“Integrable” gap probabilities for the Generalized Bessel process

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Random matrix ensemble and determinantal process

Let $X(t)$ be a $p \times n$ (assume $p \geq n$) matrix with independent standard complex Brownian entries and set

$$M(t) = X(t)^* X(t).$$

Let $\lambda(t) = (\lambda_1(t), \dots, \lambda_n(t))$ be the vector of (ordered) eigenvalues of $M(t)$, $\lambda_j(t) \geq 0$, $\forall\, t \in [0, +\infty)$.

The process $\{\lambda(t)\}_{t \geq 0}$ is a diffusion on $[0, +\infty)^n$ and it behaves like n independent BESQ^{α}, $\alpha = 2(p - n + 1)$, processes conditioned never to collide (König, O'Connell, '01).

Its transition probability density is given as

$$p_t^{\alpha}(x, y) = \frac{1}{2t} \left(\frac{y}{x} \right)^{\alpha/2} e^{-\frac{x+y}{2t}} I_{\alpha} \left(\frac{\sqrt{xy}}{t} \right).$$

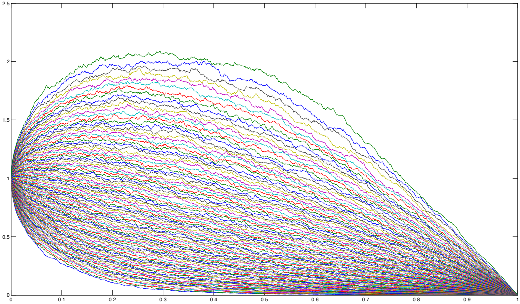
If we condition $\{\lambda(t)\}_{t\geq 0}$ in such a way that

$$\lambda(0) = (\kappa, \ldots, \kappa) \text{ and } \lambda(T) = (0, \ldots, 0)$$

for some $\kappa, T > 0$, the resulting process is a determinantal point process and the eigenvalues $\lambda(t)$ have joint probability density equal to

$$\frac{1}{n!} \det \left[K_n(\lambda_i, \lambda_j; t) \right]_{i,j=1}^{n}$$

at every time $t \in (0, T)$.



Scaling limit at the critical time

Starting from the correlation kernel K_n, Kuijlaars *et al.* ('09) performed a scaling limit in different parts of the domain of the spectrum. In particular, the sine kernel appears in the bulk, the Airy kernel at the soft edges and the Bessel kernel appears at the hard edge $x = 0$.

At a critical time t^*, there is a transition between the soft and the hard edges and the dynamics at that point is described by a critical kernel:

Theorem (Kuijlaars, Martinez-Finkelshtein, Wielonsky, '11)

$$\lim_{n\to+\infty} \frac{c^*}{n^{3/2}} K_n\left(\frac{c^*x}{n^{3/2}}, \frac{c^*y}{n^{3/2}}; t^* - \frac{c^*\tau}{\sqrt{n}}\right) = K_\alpha^{\mathrm{crit}}(x,y;\tau) \qquad x, y \in \mathbb{R}_+,$$

with

$$K_\alpha^{\mathrm{crit}}(x,y;\tau) = \int_\Gamma \frac{\mathrm{d}u}{2\pi i}\int_\Sigma \frac{\mathrm{d}v}{2\pi i} \frac{e^{\theta_\tau(u;x)-\theta_\tau(v;y)}}{v-u}\left(\frac{u}{v}\right)^\alpha$$

$$\theta_\tau(u;x) = xu + \frac{\tau}{u} + \frac{1}{2u^2}.$$

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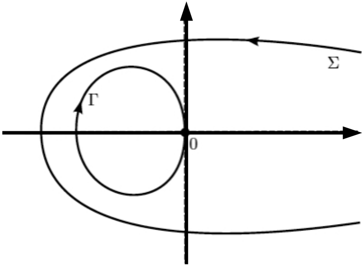
The Generalized Bessel kernel

