Explicit methods for Shimura curves

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Overview

Because of the lack of cusps on Shimura curves, there have been very few explicit methods for Shimura curves.

In this talk, we will survey recent progress on explicit methods for Shimura curves and discuss their applications.

- Realization of modular forms in terms of solutions of Schwarzian differential equations.
- Power series expansions. (Coefficients satisfy quasi-recursive relations and are related to central values of L-functions.)
- Realization of modular forms as Borcherds forms.

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Quaternion algebras

Definition

Let K be a field. A quaternion algebra B over K is a central simple algebra of dimension 4 over K.

If $\mathrm{char}\, K \neq 2$, then there exist $i, j \in B$ and $a, b \in K^*$ such that

$$i^2 = a,\ j^2 = b,\ ij = -ji$$

and $B = K + Ki + Kj + Kij$. We denote this algebra by $\left(\frac{a,b}{K}\right)$.

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Quaternion algebras over $\mathbb{Q}$

Let v be a place of $\mathbb{Q}$ and $B_v = B \otimes_{\mathbb{Q}} \mathbb{Q}_v$ be the completion of B at v. We say B splits at v if $B_v \simeq M(2, \mathbb{Q}_v)$ and B ramifies at v if B_v is a division algebra.

The number of ramified places is finite and in fact an even integer. The product of ramified finite places is the discriminant of B.

An order $\mathcal{O}$ in B is a finitely generated $\mathbb{Z}$-module that is a ring with unity containing a basis of B over $\mathbb{Q}$.

An order is maximal if it is not properly contained in another order.

An Eichler order is the intersection of two maximal orders and its level is its index in any of the two maximal orders.

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Shimura curves over $\mathbb{Q}$

Let B be a quaternion algebra of discriminant D over $\mathbb{Q}$ such that B splits at ∞. Up to conjugation, there is a unique embedding

$$\iota : B \hookrightarrow M(2,\mathbb{R}).$$

Let $\mathcal{O}$ be an Eichler order of level N in B. Let

$$\mathcal{O}_1 = \{\gamma \in \mathcal{O} : \mathrm{N}(\gamma) = 1\}, \quad N_B^+(\mathcal{O}) = \{\gamma \in N_B(\mathcal{O}) : \mathrm{N}(\gamma) > 0\},$$

and

$$\Gamma(\mathcal{O}) = \iota(\mathcal{O}_1), \quad \Gamma^*(\mathcal{O}) = \iota(N_B^+(\mathcal{O}))/\mathbb{Q}^\times.$$

The quotient space $X(\mathcal{O}) = \Gamma(\mathcal{O})\backslash\mathbb{H}$ is the Shimura curve associated to $\mathcal{O}$ and $\Gamma^*(\mathcal{O})\backslash\mathbb{H}$ is the Atkin-Lehner quotient of $X(\mathcal{O})$. Denote them by $X_0^D(N)$ and $X_0^D(N)/W_{D,N}$, respectively.

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Examples of Shimura curves

- Let $B = M(2, \mathbb{Q})$ and $\mathcal{O} = M(2, \mathbb{Z})$. Then $\Gamma(\mathcal{O}) = \mathrm{SL}(2, \mathbb{Z})$ and $X(\mathcal{O})$ is just the classical modular curve $X_0(1)$.
- Let $B = M(2, \mathbb{Q})$ and $\mathcal{O} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $\Gamma(\mathcal{O}) = \Gamma_0(N)$ and $X(\mathcal{O})$ is the modular curve $X_0(N)$.
- Let $B = \left(\frac{-1,3}{\mathbb{Q}}\right)$. Then B ramifies at 2 and 3. Let $\mathcal{O} = \mathbb{Z} + \mathbb{Z}i + \mathbb{Z}j + \mathbb{Z}(1+i+j+ij)/2$. An embedding $\iota : B \to M(2, \mathbb{R})$ is

$$i \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad j \mapsto \begin{pmatrix} \sqrt{3} & 0 \\ 0 & -\sqrt{3} \end{pmatrix}$$

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Optimal embeddings and CM-points

Let K be a quadratic number field with

$$\left(\frac{K}{p}\right) \neq 1, \quad \forall p|D,$$

so that K can be embedded in B.

