Formal groups and related topics of some Calabi-Yau threefolds

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Contents


Happy Birthday and Thank You!
Noriko

1. Introduction

1-1. One-dimensional formal groups

R: a commutative ring with identity element 1.

Definition 1. A commutative formal group (law) F of dimension one over R is a formal power series $F(x, y) \in R[[x, y]]$ satisfying

$$F(x, 0) = F(0, x) = x,$$
$$F(x, F(y, z)) = F(F(x, y), z),$$
$$F(x, y) = F(y, x).$$

$(*)$ If R has no nilpotent elements, $F(x, y) = F(y, x)$ is always true [Lazard-Serre].

Definition 2. The multiplication-by-n map $[n]_F$ is defined inductively by setting $[1]_F(x) = x$ and

$$[n]_F(x) = F([n-1]_F(x), x).$$

Example 1. (1) For the additive formal group law $F(x, y) = \mathbb{G}_a(x, y) = x + y$,

$$[n]_F(x) = nx.$$

(2) For the multiplicative formal group law $F(x, y) = \mathbb{G}_m(x, y) = x + y - xy$,

$$[n]_F(x) = 1 - (1 - x)^n.$$

1-2. Formal groups of elliptic curves

Let E be an elliptic curve in $\mathbb{P}^2_k$ over a field k:

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3.$$

The group law $\mu : E(k) \times E(k) \to E(k)$ induces the ring homomorphism

$$\mu^* : \mathcal{O}_O \to \mathcal{O}_O \otimes \mathcal{O}_O$$

on the local ring at infinity O. This extends to the completion and gives rise to a formal group law of E:

$$\Gamma(t_1, t_2) = t_1 + t_2 + \sum_{i,j \geq 1} c_{ij} t_1^i t_2^j.$$

1-3. Formal groups in positive characteristic

Suppose that k is a field of characteristic $p > 0$. Then the multiplication-by-p map $[p]_F$ becomes as follows:

- If $[p]_F(x) \neq 0$, then

$$[p]_F(x) = cx^{p^h} + \text{higher-degree terms}, \quad (c \neq 0).$$

The integer h (≥ 1) is called the height of F and denoted by $\mathrm{ht}\,(F)$.

- If $[p]_F(x) = 0$, then F is said to have infinite height.

Example 2. (1) $\mathrm{ht}\,(\mathbb{G}_a) = \infty$, as $[p]_{\mathbb{G}_a}(x) = px = 0$.

(2) $\mathrm{ht}\,(\mathbb{G}_m) = 1$, as $[p]_{\mathbb{G}_m}(x) = 1 - (1-x)^p \equiv x^p (\mathrm{mod}\ p)$.

Properties

Let F and G be (one-dimensional) formal groups over an algebraically closed field k.

- If $\text{ht}\,(F) = \infty$, then F is isomorphic to $\mathbb{G}_a$ over k.

- If $\text{ht}\,(F) < \infty$ and $\text{ht}\,(G) < \infty$, then F and G are isomorphic over k if and only if $\text{ht}\,(F) = \text{ht}\,(G)$.

$\Longrightarrow$ The height classifies the isomorphism classes of formal groups over an algebraically closed field k.

1-4. Formal groups of Calabi-Yau varieties

k: an algebraically closed field of characteristic $p > 0$

Definition 3. X is a Calabi-Yau variety over k of dimension n if it is a projective variety over k of dim $= n$ with $\omega_X \cong \mathcal{O}_X$ and $h^{0,i} = \dim H^i(X, \mathcal{O}) = 0$ for $0 < i < n$.

We consider infinitesimal deformation of X.

Definition 4. For a finite local k-algebra A with residue field k, the functor

$$\Phi_X(A) = \ker(H^n_{et}(X_A, \mathbb{G}_m) \longrightarrow H^n_{et}(X, \mathbb{G}_m))$$

is called the Artin-Mazur functor of X, where $X_A = X \times \operatorname{Spec} A$ and $\mathbb{G}_m$ is the sheaf of multiplicative groups.

The formal group of an elliptic curve can be generalized to Calabi-Yau varieties.

Theorem (M. Artin and Mazur) If X is Calabi-Yau, Φ_X is representable by a commutative formal group of dimension $P_g = 1$, called the formal group of X.

Functor Φ_X

$\downarrow$

Dieudonné module $\Longrightarrow$ Crystalline cohomology

$H^3(X, W(\mathcal{O}))$ (Illusie) $H^3_{cris}(X/W) \otimes K$

(slopes)

$\uparrow$

"Zeta-function"

Height of Φ_X can be computed from slopes of the Newton polygon of the zeta-function of X (Artin-Mazur).

1-5. Questions

- Is height h bounded?

- Does h take every value within its range?

- Find a concrete model of X for each h.

