Vector Valued Modular Forms in Vertex Operator Algebras

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Overview

Vertex Operator Algebra = VOA

Origins in deep physics theories that aim beyond QM + GR

Philosophy : The relevance of a VOA is found in its rep theory.

Overview : modular objects

In the VOA theory...

'C_2-cofiniteness'	vs	'finite # of simple modules'
	vs	modularity of characters

Following work by Y.Zhu, M.Miyamoto proved that the linear span of trace & 'pseudo-trace' functions of such VOAs is **a representation of the modular group**.

Overview : broad aim

An obstacle to non-ss settings : the lack of examples...

To this date, a single family of VOAs with

- C_2-cofiniteness
- non-semisimple rep theory

has been known... : the $W(p)$-triplet VOAs.

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Broad aim

To find new examples of VOAs that are as such.

Overview : local aim

Several people have been looking for candidate VOAs including D.Adamović, T.Creutzig, A.Milas, D.Ridout, S.Wood.

Some of the more accessible candidates with

- C_2-cofiniteness
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are constructed **out of affine VOAs**.

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Local aim

To expose the character modular invariance property for the most accessible candidate !!

Overview : a candidate

The VOA $\mathcal{D}_k$ from the following diagram :

$$L_k(\mathfrak{sl}_2) \xrightarrow{\text{Coset}} \mathcal{C}_k = \mathrm{Com}\left(\mathcal{H}, L_k(\mathfrak{sl}_2)\right) \xrightarrow{\text{Extension}} \mathcal{D}_k$$

where

- $k < 0 \quad \& \quad k + 2 = \frac{u}{v} \in \mathbb{Q}_{>0} \setminus \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$
- $\mathcal{H}$ = the Heisenberg subalgebra of $L_k(\mathfrak{sl}_2)$

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Then under a suitable assumption on $\mathcal{C}_k$...

'Schur-Weyl' $+$ Extension process $\Rightarrow$ $\mathcal{D}_k$ is promising

Overview : 'Schur-Weyl' duality

Assuming that the vertex tensor theory of HLZ applies for $\mathcal{C}_k$...

Theorem [T.Creutzig, S.Kanade, A.R.Linshaw, D.Ridout]

Then for any a simple $L_k(\mathfrak{sl}_2)$-module M on which $\mathcal{H}$ acts semisimply, we have a decomposition :

$$M = \bigoplus_{y \in v^M + lattice} F_y \otimes C_y^M$$

as a $(\mathcal{H} \otimes \mathcal{C}_k)$-module where the F_y's are Fock spaces and the C_y^M are simple $\mathcal{C}_k$-modules.

+ a few technical properties.

Note : $\mathcal{H} = \mathrm{Com}\left(\mathcal{C}_k, L_k(\mathfrak{sl}_2)\right)$.

One defines characters as : $\mathrm{tr}_M(y^k z^{h_0} q^{L_0-\frac{c}{24}})$.

We should think : $q = e^{2\pi i\tau}$.

By some classification work, it is sufficient to consider characters of two types of $L_k(\mathfrak{sl}_2)$-modules...

$$\sigma^\ell \mathcal{E}_{\lambda,\Delta_{r,s}} \qquad \sigma^\ell \mathcal{L}_{r,0}$$

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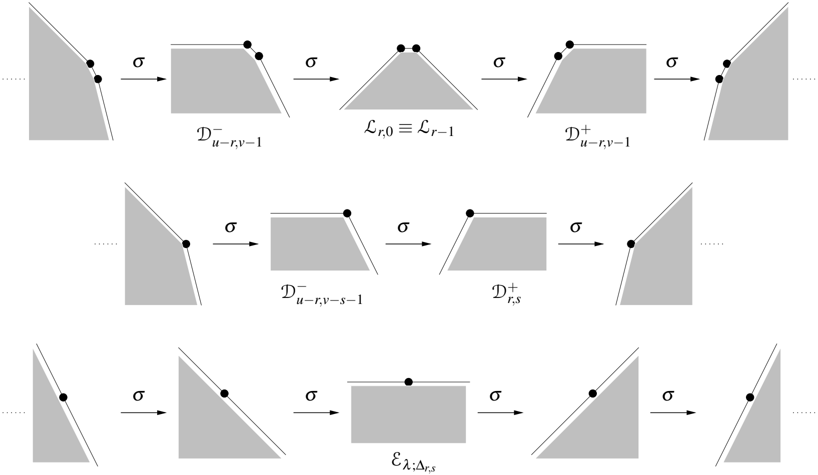
where...