Let $\phi : K \hookrightarrow B$ be an embedding. If R is the order in K such that

$$\phi(K) \cap \mathcal{O} = \phi(R),$$

then we say ϕ is an optimal embedding relative to $(\mathcal{O}, R)$, and let disc R be the discriminant of ϕ.

If $d = \operatorname{disc} R < 0$, there is a unique fixed point of $\iota(\phi(R))$ on $\mathbb{H}$, called a CM-point of discriminant d.

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Canonical models of Shimura curves

Shimura:

- $X_0^D(N)$ parameterizes

$$\{(A,\Theta,\iota):\ (A,\Theta)\text{ principally polarized abelian surface},\ \iota:\mathcal{O}\hookrightarrow \operatorname{End}(A)\}.$$

- Canonical models for $X_0^D(N)$ over $\mathbb{Q}$ exist.
- The field of moduli of a CM-point of discriminant $d = \operatorname{disc} R_d$ is contained in the ray class field H_{R_d} of R_d, and there is an explicit description how $\operatorname{Gal}(H_{R_d}/\mathbb{Q}(\sqrt{d}))$ acts on the CM-points of discriminant d. (Shimura reciprocity law.)
- For $D > 1$, $X_0^D(N)(\mathbb{R}) = \emptyset$.

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Modular forms on Shimura curves

Definition.
A modular form of weight k on $X_0^D(N)$ is a holomorphic function $f:\mathbb{H}\to\mathbb{C}$ such that

$$f\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^k f(\tau)$$

for all $\tau\in\mathbb{H}$ and all $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma(\mathcal{O})$.

If f is meromorphic and $k=0$, then f is a modular function. (If $B=M(2,\mathbb{Q})$, we also need conditions at cusps.)

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Hecke operators

For $n > 0$ with $(n, DN) = 1$, we let α be an element of norm n in $\mathcal{O}$. Then the Hecke operator T_n on $S_k(X_0^D(N))$ is defined by

$$T_n : f \longmapsto n^{k/2-1} \sum_{\gamma \in \Gamma(\mathcal{O}) \backslash \Gamma(\mathcal{O})\iota(\alpha)\Gamma(\mathcal{O})} f\big|_k \gamma.$$

As in the case of classical modular curves, there exists a basis of $S_k(X_0^D(N))$ consisting of simultaneous eigenforms for all T_n, $(n, DN) = 1$.

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Jacquet-Langland correspondence

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Let

$$S_k^{D\text{-new}}(DN) = \bigoplus_{d|N} \bigoplus_{m|N/d} S_k^{\text{new}}(dD)^{[m]},$$

where

$$S_k^{\text{new}}(dD)^{[m]} = \{f(m\tau) : \ f(\tau) \in S_k^{\text{new}}(dD)\}.$$

Then

$$S_k^{D\text{-new}}(DN) \simeq S_k(X_0^D(N))$$

as Hecke modules. (In other words, a Hecke eigenform in $S_k(X_0^D(N))$ shares the same Hecke eigenvalues as some Hecke eigenform in $S_k^{D\text{-new}}$.)

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Difficulties in explicit methods for Shimura curves

Classical modular curves.

- Many problems reduce to computation of q-expansions of modular forms and modular functions.
- There are many methods to construct modular forms and modular functions.
- For normalized eigenforms, Fourier coefficients are the same as Hecke eigenvalues.

Shimura curves.

- A Shimura curve has no cusps. It is not easy to determine Taylor coefficients of quaternionic modular forms and functions.
- Few explicit methods to construct quaternionic modular forms and functions.
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Modular differential equation

Theorem (Folklore)

If $F(\tau)$ is a meromorphic modular form of weight k and $t(\tau)$ is a nonconstant modular function on a Shimura curve X, then $F, \tau F, \ldots, \tau^k F$, as functions of t, satisfy a $(k+1)$-st order linear ODE

$$\theta^{k+1} F + r_k(t)\theta^k F + \cdots r_0(t) F = 0, \quad \theta = t\frac{d}{dt},$$

with algebraic functions as coefficients $r_j(t)$.

We call the differential equation above a modular differential equation.