$$K_\alpha^{\mathrm{crit}}(x,y;\tau) = \int_\Gamma \frac{\mathrm{d}u}{2\pi i}\int_\Sigma \frac{\mathrm{d}v}{2\pi i}\,\frac{e^{xu+\frac{\tau}{u}+\frac{1}{2u^2}-yv-\frac{\tau}{v}-\frac{1}{2v^2}}}{v-u}\left(\frac{u}{v}\right)^\alpha$$

$$= \frac{\left[q''(y)-(\alpha-2)q'(y)-\tau q(y)\right]p(x)+\left[-yq'(y)+(\alpha-1)q(y)\right]p'(x)+yq(y)p''(x)}{2\pi i(x-y)}$$



Gap probabilities of the Generalized Bessel process

Our object of study are the "gap probabilities", meaning the probability of finding no points in a given domain.

For a generic determinantal process with kernel K_n on $\mathbb{R}_+$, the smallest particle $\lambda_{\min}$ has a distribution

$$\mathbb{P}\left(\lambda_{\min} > s\right) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{[0,s]^k} \det\left[K_n(x_i, x_j)\right]_{i,j=1,\dots,k} \mathrm{d}x_1 \dots \mathrm{d}x_k =$$

$$= \det\left(\mathrm{Id}_{L^2(\mathbb{R}_+)} - K_n\Big|_{[0,s]}\right).$$

and, thanks to the double scaling result above,

$$\det\left(\mathrm{Id}_{L^2(\mathbb{R}_+)} - K_n\Big|_{\left[0, \frac{c^* s}{n^{3/2}}\right]}\right) \to \det\left(\mathrm{Id}_{L^2(\mathbb{R}_+)} - K_\alpha^{\mathrm{crit}}\Big|_{[0,s]}\right) \qquad \text{as } n \nearrow +\infty.$$

Differential identity

Theorem (Girotti, '14)

Let $s > 0$ *and* K_α^{crit} *be the integral operator acting on* $L^2(\mathbb{R}_+)$ *with kernel defined above.*
Then, the following differential formula for gap probabilites holds

$$\mathrm{d}_{s,\tau} \ln\det\left(\mathrm{Id}_{L^2(\mathbb{R}_+)} - K_\alpha^{\text{crit}}\Big|_{[0,s]}\right) = (Y_1)_{2,2}\,\mathrm{d}s - \left(\tilde{Y}_0^{-1}\tilde{Y}_1\right)_{2,2}\mathrm{d}\tau$$

where $(Y_1)_{2,2}$ *is the* $(2,2)$*-entry of the residue matrix appearing in the asymptotic expansion at infinity of the matrix-valued function* Y

$$Y(\lambda; s,\tau) = I + \frac{Y_1(s,\tau)}{\lambda} + \mathcal{O}\left(\frac{1}{\lambda^2}\right) \qquad \text{as } \lambda \to \infty,$$

$\tilde{Y}_j$ *are the coefficients appearing in the asymptotic expansion of* Y *in a neighbourhood of zero*

$$Y(\lambda; s) = \tilde{Y}_0(s,\tau) + \tilde{Y}_1(s,\tau)\lambda + \mathcal{O}\left(\lambda^2\right) \qquad \text{as } \lambda \to 0,$$

and Y *is the solution to a RH problem that will be described below.*

The Riemann-Hilbert problem

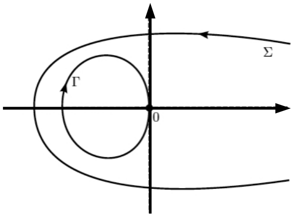
Find a matrix-valued function $Y = Y(\lambda; s, \tau)$ such that