- $\ell \in \mathbb{Z}$ & σ is an automorphism of $L_k(\mathfrak{sl}_2)$
- $r \in \{1,\dots,u-1\}$ & $s \in \{0,\dots,v-1\}$
- $\lambda \in \frac{1}{v}\mathbb{Z}$

Modular invariance $L_k(\mathfrak{sl}_2) \to \mathcal{C}_k \to \mathcal{D}_k$



Source : T.Creutzig, D.Ridout, *Modular Data and Verlinde Formulae for Fractional Level WZW Models II*, Nucl. Phys. B 875 (2013) 423. **I thank the authors for allowing me to use this picture.**

Decomposing the relevant characters accordingly to the 'Schur-Weyl' result, we get :

$$\mathrm{ch}\,\sigma^{\ell}\mathcal{E}_{\lambda,\Delta_{r,s}} = \sum_{n\in\mathbb{Z}} \big(\mathrm{ch}\,F_{\lambda+2n+k\ell}\big)\cdot\big(\mathrm{ch}\,C^{\mathcal{E}}_{r,s,\lambda+2n}(q)\big)$$

$$\mathrm{ch}\,\sigma^{\ell}\mathcal{L}_{r,0} = \sum_{n\in\mathbb{Z}} \big(\mathrm{ch}\,F_{r-1+2n+k\ell}\big)\cdot\big(\mathrm{ch}\,C^{\mathcal{L}}_{r,r-1+2n}(q)\big)$$

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where...

$$\mathrm{ch}\,C^{\mathcal{E}}_{r,s,x}(q) = \frac{\chi^{Vir}_{r,s}(q)}{\eta(q)} q^{-\frac{1}{4k}x^2}$$

$$\mathrm{ch}\,C^{\mathcal{L}}_{r,x}(q) = \sum_{d=1}^{v-1} (-1)^{d-1} \frac{\chi^{Vir}_{r,d}(q)}{\eta(q)} \cdot \sum_{a=0}^{\infty} q^{-\frac{1}{4k}(x-k(2av+d))^2} - q^{-\frac{1}{4k}\left(x-k\left(2(a+1)v-d\right)\right)^2}$$

Set $p = -kv^2$ and $\Gamma = \sqrt{2p}\,\mathbb{Z}$.

Lifting the $\mathcal{C}_k$-modules $C^{\mathcal{E}}_{r,s,x}$ and $C^{\mathcal{L}}_{r,x}$ results in the apparition of lattice Θ-functions and derivatives :

$$\underbrace{D^{\mathcal{E},0}_{r,s,\omega}(q)}_{\Theta} + \underbrace{0}_{\Theta'} \qquad\qquad \underbrace{D^{\mathcal{L},0}_{r,t}(q)}_{\Theta} + \underbrace{D^{\mathcal{L},1}_{r,t}(q)}_{\Theta'}$$

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where...

$$D^{\mathcal{E},0}_{r,s,\omega}(q) = \frac{\chi^{Vir}_{r,s}(q)}{\eta(q)}\ \Theta_{\frac{\omega}{\sqrt{2p}}+\Gamma}(1,q)$$

$$D^{\mathcal{L},0}_{r,t}(q) = \text{a linear combination of expressions of the form } D^{\mathcal{E},0}_{r,s,\omega}(q)$$

$$D^{\mathcal{L},1}_{r,t}(q) = \sum_{d=1}^{v-1}(-1)^{d-1}\frac{\chi^{Vir}_{r,d}(q)}{\eta(q)}\left(\Theta'_{\frac{(r-1+2t)v+kvd}{\sqrt{2p}}+\Gamma}(1,q) - \Theta'_{\frac{(r-1+2t)v-kvd)}{\sqrt{2p}}+\Gamma}(1,q)\right)$$

$$\boxed{\frac{\chi^{Vir}_{r,s}(q)}{\eta(q)}\ \Theta_{\frac{\omega}{\sqrt{2p}}+\Gamma}(1,q)}$$

