New Results on Kirillov-Reshetikhin Modules and Macdonald Polynomials

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Joint work with Satoshi Naito, Daisuke Sagaki, Anne Schilling, and Mark Shimozono.

Summary

- Background:
  - (level 0) extremal weight modules for affine Lie algebras
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  - Kirillov-Reshetikhin modules
  - (symmetric and non-symmetric) Macdonald polynomials
- The relationship between the objects above
- Two combinatorial models
- Applications (including Whittaker functions)

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Remark. For $w = w_\circ$ (the longest element of the finite Weyl group W), $V_{w_\circ}^{+}(\lambda)$ is the global Weyl module over the current algebra $\mathfrak{g} \otimes \mathbb{C}[x]$.

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Remarks. (1) $U^+_{w_\circ}(\lambda)$ is a local Weyl module over the current algebra (unique maximal finite-dimensional quotient of a global Weyl module).

(2) For $\lambda = \sum_{i\in I} m_i\omega_i$, we have, as $U_q(\mathfrak{g})$-modules:

$$U^+_{w_\circ}(\lambda) \simeq \bigotimes_{i\in I} (W^{i,1})^{\otimes m_i} \,,$$

where $W^{i,1}$ are the (column shape) Kirillov-Reshetikhin (KR) modules of the affine Lie algebra without the derivation (finite-dimensional, not of highest weight).

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Graded character:

$$\mathrm{gch}\, U_w^+(\lambda) := \sum_{\gamma \in Q,\, k \in \mathbb{Z}} \dim U_w^+(\lambda)_{\lambda-\gamma+k\delta}\, x^{\lambda-\gamma}\, q^k, \quad \text{where } q = x^\delta.$$

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Remark. Under the isomorphism

$$U_{w_\circ}^+(\lambda) \simeq \bigotimes_{i \in I} (W^{i,1})^{\otimes m_i},$$

the grading is the one by the energy function (originates in the theory of exactly solvable lattice models).

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Fact. All the above modules have crystal bases.

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$E_\mu(q,t)$ defined in the double affine Hecke algebra (DAHA) setup, as common eigenfunctions of the Cherednik operators.

Main result

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- the character of a tensor product of one-column KR modules/crystals, graded by the energy function (LNSSS, previous work);
- the graded character of a local Weyl module for the current algebra (Chari-Ion, based on our work).

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The bijection to QLS paths is a forgetful map, but the inverse map is highly non-trivial.

The quantum alcove model

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The quantum Bruhat graph $\mathrm{QB}(W)$ is the directed graph on W with labeled edges

$$w \xrightarrow{\alpha} ws_\alpha \, ,$$

where

$$\ell(ws_\alpha) = \ell(w) + 1 \quad \text{(covers of strong Bruhat order)}, \qquad \text{or}$$

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Comes from the multiplication of Schubert classes in the quantum cohomology of flag varieties $QH^*(G/B)$ (Fulton and Woodward).

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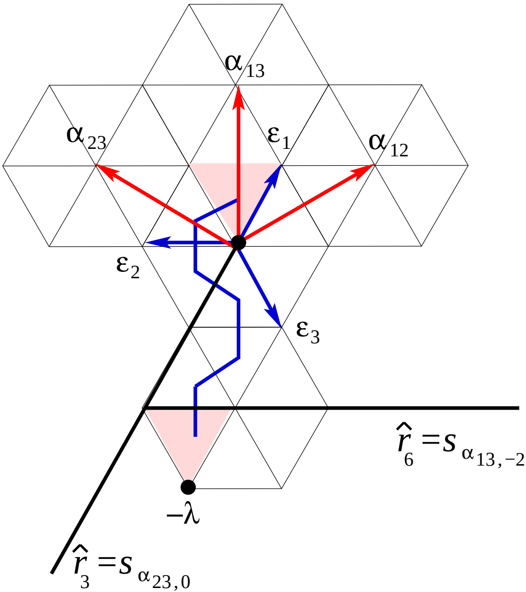
Fact. The construction of a λ-chain is based on a reduced decomposition of the translation by λ, as an element of the affine Weyl group. This corresponds to a sequence of alcoves.

Example. Type A_2, $\lambda = (3,1,0) = 3\varepsilon_1 + \varepsilon_2$,
$\Gamma = (\ (1,2),\ (1,3),\ (2,3),\ (1,3),\ (1,2),\ (1,3)\ )$.