- Y is analytic on $\mathbb{C} \setminus (\Gamma \cup \Sigma)$
- Y possesses a limit when approaching the contours from the left Y_+ or from the right Y_- (according to their orientation); moreover,

$$Y_+(\lambda) = Y_-(\lambda) \begin{cases} \begin{bmatrix} 1 & -e^{-\lambda s - \frac{\tau}{\lambda} - \frac{1}{2\lambda^2} - \alpha \ln \lambda} \\ 0 & 1 \end{bmatrix} & \lambda \in \Sigma \\ \begin{bmatrix} 1 & 0 \\ -e^{\lambda s + \frac{\tau}{\lambda} + \frac{1}{2\lambda^2} + \alpha \ln \lambda} & 1 \end{bmatrix} & \lambda \in \Gamma \end{cases}$$

- Y has the following (normalized) behaviour at ∞:

$$Y(\lambda) = I + \frac{Y_1}{\lambda} + \mathcal{O}\left(\frac{1}{\lambda^2}\right) \qquad \lambda \to \infty.$$



Sketch of the proof

Proposition

The following identity holds

$$\det\left(\mathrm{Id}_{L^2(\mathbb{R}_+)} - K_\alpha^{\mathrm{crit}}\Big|_{[0,s]}\right) = \det\left(\mathrm{Id}_{L^2(\Sigma\cup\Gamma)} - \mathbb{H}\right)$$

where $\mathbb{H}$ *is an IIKS operator with kernel*

$$\mathbb{H} = \frac{\boldsymbol{f}(\lambda)^T \boldsymbol{g}(\mu)}{\lambda - \mu}$$

$$\boldsymbol{f}(\lambda) = \frac{1}{2\pi i}\begin{bmatrix} e^{-\frac{\lambda s}{2}}\chi_\Sigma(\lambda) \\ \chi_\Gamma(\lambda) \end{bmatrix} \qquad \boldsymbol{g}(\mu) = \begin{bmatrix} e^{\mu s + \frac{\tau}{\mu} + \frac{1}{2\mu^2} + \alpha \ln \mu}\chi_\Gamma(\mu) \\ e^{-\frac{\mu s}{2} - \frac{\tau}{\mu} - \frac{1}{2\mu^2} - \alpha \ln \mu}\chi_\Sigma(\mu) \end{bmatrix}.$$

The result can be proved noticing that $K_\alpha^{\mathrm{crit}}\Big|_{[0,s]}$ is unitarily equivalent (via Fourier transform) to a certain integral operator that can be decomposed as the above operator $\mathbb{H}$.

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Integrable kernels: Its-Izergin-Korepin-Slavnov (IIKS) theory

Given an integral operator $\mathbb{H}$ acting on $L^2(\Sigma)$, with $\Sigma \subset \mathbb{C}$ a collection of oriented contours and with kernel

$$\mathbb{H}(\lambda, \mu) = \frac{\mathbf{f}(\lambda)^T \mathbf{g}(\mu)}{\lambda - \mu}, \qquad \mathbf{f}, \mathbf{g} \in \mathrm{Vec}_m(\mathbb{C}) \qquad \mathbf{f}(\lambda)^T \mathbf{g}(\lambda) = 0,$$

it is possible to define a RH problem as following: find an $m \times m$ matrix-valued function Y such that

- Y is analytic on $\mathbb{C} \backslash \Sigma$
- a jump condition holds: $Y_+(\lambda) = Y_-(\lambda) \left[I - 2\pi i \, \mathbf{f}(\lambda) \mathbf{g}(\lambda)^T \right]$ for $\lambda \in \Sigma$
- $Y(\lambda)$ behaves like $I + \mathcal{O}\left(\frac{1}{\lambda}\right)$ as $\lambda \to \infty$.

Then,

Theorem (Its, Izergin, Korepin, Slavnov, '90)

The RH problem above has solution if and only if the operator $\mathrm{Id} - \mathbb{H}$ *is invertible.*

Furthermore, the corresponding resolvent operator $\mathbb{R} := \mathbb{H}(\mathrm{Id} - \mathbb{H})^{-1}$ *is of IIKS type as well and its kernel can be explicitly built out of the functions* $\boldsymbol{f}$, $\boldsymbol{g}$ *and* Y.