2. Formal groups of $K3$ surfaces

2-1. Result of M. Artin and Mazur

For $K3$ surfaces, Φ is also called the formal Brauer group of X.

Theorem (M. Artin and Mazur). For $K3$ surfaces X,

$$h := \mathrm{ht}\,(\Phi) = 1, 2, \cdots, 10 \quad \text{or} \quad \infty.$$

If h is finite, then $\rho(X) \leq 22 - 2h$, where $\rho(X) = \mathrm{rank}\, NS(X) \quad (= \mathrm{rank}\, Pic(X)$ for $K3$ surfaces) is the Picard number of X.

[M. Artin] X is called supersingular if $h = \infty$ (i.e. $\Phi \cong \hat{\mathbb{G}}_a$).

[Shioda] X is called supersingular if $\rho(X) = 22$

Theorem (M. Artin, Shioda; Maulik, Charles et. al.) (1) $\rho(X) = 22$ and $h = \infty$ are equivalent.

(2) If $\rho(X) = 22$, then $\operatorname{disc} NS(X) = -p^{2\sigma_0}$ with $1 \leq \sigma_0 \leq 10$, and σ_0 is caled the Artin invariant of X.

h and σ_0 give a stratification on the moduli space $\mathcal{M}$ of polarized $K3$ surfaces over k. Let

$$\{h \geq i\} := \{X \mid \mathrm{ht}\,\Phi \geq i\}$$

$$\{\sigma_0 \leq j\} = \{X \mid \rho(X) = 22 \text{ and } \sigma_0 \leq j\}.$$

dim 19 $\qquad$ dim 9

$$\mathcal{M} = \{h \geq 1\} \supset \cdots \supset \{h \geq 10\} \supset \{h \geq 11\} = \{h = \infty\}$$

$$= \{\sigma_0 \leq 10\} \supset \{\sigma_0 \leq 9\} \supset \cdots \supset \{\sigma_0 = 1\}$$

dim 0

2-2. Concrete models

Theorem (Yui [1999]) Using weighted diagonal or quasi-diagonal $K3$ surfaces, Yui gave concrete examples of $K3$ surfaces for $h = 1, 2, 3, 4, 6$ or 10 (in some characteristic).

Theorem (G. [2002]) Using weighted $K3$ surfaces of Delsarte type, we find concrete examples of $K3$ surfaces for $h = 5, 8$ or 9 (in some characteristic).

Note: Examples of $h = 7$ are still open.

Example 3.

$$S: \; x_0^8 x_1 + x_1^6 x_2 + x_2^3 + x_3^2 x_0 = 0 \quad \subset \mathbb{P}^3(1,1,3,4)$$

with $p \neq 2, 3$. Then

$$h = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{32} \\ 2 & \text{if } p \equiv \pm 15 \pmod{32} \\ 4 & \text{if } p \equiv \pm 7, \; \pm 9 \pmod{32} \\ 8 & \text{if } p \equiv \pm 3, \; \pm 5, \; \pm 11, \; \pm 13 \\ \infty & \text{otherwise.} \pmod{32} \end{cases}$$

3. Formal groups of Calabi-Yau threefolds

3-1. Height of formal groups

X: Calabi-Yau threefold over an algebraically closed field k of characteristic $p > 0$ with $h := \mathrm{ht}\,(\Phi_X)$.

Theorem (van der Geer and Katsura) If $h \neq \infty$, then $h \leq h^{1,2} + 1$.

- Is h bounded?
- Is Hodge number $h^{1,2} = \dim H^2(X, \Omega_X)$ bounded?

$\Longrightarrow$ **We will see what values we find for h.**

3-2. Weighted threefolds of Delsarte type

• $\mathbb{P}^4(Q)$: Weighted projective 4-space of weight Q, where $Q = (w_0, w_1, w_2, w_3, w_4)$ and $\deg x_i = w_i$.

There are 7555 weighted projective 4-spaces containing quasi-smooth Calabi-Yau hypersurfaces.

• X_A: weighted projecive 3-fold of Delsarte type with matrix $A = (a_{ij})$

$$X_A : \sum_{i=0}^{4} x_0^{a_{i0}} x_1^{a_{i1}} x_2^{a_{i2}} x_3^{a_{i3}} x_4^{a_{i4}} = 0$$

of degree $m := \sum_{j=0}^{4} w_j a_{ij}$ in 5 monomials.

Properties

Let $d = |\det A|$ (assume $p \nmid d$).

• If X_A is quasi-smooth, then X_A has only cyclic quotient singularities.