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Normal form of a modular differential equation

Observation. $t'(\tau)$ is a (meromorphic) modular form of weight 2, so that $t'(\tau)^{1/2}$ and $t(\tau)$ satisfy a second-order ODE.

Proposition

Let $F(\tau)$ be a modular form of weight 1 and $t(\tau)$ be a nonconstant modular function on X. Assume that

$$\theta^2 F + r_1(t)\theta F + r_0(t)F = 0, \qquad \theta = \frac{d}{dt},$$

then the DE satsified by $t'(\tau)^{1/2}$ and $t(\tau)$ is

$$\frac{d^2}{dt^2}G + Q(t)G = 0, \quad Q(t) = \frac{1 + 4r_0 - 2t(dr_1/dt) - r_1^2}{4t^2}.$$

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Schwarzian differential equation

Proposition

The function $Q(t)$ above satisfies

$$Q(t) = -\frac{\{t,\tau\}}{2t'(\tau)^2}, \qquad \{t,\tau\} = \frac{t'''(\tau)}{t'(\tau)} - \frac{3}{2}\left(\frac{t''(\tau)}{t'(\tau)}\right)^2.$$

Definition

The function $\{t,\tau\}$ is the Schwarzian derivative of t and τ. It is a meromorphic modular form of weight 4 on X.

We call the DE satisfied by $t'(\tau)^{1/2}$ and $t(\tau)$ the Schwarzian differential equation associated to t. If X has genus zero, then Schwarzian differential equations associated to Hauptmoduls are linear fractional transformations of each other, and we may talk about the Schwarzian differential equation of X.

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A basis for $S_k(X)$

Proposition

Assume that X has genus 0 with signature $(0; e_1, \ldots, e_r)$ and the corresponding elliptic points τ_i. Let $t(\tau)$ be a Hauptmodul and set $a_i = t(\tau_i)$. For a positive even integer $k \geq 4$, let

$$d_k = \dim S_k(\mathcal{O}) = 1 - k + \sum_{i=1}^{r} \left\lfloor \frac{k}{2} \left(1 - \frac{1}{e_i} \right) \right\rfloor$$

be the dimension of the space of modular forms of weight k on X. Then a basis for $S_k(X)$ is

$$t(\tau)^j t'(\tau)^{k/2} \prod_{i=1, a_i \neq \infty}^{r} (t(\tau) - a_i)^{-\lfloor k(1-1/e_i)/2 \rfloor}, \quad j = 0, \ldots, d_k - 1.$$

A basis for $S_k(X)$

Corollary

With assumptions be given as above, let $F_1(t)$ and $F_2(t)$ be two linearly independent solutions of its Schwarzian differential equation. Then there exist constants C_1 and C_2 such that a basis for $S_k(X)$ is

$$t(\tau)^j(C_1F_1(t)+C_2F_2(t))^k \prod_{i=1,a_i\neq\infty}^{r} (t(\tau)-a_i)^{-\lfloor k(1-1/e_i)/2\rfloor}, \quad j=0,\ldots,d_k-$$

Determining $Q(t)$

The function $Q(t)$ can be determined using the following proposition and properties of $D(t,\tau) := \{t,\tau\}/t'(\tau)^2$.

Proposition

1. We have
$$Q(t) = \frac{1}{4}\left(\sum \frac{1-1/e_i^2}{(t-a_i)^2} + \sum \frac{B_i}{t-a_i}\right)$$
for some complex numbers B_i, where the sums run over finite singularities.

2. If $\infty = a_r$ is a singularity, then
$$\sum_{i=1}^{r-1} B_i = 0, \qquad \sum_{i=1}^{r-1} a_i B_i + \sum_{i=1}^{r-1}(1-1/e_i^2) = 1 - 1/e_r^2.$$
(Similar relations for the case $a_i \neq \infty$ for all i.)

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Examples

- We have

$$E_4(\tau) = {}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; \frac{1728}{j(\tau)}\right)^4,$$

where $E_4(\tau)$ is the Eisenstein series of weight 4 on $\mathrm{SL}(2,\mathbb{Z})$ and $j(\tau)$ is the elliptic j-function.