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The next fundamental step is to link the Fredholm determinant with the RH problem that we just defined. We make use of a major (and more general) result due to Bertola ('10) and Bertola-Cafasso ('11) which, if applied to our case, reads as follows

Theorem (Bertola-Cafasso, '11)

Define the quantity for $\rho = s, \tau$

$$\omega(\partial_\rho) := \int_{\Sigma\cup\Gamma} \mathrm{Tr}\left[Y_-^{-1} Y_-' \,\partial_\rho J^{\mathrm{crit}} \left(J^{\mathrm{crit}}\right)^{-1}\right] \frac{\mathrm{d}\lambda}{2\pi i}.$$

Then, we have the equality

$$\omega(\partial_\rho) = \partial_\rho \ln\det\left(\mathrm{Id}_{L^2(\Sigma\cup\Gamma)} - \mathbb{H}\right).$$

By expanding the solution Y at infinity and at zero, this identity can be further simplified and explicitly calculated and it yields the final result:

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A few more words on $\omega(\partial)$

In general, the quantity $\omega(\partial)$ can be defined for any vector field ∂ in the space of the parameters S on which the integrable kernel (and, thus, its associated RH problem) depends.

Additionally, assume that the solution to our RH problem solves a rational ODE. If we restrict $\omega(\partial)$ to the (sub)-manifold of isomonodromic deformation, then

$$\mathrm{d}\omega = 0 \quad \text{and} \quad e^{\int \omega} = \tau_{\mathrm{JMU}}$$

ω is equal to the logarithmic total differential of the isomonodromic τ function of Jimbo-Miwa-Ueno (Bertola, '10).

Conclusion

In our case, it is easy to show that Y solves a rational ODE (up to a gauge transformation) and therefore we give a specific geometrical meaning to a probabilistic quantity:

$$\tau_{\mathrm{JMU}} = \det\left(\mathrm{Id}_{L^2(\mathbb{R}_+)} - K_\alpha^{\mathrm{crit}}\Big|_{[0,s]}\right) = \left\{\begin{array}{l}\text{infinitesimal fluctuation of} \\ \text{smallest eigenvalue of } \mathrm{BESQ}^\alpha \\ \text{at the critical time}\end{array}\right\}$$

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What now?

Given

$$\mathrm{d}_{s,\tau} \ln\det\left(\mathrm{Id}_{L^2(\Sigma\cup\Gamma)} - \mathbb{H}\right) = (Y_1)_{2,2}\,\mathrm{d}s - \left(\tilde{Y}_0^{-1}\tilde{Y}_1\right)_{2,2}\mathrm{d}\tau$$

we can further study our RH problem to draw some interesting conclusions:

- asymptotic behaviour of gap probability (large/small gap, degeneration regimes) → Deift-Zhou steepest descent method
- integrability and differential equations (Tracy-Widom) → Lax pair, hamiltonian formalism

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Inspiration: the Bessel process

Consider the Bessel process with kernel

$$K_{\mathrm{Bessel}}(x,y) = \frac{J_\alpha(\sqrt{x})\sqrt{y}J_{\alpha+1}(\sqrt{y}) - J_{\alpha+1}(\sqrt{x})\sqrt{x}J_\alpha(\sqrt{y})}{2(x-y)} \tag{1}$$

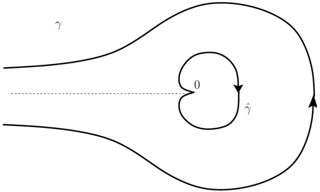
Then, applying the same procedure as described before, we have