Consider the generating modular transformations

$$S:\ \tau \mapsto -\frac{1}{\tau} \qquad\qquad T:\ \tau \mapsto \tau+1$$

$\mathrm{Span}_{\mathbb{C}}\left\{D^{\mathcal{E},0}_{r,s,\omega}(q)\right\}$ is then **automatically** a representation of $PSL(2,\mathbb{Z})$!

$$\boxed{\sum_{d=1}^{v-1}(-1)^{d-1}\frac{\chi^{Vir}_{r,d}(q)}{\eta(q)}\left(\Theta'_{\frac{(r-1+2t)v+kvd}{\sqrt{2p}}+\Gamma}(1,q)-\Theta'_{\frac{(r-1+2t)v-kvd)}{\sqrt{2p}}+\Gamma}(1,q)\right)}$$

Fix parameters r, t and write

$$D^{\mathcal{L},1}_{r,t}\left(-\frac{1}{\tau}\right)=\sum \mathrm{Coeff}_{(r',s'),\,\omega}\cdot\left(\frac{\chi^{Vir}_{r',s'}(\tau)}{\eta(\tau)}\;\Theta'_{\frac{\omega}{\sqrt{2p}}+\Gamma}(\tau)\right)$$

$$\boxed{\sum_{d=1}^{v-1}(-1)^{d-1}\frac{\chi_{r,d}^{Vir}(q)}{\eta(q)}\left(\Theta'_{\frac{(r-1+2t)v+kvd}{\sqrt{2p}}+\Gamma}(1,q)-\Theta'_{\frac{(r-1+2t)v-kvd)}{\sqrt{2p}}+\Gamma}(1,q)\right)}$$

Fix parameters r, t and write

$$D_{r,t}^{\mathcal{L},1}\left(-\frac{1}{\tau}\right)=\sum \text{Coeff}_{(r',s'),\,\omega}\cdot\left(\frac{\chi_{r',s'}^{Vir}(\tau)}{\eta(\tau)}\ \Theta'_{\frac{\omega}{\sqrt{2p}}+\Gamma}(\tau)\right)$$

Fix d. Then for any r', t', one can find that

$$(-1)^{d-1}\,\text{Coeff}_{(r',d),\,(r'-1+2t')v\pm kvd}\ =\ \pm\left[\#(r,t,r',t')\right]$$

... and that the irrelevant 'Coeffs' vanish !

Modular invariance

Result

The vector space

$$V = \mathrm{Span}_{\mathbb{C}} \left\{ \, D^{\mathcal{E},0}_{r,s,\omega}(q) + 0 \ \ , \ D^{\mathcal{L},0}_{r,t}(q) + D^{\mathcal{L},1}_{r,t}(q) \, \right\}$$

is a representation of $PSL(2,\mathbb{Z})$!

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is a representation of $PSL(2,\mathbb{Z})$!

More interestingly $\mathrm{Span}_{\mathbb{C}} \left\{ D^{\mathcal{L},1}_{r,t}(q) \right\}$ also is ;

$$D^{\mathcal{L},1}_{r,t}\left(-\tfrac{1}{\tau}\right) = \sum \; S^{\mathcal{L},1}_{(r,t),(r',t')} \cdot \; D^{\mathcal{L},1}_{r',t'}(\tau)$$

where

$$S^{\mathcal{L},1}_{(r,t),(r',t')} = \underbrace{X_{(r'\,t')}}_{1 \text{ or } 1/2} \cdot \frac{4i\tau}{\sqrt{u}\sqrt{2v-u}} \sin\left(\pi \frac{v}{u} r r'\right) \cos\left(\pi \frac{(r-1+2t)(r'-1+2t')v}{2v-u}\right)$$