- Let $X = X_0^6(1)/W_6$ with signature $(0; 2, 4, 6)$. Let t be the Hauptmodul with values 0, 1, and ∞ at the elliptic points of orders 6, 2, and 4. Then the $S_{12}(X)$ is spanned by

$$\left({}_2F_1\left(\frac{1}{24}, \frac{7}{24}; \frac{5}{6}; t\right) - Ct^{1/6}{}_2F_1\left(\frac{5}{24}, \frac{11}{24}; \frac{7}{6}; t\right)\right)^{12}$$

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Applications

- Compute Hecke operators with respect to an explicitly given basis of modular forms. An interesting byproduct is the evaluation

$$ {}_2F_1\left(\frac{1}{24},\frac{7}{24};\frac{5}{6};-\frac{2^{10}\cdot 3^3\cdot 5}{11^4}\right)=\sqrt{6}\sqrt[6]{\frac{11}{5^5}}. $$

(Y., 2013)

- Obtain algebraic transformations of hypergeometric functions such as

$$ {}_2F_1\left(\frac{1}{20},\frac{1}{4};\frac{4}{5};\frac{64z(1-z)(1-3z+z^2)^5}{(1-2z)(1+2z-4z^2)^5}\right) $$
$$ =(1-2z)^{1/20}(1+2z-4z^2)^{1/4}{}_2F_1\left(\frac{3}{10},\frac{2}{5};\frac{4}{5};4z(1-z)\right). $$

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Applications

Ramanujan-type identities, such as

$$\sum_{n=0}^{\infty} \frac{(1/12)_n(1/4)_n(5/12)_n}{(1/2)_n(3/4)_n n!}(R_1 n + R_2)\left(\frac{M}{N}\right)^n = R_3^{1/2}|M|^{3/4}N^{1/4}C,$$

with

$$M = -7^4, \quad N = 15^3, \quad R_1 = 74480, \quad R_2 = 6860/3, \quad R_3 = 5,$$

and

$$C = \frac{4}{\sqrt[4]{12}}\frac{\pi}{\Omega_{-4}^2},$$

where $\Omega_{-4} = \sqrt{\pi}\Gamma(1/4)/\Gamma(3/4)$ is the period of certain elliptic curve over $\overline{\mathbb{Q}}$ with CM by $\mathbb{Q}(i)$. (Y., 2016)

Borchards forms

Idea. The set of elements of trace 0 in $\mathcal{O}$ forms a lattice L of signature $(1, 2)$.

For each suitable weakly holomorphic vector-valued modular form $f : \mathbb{H} \to \mathbb{C}[L^\vee/L]$, there corresponds a modular form Φ_f on the orthogonal group O_L^+, called a Borcherds form.

Since O_L^+ is essentially just $N_B^+(\mathcal{O})/\mathbb{Q}^\times$, such a Borcherd forms is a modular form on the Shimura curve $X_0^D(N)/W_{D,N}$.

Schofer's formula + Kudla-Rapoport-T. Yang's formula gives values of a Borcherds form at CM-points.

To construct Borcherds forms, we find suitable eta-products and lift them to vector-valued modular forms and then to Borcherds forms. To find suitable eta-products, we solve certain integer programming problem using AMPL + Gurobi solver.

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Applications

- Complete list of equations hyperelliptic Shimura curves, such as

$$X_0^{111}(1) : y^2 = -(x^8 + 3x^5 - x^4 - 3x^3 + 1)$$
$$(19x^8 + 44x^7 - 16x^6 - 55x^5 + 37x^4 + 55x^3 - 16x^2 - 44x +$$
$$X_0^6(37) : y^2 = -4096x^{12} - 18480x^{10} - 40200x^8 - 51595x^6$$
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- Determination of quaternionic loci in Siegel's modular threefold. (Joint work with Lin, in preparation.)
- Height of a CM-divisor on $J(X_0^D(N)(\mathbb{Q}))$.