Theorem (Girotti, '15)

$$\frac{\mathrm{d}}{\mathrm{d}s}\ln\det\left(\mathrm{Id}_{L^2(\mathbb{R}_+)} - K_{\mathrm{Bessel}}\Big|_{[0,s]}\right) = \frac{1}{2}\left(Y_1\right)_{2,2} + \frac{1}{2}\left(\tilde{Y}_0^{-1}\tilde{Y}_1\right)_{2,2}$$

where Y_1, $\tilde{Y}_0$ and $\tilde{Y}_1$ come from the expansions at infinity and zero of the solution Y to a RH problem (normalized at infinity) with jumps

$$Y_+(\lambda) = Y_-(\lambda)\begin{cases}\begin{bmatrix}1 & e^{\frac{s}{2}\left(\lambda-\frac{1}{\lambda}\right)-\alpha\ln\lambda}\\ 0 & 1\end{bmatrix} & \lambda\in\gamma\\ \begin{bmatrix}1 & 0\\ -e^{-\frac{s}{2}\left(\lambda-\frac{1}{\lambda}\right)+\alpha\ln\lambda} & 1\end{bmatrix} & \lambda\in\tilde{\gamma}\end{cases}$$



From the RH problem for Y, we can derive the Lax pair

$$\mathcal{A} = A_0 + \frac{A_{-1}}{\lambda} + \frac{A_{-2}}{\lambda^2} \qquad \mathcal{B} = \lambda B_1 + B_0 + \frac{B_{-1}}{\lambda}$$

with coefficients

$$A_0 = \frac{s}{4}\sigma_3, \qquad A_{-1} = \begin{bmatrix} -\frac{\alpha}{2} & p(s) \\ q(s) & \frac{\alpha}{2} \end{bmatrix}, \qquad A_{-2} = \begin{bmatrix} \frac{s}{4} - v(s) & -v(s)w(s) \\ \frac{v(s)-\frac{s}{2}}{w(s)} & -\frac{s}{4} + v(s) \end{bmatrix}$$

$$B_1 = \frac{1}{4}\sigma_3, \qquad B_0 = \begin{bmatrix} 0 & \frac{p(s)}{s} \\ \frac{q(s)}{s} & 0 \end{bmatrix}, \qquad B_{-1} = -\frac{1}{s}A_{-2}.$$

This is the well-known Painlevé III Lax pair (Jimbo, Miwa, Ueno, ’81). By setting

$$u(s) := -\frac{p(s)}{w(s)v(s)}$$

and calculating the compatibility equations, we have that u is a solution to the PIII equation:

$$u_{ss} = \frac{(u_s)^2}{u} - \frac{u_s}{s} - \frac{4}{s}\left(\Theta_0 u^2 + \frac{\alpha+1}{2}\right) + u^3 - \frac{1}{u}.$$

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Moreover, we can identify the above quantities with the RH problem Y and link them to the gap probabilities:

Theorem (Girotti, '15)

$$\frac{\mathrm{d}}{\mathrm{d}s} \ln\det\left(\mathrm{Id}_{L^2(\mathbb{R}_+)} - K_{\mathrm{Bessel}}\Big|_{[0,s]}\right) = H_{\mathrm{III}}(u,v;s)$$

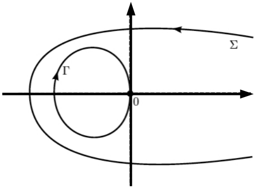
where H_{III} is the Hamiltonian associated to the Painlevé III *equation*

$$H_{\mathrm{III}}\left(u(s),v(s);s\right) = \frac{1}{s}\left[-2u^2v^2 + \left(su^2 + 2\alpha u + s\right)v - \frac{s^2}{4}\right].$$