• X_A is birational to a finite quotient of Fermat 3-fold F_d defined by $y_0^d + \cdots + y_4^d = 0 \subset \mathbb{P}^4$:

$$\begin{array}{ccl} F_d & & \\ \downarrow & & \\ X_A & \longleftarrow & X \text{ (crepant resolution)} \end{array}$$

Cohomology group

There exists a finite group action Γ_A such that X_A is birational to F_d/Γ_A, where Γ_A acts on F_d coordinatewise and

$$H^3(F_d/\Gamma_A) = \oplus_{\underline{\boldsymbol{\alpha}} \in \mathcal{A}} V(\underline{\boldsymbol{\alpha}})$$

where $\dim V(\underline{\boldsymbol{\alpha}}) = 1$ and $\mathcal{A}$ is an index set of vectors:

$$\mathcal{A} = \{\underline{\boldsymbol{\alpha}} = (\alpha_0, \cdots, \alpha_4) \in \mathbb{Z}/d\mathbb{Z} \times \cdots \times \mathbb{Z}/d\mathbb{Z} \mid \cdots\}.$$

One can compute the slopes of the zeta-function of F_d/Γ_A from vectors $\underline{\boldsymbol{\alpha}} = (\alpha_0, \cdots, \alpha_4)$.

Then there is a unique vector $\underline{\boldsymbol{\alpha}}_X = (\alpha_0, \cdots, \alpha_4)$ such that

- $(\alpha_0, \cdots, \alpha_4)A \equiv (0, \cdots, 0) \pmod{d}$
- $\alpha_0 + \cdots + \alpha_4 = d$.

Put

$$e_X := \frac{d}{\gcd(\alpha_0, \cdots, \alpha_4)}$$

$\Longrightarrow e_X$ is the smallest modulus that describes the main part of $H^3(\widetilde{X})$. Roughly, e_X is the smallest degree of Fermat threefolds that birationally cover X_A.

- For each $\underline{\boldsymbol{\alpha}} = (\alpha_0, \cdots, \alpha_4)$, define an integer

$$\|\underline{\boldsymbol{\alpha}}\| = \sum_{i=0}^{4} \left\langle \frac{\alpha_i}{d} \right\rangle - 1$$

where $< \alpha_i/d >$ denotes the fractional part of α_i/d.

(It takes four values $\|\underline{\boldsymbol{\alpha}}\| = 0, 1, 2$ or 3.)

- Let f be the order of p modulo m. Put

$$H = \{ \; p^i \; (\text{mod } m) \;\; | \;\; 0 \leq i < f \; \}$$

and define

$$A_H(\underline{\boldsymbol{\alpha}}) = \sum_{t \in H} \|t\underline{\boldsymbol{\alpha}}\|$$

3-3. Calculations of height (Fermat type)

As a special case of X_A, consider a Calabi-Yau threefold of weighted Fermat type of degree m:

$$X_A : \ x_0^{m_0} + x_1^{m_1} + x_2^{m_2} + x_3^{m_3} + x_4^{m_4} = 0 \subset \mathbb{P}^4(Q)$$

There are 147 weights giving such threefolds.

Theorem Let X be a crepant resolution of X_A as above. Then $e_X = m$. Let f be the order of p modulo m. Then

(1) $h := \mathrm{ht}\,\Phi_X < \infty$ if and only if $\|p^i \underline{\boldsymbol{\alpha}}_X\| = 1$ for all i $(0 < i < f)$.

(2) If h is finite, then $h = f$.

Proof. (1) Write K for the quotient field of the ring $W(k)$ of Witt vectors over k. Then

$$\begin{aligned}
\text{ht}\,\Phi_{\widetilde{X}} < \infty \quad & \Leftrightarrow \dim_K(H^3_{cris}(\widetilde{X}) \otimes K_{[0,1[}) \geq 1 \\
& \Leftrightarrow \#\{\underline{\boldsymbol{\alpha}} \in \mathfrak{A} | A_H(\underline{\boldsymbol{\alpha}}) < f\} \geq 1 \\
& \Leftrightarrow A_H(\underline{\boldsymbol{\alpha}}_X) < f \\
& \Leftrightarrow \|p^i \underline{\boldsymbol{\alpha}}_X\| = 1 \text{ for all } i \ (0 < i < f)
\end{aligned}$$

(2)

$$\begin{aligned}
h \quad & = \dim_K(H^3_{cris}(\widetilde{X}) \otimes K_{[0,1[}) \\
& = \#\{\underline{\boldsymbol{\alpha}} \in \mathfrak{A} | A_H(\underline{\boldsymbol{\alpha}}) < f\} \\
& = \text{the length of the } H\text{-orbit of } \underline{\boldsymbol{\alpha}}_X \\
& = f
\end{aligned}$$

□

Proposition If X is a Calabi-Yau threefold of weighted Fermat type, then following are the values for $\mathrm{ht}\,\Phi_X$:

