use the following formula due to Tsuyumine

\[ \dim S_k^{\nu} (\Gamma_K) \geq (2^{-2n+1} \pi^{-2n} d_K^{3/2} \zeta_K(2) - 2^{n-1} \nu^n n^{-n} d_K^{1/2} hR) k^n + O(k^{n-1}) \]

for even \( k \geq 0 \), where \( h, d_K, R, \zeta_K \) denote the class number of \( K \), the absolute value of the discriminant, the positive regulator and the zeta function of \( K \).

In particular, there is a modular form \( F \) with \( \operatorname{ord}(f)/\operatorname{weight}(f) \geq \nu \), if

\[ \nu < 2^{-3} \pi^{-2} n \left( \frac{4d_K \zeta_K(2)}{hR} \right)^{1/n}. \]

**Corollary**

*For* \( n \) *fixed, except for a finite number of* \( K \), *there is a Hilbert modular form* \( F \) *such that* \( \|s\|^{2(1-n\epsilon)/n} \) *extends as a pseudo-metric over cusps. Moreover as* \( d_K \) *tends to infinity, the number of such forms grows at least with order* \( O(d_K^{3/2}) \).
Corollary

If the number of elliptic fixed points is $O(d_K^\epsilon)$ for $0 < \epsilon < 3/2$, then with finite exceptions, Hilbert modular varieties of dimension $n$ satisfy the strong Green-Griffiths-Lang conjecture.
Corollary

If the number of elliptic fixed points is $O(d_K^\epsilon)$ for $0 < \epsilon < 3/2$, then with finite exceptions, Hilbert modular varieties of dimension $n$ satisfy the strong Green-Griffiths-Lang conjecture.

Proposition

For fixed $n$, the number of equivalence classes of elliptic fixed points is $O(d_K^{\frac{1}{2}+\epsilon})$ for every $\epsilon > 0$. 
Corollary

If the number of elliptic fixed points is $O(d_K^\varepsilon)$ for $0 < \varepsilon < 3/2$, then with finite exceptions, Hilbert modular varieties of dimension $n$ satisfy the strong Green-Griffiths-Lang conjecture.

Proposition

For fixed $n$, the number of equivalence classes of elliptic fixed points is $O(d_K^{1/2+\varepsilon})$ for every $\varepsilon > 0$.

With finite exceptions, Hilbert modular varieties of dimension $n$ satisfy the strong Green-Griffiths-Lang conjecture.