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Applications

Combining the method of Schwarzian DE and the method of Borcherds forms, we get special value formulas for hypergeometric functions, such as

$$ {}_2F_1\left(\frac{1}{24},\frac{7}{24};\frac{5}{6};-\frac{5^3}{3^7}\right) = \sqrt[12]{\frac{4}{3}}\sqrt{2\sqrt{3}+\sqrt{10}}\,\frac{\Omega_{-40}}{\Omega_{-3}}, $$

and

$$ {}_3F_2\left(\frac{1}{4},\frac{1}{2},\frac{3}{4};\frac{5}{6},\frac{7}{6};-\frac{5^3}{3^7}\right) = \frac{6}{\sqrt{5}}\Omega_{-40}^2, $$

where

$$ \Omega_d = \frac{1}{\sqrt{|d|}}\prod_{a=1}^{|d|-1}\Gamma\left(\frac{a}{|d|}\right)^{\chi_d(a)w_d/4h_d}. $$

(Y., 2015)

Weil representation associated to a lattice

Let L be a lattice of signature (b^+, b^-), and e_η, $\eta \in L^\vee/L$, be the standard basis for $\mathbb{C}[L^\vee/L]$.

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$$\widetilde{\mathrm{SL}}(2,\mathbb{Z}) = \left\{ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \pm\sqrt{c\tau + d} \right) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \right\}$$

be the metaplectic double cover of $\mathrm{SL}(2,\mathbb{Z})$ generated by

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Weil representation and vector-valued modular forms

Define the Weil representation ρ_L associated to L by

$$\rho_L(T)e_\eta = e^{2\pi i\langle\eta,\eta\rangle/2}e_\eta,$$
$$\rho_L(S)e_\eta = \frac{e^{2\pi i(b^- - b^+)/8}}{\sqrt{|L^\vee/L|}}\sum_{\delta\in L^\vee/L} e^{-2\pi i\langle\eta,\delta\rangle}e_\delta.$$

If a holomorphic function $f:\mathbb{H}\to\mathbb{C}[L^\vee/L]$ satisfies

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for all $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\in\mathrm{SL}(2,\mathbb{Z})$, we then say f is a vector-valued modular form of type ρ_L and weight k.

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A vector-valued modular form admits a Fourier expansion

$$f(\tau) = \sum_{\eta \in L^\vee/L} \sum_{m \in \mathbb{Q}} c_\eta(m) q^m e_\eta, \qquad q = e^{2\pi i \tau}.$$

We say f is weakly holomorphic if there are only a finite number of $c_\eta(m)$, $m < 0$, such that $c_\eta(m) \neq 0$.

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Orthogonal groups

For $k = \mathbb{Q}, \mathbb{R}, \mathbb{C}$, let $V(k) = L \otimes k$, and

$$O_V(\mathbb{R}) = \{\sigma \in \mathrm{GL}(V(\mathbb{R})) : \langle \sigma x, \sigma y\rangle = \langle x, y\rangle \text{ for all } x, y \in V(\mathbb{R})\},$$
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Modular forms on orthogonal groups

Assume the signature of L is $(b,2)$. Let

$$K = \{z \in V(\mathbb{C}) : \langle z, z\rangle = 0, \langle z, \overline{z}\rangle < 0\}/\mathbb{C}^\times$$

be a symmetric space for $O_V(\mathbb{R})$.

Pick one of the two connected components as K^+ and let $\widetilde{K}^+ = \{z \in V(\mathbb{C}) : [z] \in K^+\}$.

A meromorphic function $F : \widetilde{K}^+ \to \mathbb{C}$ is a meromorphic modular form of weight k and character χ on $\Gamma < O_L^+$ if

- $F(cz) = c^{-k}F(z)$ for all $c \in \mathbb{C}^\times$,
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Theorem (Borcherds). If $f=\sum_\eta f_\eta e_\eta=\sum_\eta\sum_m c_\eta(m)q^m e_\eta$ is a weakly holomorphic modular form of weight $1-b/2$ and type ρ_L with $c_\eta(m)\in\mathbb{Z}$ for $m\le 0$, then there exists a meromorphic modular form $\Psi(z,f)$, called the Borcherds form associated to f, on

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Borcherds forms in the setting of Shimura curves

Let $L = \{\alpha \in \mathcal{O} : \mathrm{Tr}\,\alpha = 0\}$ with $\langle \alpha, \beta \rangle = \mathrm{Tr}(\alpha\beta')$ and signature $(1, 2)$.