The Lax triplet

Getting back to the K_α^{crit} case, we are able to derive from the RH problem Y

$$Y_+(\lambda) = Y_-(\lambda) \begin{cases} \begin{bmatrix} 1 & -e^{-\lambda s - \frac{\tau}{\lambda} - \frac{1}{2\lambda^2} - \alpha \ln \lambda} \\ 0 & 1 \end{bmatrix} & \lambda \in \Sigma \\ \begin{bmatrix} 1 & 0 \\ -e^{\lambda s + \frac{\tau}{\lambda} + \frac{1}{2\lambda^2} + \alpha \ln \lambda} & 1 \end{bmatrix} & \lambda \in \Gamma \end{cases}$$



the following Lax triplet

$$\mathcal{A} = \mathcal{A}^{(\lambda)} = A_0 + \frac{A_{-1}}{\lambda} + \frac{A_{-2}}{\lambda^2} + \frac{A_{-3}}{\lambda^3},$$
$$\mathcal{B} = \mathcal{B}^{(s)} = \lambda B_1 + B_0,$$
$$\mathcal{C} = \mathcal{C}^{(\tau)} = \frac{C_{-1}}{\lambda}.$$

Now, we focus on the couple $\{\mathcal{A}, \mathcal{C}\}$. Performing the change of variables

$$\lambda \mapsto \frac{1}{\lambda}$$

the resulting Lax pair is

$$\mathcal{A} = A_1\lambda + A_0 + \frac{A_{-1}}{\lambda} + \frac{A_{-2}}{\lambda^2} \qquad \mathcal{C} = C_1\lambda + C_0$$

with coefficients

$$A_1 = \frac{1}{2}\sigma_3, \qquad A_0 = \begin{bmatrix} \frac{\tau}{2} & uw \\ -\frac{1}{w}\left[v_\tau + u\left(v^2 - \Theta\right)\right] & -\frac{\tau}{2} \end{bmatrix},$$

$$A_{-1} = \begin{bmatrix} a_{-1} & b_{-1} \\ c_{-1} & -a_{-1} \end{bmatrix}, \qquad A_{-2} = \begin{bmatrix} v & w \\ -\frac{1}{w}\left(v^2 - \Theta\right) & -v \end{bmatrix},$$

$$C_1 = \frac{1}{2}\sigma_3, \qquad C_0 = \begin{bmatrix} 0 & uw \\ -\frac{1}{w}\left[v_\tau + u\left(v^2 - \Theta\right)\right] & 0 \end{bmatrix}.$$

We can recognize the Lax pair associated to the second member of the Painlevé III hierarchy defined by Sakka ('09).

Throught the compatibility condition

$$\partial_\tau \mathcal{A} - \partial_\lambda \mathcal{C} + [\mathcal{A}, \mathcal{C}] = 0,$$

it is possible to derive a system of two coupled 2^{nd}-order differential equations in τ for u and v (which can be further reduced to a 4^{th}-order equation for the function u):

$$\begin{cases} u_{\tau\tau} = (6uv - \tau)u_\tau - 6u^3v^2 + 2\tau u^2 v + 2\Theta u^3 - (\alpha + 1)u + 1 \\ v_{\tau\tau} = -(6uv - \tau)v_\tau - 2u(3uv - \tau)(v^2 - \Theta) - \alpha v + \tilde{\Theta}. \end{cases}$$

Garnier systems

As in the classical Painlevé theory (Jimbo, Miwa, Ueno, '81), we would like to find a completely integrable hamiltonian system associated with the Lax triplet $\{\mathcal{A}, \mathcal{B}, \mathcal{C}\}$.

In this case, we have two independent parameters that describe the flow,

the "time" τ and the "space" s,

and two sets of differential equations (from the compatibility conditions).

Garnier systems

The 2-dimensional Garnier system (Okamoto-Kimura, '86) is a completely integrable system for the canonical coordinates $(\mu_1, \mu_2; \lambda_1, \lambda_2)$:

$$\begin{cases} \frac{\partial \lambda_j}{\partial t} = \frac{\partial H_t}{\partial \mu_j} \\ \frac{\partial \mu_j}{\partial t} = -\frac{\partial H_t}{\partial \lambda_j} \end{cases} \qquad \begin{cases} \frac{\partial \lambda_j}{\partial s} = \frac{\partial H_s}{\partial \mu_j} \\ \frac{\partial \mu_j}{\partial s} = -\frac{\partial H_s}{\partial \lambda_j} \end{cases}$$

with rational Hamiltonians $H_t = H_t(\lambda_j, \mu_j; s, t)$ and $H_s = H_s(\lambda_j, \mu_j; s, t)$.