1	2	3	4	5	6	7	8	9	10
11	12		14		16		18		20
21	22								
	42								

If X is a weighted hypersurface of Delsarte type in $\mathbb{P}^4(Q)$ with a finite group action G such that $(\widetilde{X}, \widetilde{Y})$ is a mirror pair.

$$\begin{array}{ccc} X & \longleftarrow & \widetilde{X} \\ \downarrow & & \\ Y := X/G & \longleftarrow & \widetilde{Y} \end{array}$$

Then by [van der Geer and Katsura],

$$\text{ht}\,(\widetilde{X}) \le h^{2,1}(\widetilde{X}) + 1 \quad \text{and} \quad \text{ht}\,(\widetilde{Y}) \le h^{2,1}(\widetilde{Y}) + 1 = h^{1,1}(\widetilde{X}) + 1$$

and by [Stienstra], $\text{ht}\,(\widetilde{X}) = \text{ht}\,(\widetilde{Y})$. Hence

$$\text{ht}\,(\widetilde{X}) \le \min\{h^{2,1}(\widetilde{X}), h^{1,1}(\widetilde{X})\} + 1.$$

To obtain a big h, better to have $h^{2,1}(\widetilde{X}) \approx h^{1,1}(\widetilde{X})$.

Example 4.

$$X:\ x_0^{1806}+x_1^{43}+x_2^{7}+x_3^{3}+x_4^{2}=0 \quad \subset \mathbb{P}^4(1,42,258,602,903)$$

of degree 1806 ($p \nmid 1806$). Then $e_X = 1806$, $h^{1,1}=h^{2,1}=251$ and

$$h=\begin{cases} 1 & \text{if } p\equiv 1 \pmod{1806} \\ 2 & \text{if } p\equiv 85,\cdots \pmod{1806} \\ & \cdots \\ 21 & \text{if } p\equiv 169,\cdots \pmod{1806} \\ 42 & \text{if } p\equiv 421,\cdots \pmod{1806} \\ \infty & \text{otherwise} \end{cases}$$

Remark. Compared with $K3$ surfaces, less frequent to get finite h.

3-4. Calculations of height (quasi-diagonal type)

As another special case of X_A, consider a Calabi-Yau threefold of weighted quasi-diagonal type of degree m:

$$X_A: \ x_0^{m_0}x_1 + x_1^{m_1} + x_2^{m_2} + x_3^{m_3} + x_4^{m_4} = 0 \subset \mathbb{P}^4(Q).$$

There are 137 weights giving such threefolds.

Remark. We also computed with the following quasi-diagonal threefolds, but could not find new values for h:

$$x_0^{m_0}x_2 + x_1^{m_1} + x_2^{m_2} + x_3^{m_3} + x_4^{m_4} = 0$$

$$x_0^{m_0} + x_1^{m_1}x_2 + x_2^{m_2} + x_3^{m_3} + x_4^{m_4} = 0$$

$$x_0^{m_0} + x_1^{m_1} + x_2^{m_2} + x_3^{m_3} + x_3x_4^{m_4} = 0$$

Theorem. Let X be a crepant resolution of X_A of degree m as above. Set

$$M = \mathrm{lcm}\,(m_0, m_2, m_3, m_4),$$

$M_i = M/m_i$ $(i = 0, 2, 3, 4)$ and $M_1 = M - M_0 - M_2 - M_3 - M_4$. Let f be the order of p modulo M. Then

(1) $e_X = M$ and $\underline{\boldsymbol{\alpha}}_X = (M_0, M_1, M_2, M_3, M_4)$.

(2) $h := \mathrm{ht}\,\Phi_X < \infty$ if and only if $\|p^i \underline{\boldsymbol{\alpha}}_X\| = 1$ for all i $(0 < i < f)$.

(3) If h is finite, then $h = f$.

Proposition Following are the values for the height of Calabi-Yau threefolds of weighted quasi-diagonal type for some characteristic:

1	2	3	4	5	6	7	8	9	10
11	12		14	15	16		18		20
21	22	23	24			27	28		
30									
41	42			46					
82									

Remark: Examples of $h = 46$ or $h = 82$ are not self-mirror.

Example 5.

$$X:\ x_0^{83}x_1 + x_1^{84} + x_2^7 + x_3^3 + x_4^2 = 0 \qquad \subset \mathbb{P}^4(1,1,12,28,42)$$

of degree 84 ($p \nmid 84$). Then $e_X = 3486$, $h^{1,1} = 11$, $h^{2,1} = 491$ and

$$h = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{3486} \\ 2 & \text{if } p \equiv 1163, 3319 \pmod{3486} \\ & \cdots \\ 41 & \text{if } p \equiv 127, 169, 253, \cdots \pmod{3486} \\ 82 & \text{if } p \equiv 43, 85, 211, \cdots \pmod{3486} \\ \infty & \text{otherwise} \end{cases}$$