We have

$$O_L^+ = \{\sigma_\alpha : \eta \mapsto \alpha\eta\alpha^{-1} \,|\, \alpha \in N_B^+(\mathcal{O})\} \times \{\pm 1\}$$

and K can be identified with $\mathbb{H}^\pm$ through

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commutes.

Thus, if $O_{L,f}^+ = O_L^+$, then $\psi_f(\tau) = \Psi(z(\tau), f)$ is a meromorphic modular form on $X_0^D(N)/W_{D,N}$ of weight $c_0(0)$.

Its divisor is supported on CM-points since $\lambda^\perp$ in Borcherds' theorem is $z(\tau_\lambda)$, where τ_λ is the CM-point fixed by $\iota(\lambda)$.

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Schofer's formula for singular moduli

Theorem (Schofer). Let $\mathrm{CM}(d)$ denote the set of CM-points of discriminant d on $X_0^D(N)/W_{D,N}$. Then

$$\sum_{\tau\in\mathrm{CM}(d)} \log\left|\psi_f(\tau)(\mathrm{Im}\,\tau)^{c_0(0)/2}\right| = -\frac{1}{4}|\mathrm{CM}(d)| \sum_{\eta\in L^\vee/L}\sum_{m>0} c_\eta(-m)\kappa_\eta(m).$$

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Construction of Borcherds forms

Lemma (???). Let M be the level of L. Suppose that f is a scalar-valued modular form of weight k with character χ_θ on $\widetilde{\Gamma}_0(M)$. Then

$$F_f(\tau) = \sum_{\gamma \in \widetilde{\Gamma}_0(M) \backslash \widetilde{\mathrm{SL}}(2,\mathbb{Z})} f(\tau)\big|_k \gamma \rho_L(\gamma^{-1}) e_0$$

is a modular form of weight k and type ρ_L.

Moreover, if $\mathrm{N}(\eta) = \mathrm{N}(\eta')$, then the e_η-component and $e_{\eta'}$-component of F_f are equal.

Corollary. If f is weakly holomorphic of weight $1/2$ and character χ_θ, then $\Psi(z, F_f)$ is a modular form on O_L^+ and the function $\psi_f(\tau) = \Psi(z(\tau), F_f)$ is a modular form on $X_0^D(N)/W_{D,N}$.

Lemma. If f has a pole only at the cusp ∞ of $X_0(M)$, then $c_\eta(m) = 0$ for $m < 0$ and $\eta \neq 0$, where $c_\eta(m)$ are the Fourier coefficients of $F_f = \sum_\eta \sum_m c_\eta(m) q^m e_\eta$.

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Construction using the Dedekind eta function

Lemma (Borcherds). If r_d, $d|N$, are integers such that

- $\sum_{d|N} r_d = 1$,
- $2\prod_{d|N} d^{r_d}$ is a rational square,
- $\sum_{d|N} r_d d \equiv 0 \bmod 24$, and
- $\sum_{d|N} r_d N/d \equiv 0 \bmod 24$,

then f is a weakly holomorphic modular form of weight $1/2$ on $\widetilde{\Gamma}_0(N)$ with character χ_θ.

If we wish f to have a pole only at the cusp ∞, this becomes an integer programming problem.

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An integer program problem

For $D=6$, we have $M=12$, and we need to find integer solutions to

$$\begin{aligned}
r_1 + r_2 + r_3 + r_4 + r_6 + r_{12} &= 1 \\
r_2 + r_6 &= 1+2\delta_2 \\
r_3 + r_6 + r_{12} &= 2\delta_3 \\
r_1 + 2r_2 + 3r_3 + 4r_4 + 6r_6 + 12r_{12} &= 24\epsilon_1 \\
12r_1 + 6r_2 + 4r_3 + 3r_4 + 2r_6 + r_{12} &= 24\epsilon_2
\end{aligned}$$

If we wish f to have a pole only at ∞ of order $\le k$, then we also need

$$\begin{aligned}
r_1 + 2r_2 + 3r_3 + 4r_4 + 6r_6 + 12r_{12} &\ge -24k \\
r_1 + 2r_2 + 3r_3 + r_4 + 6r_6 + 3r_{12} &\ge 0 \\
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