Remarks:

- Garnier systems can be defined for arbitrary N dimension. For $N = 1$, the system reduces to the Painlevé VI equation.
- In the case $N = 2$, it is possible to obtain several Hamiltonian systems from the original Garnier system by the process of step by step degeneration.

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Theorem (Okamoto, Kimura, '86)

There exists a particular solution to the Hamiltonian system of the form

$$(\mu_1, \mu_2; \lambda_1, \lambda_2) = (0, 0; \lambda_1(s,t), \lambda_2(s,t))$$

and for $j = 1, 2$

$$\lambda_1(s,t) = \kappa(t)\, \partial_t \ln u(s,t) \qquad \lambda_2(s,t) = \gamma(s)\, \partial_s \ln u(s,t)$$

with $u(s)$ *a function satisfying a system of linear PDEs.*

Hint for our case: starting from the IIKS-integral representation of the kernel,

$$K_\alpha^{\text{crit}}(x,y;\tau) = \frac{\left[q''(y) - (\alpha-2)q'(y) - \tau q(y)\right] p(x) + \left[-yq'(y) + (\alpha-1)q(y)\right] p'(x) + yq(y)p''(x)}{2\pi i(x-y)},$$

we noticed that the function p is a particular solution of the 2-dimensional Garnier system $LH(2+3)$.

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Our Hamiltonians

Action plan: making use of the compatibility equations of our Lax triplet, we reconstruct the Hamiltonians and the Hamiltonian system for the set of coordinates $(\mu_1, \mu_2; \lambda_1, \lambda_2)$.

By setting

$$\{\lambda_j\}_{j=1,2} \text{ as the solutions of the equation } (\mathcal{A}(\lambda; s, \tau))_{1,2} = 0$$
$$\{\mu_j\}_{j=1,2} \text{ as } \mu_j = (\mathcal{A}(\lambda_j; s, \tau))_{1,1}$$

we get the systems

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These Hamiltonians are different from the Hamiltonians of the $LH(2+3)$ system defined in Okamoto-Kimura, '86. The identification process is on-going...

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Conclusive remarks

This method for studying gap probabilities was first introduced by Bertola-Cafasso ('11) to study the well-known Airy and Pearcey processes.

It has been successfully applied later on for other processes: tacnode process (Girotti, '14), hard-edge processes for product of Ginibre or truncated unitary matrices and Muttalib-Borodin process (Claeys, Girotti, Stivigny, '16).

The key point is that the restriction of the given operator to an interval $K|_I$ is isometrically equivalent to an IIKS operator. The main clue is the double-contour integral representation of the type:

$$K(x,y) = \int_{\Sigma_1} \frac{\mathrm{d}u}{2\pi i} \int_{\Sigma_2} \frac{\mathrm{d}v}{2\pi i} \frac{\mathcal{F}(u;x)\mathcal{F}^{-1}(v;y)}{u-v}$$

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New horizons

past

future

Explicit connection between Hamiltonians and gap probabilities/RH problem for K_α^{crit}?

Quantization:

- Okamoto/Nagoya style: polynomial Hamiltonians
- Zabrodin/Zotov style: Hamiltonians of the form $p^2 + V(q)$ (Painlevé-Calogero correspondence).

Further work:

- what will the Lax pair $\{\mathcal{A}, \mathcal{B}\}$ yield?
- asymptotic behaviour?
  Conjecture: degeneration of the gap probabilities of K_α^{crit} into gap probabilities of the Airy process (for $\tau \searrow -\infty$) or the Bessel process (for $\tau \nearrow +\infty$).

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Thanks for your attention!