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Definition. A subset $J = \{j_1 < j_2 < \ldots < j_s\}$ is admissible if we have a path in the quantum Bruhat graph

$$Id = w_0 \xrightarrow{\beta_{j_1}} r_{j_1} \xrightarrow{\beta_{j_2}} r_{j_1} r_{j_2} \ldots \xrightarrow{\beta_{j_s}} r_{j_1} \ldots r_{j_s} =: \phi(J) \quad \text{(final direction)}\,.$$

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The objects of the model: $\mathcal{A}(\Gamma)$ − the collection of all admissible subsets.

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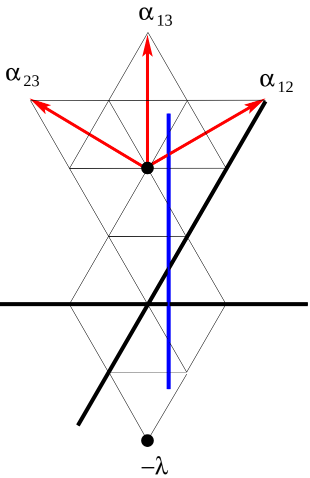

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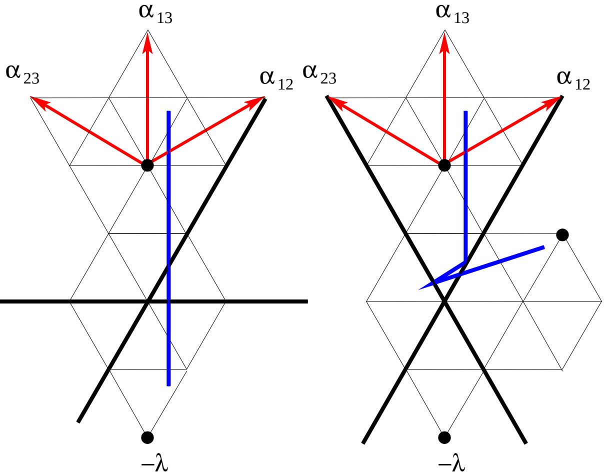

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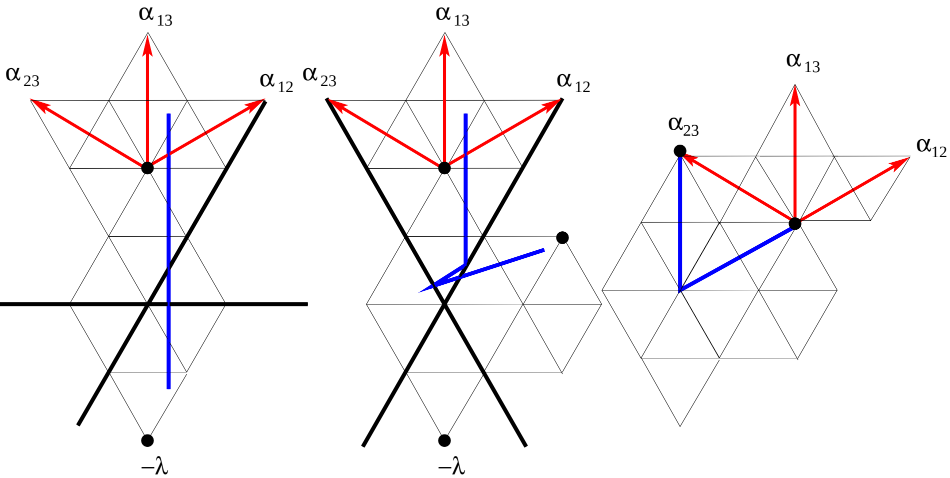

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Definition. A QLS path of shape λ is a pair

$$\eta = (w_1, \, w_2, \, \ldots, \, w_s \, ; \, b_0, \, b_1, \, \ldots, \, b_s) \quad \text{with}$$

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Let $w_1 =: \iota(\eta)$ (initial direction).

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Remark. The above theorem generalizes to KR crystals the description of Demazure subcrystals inside highest weight crystals in terms of LS paths (Littelmann) and the alcove model (L.).

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(2) define a statistic which efficiently computes the energy function (LNSSS, previous work);

(3) give an explicit construction of the combinatorial R-matrix, i.e., the (unique) affine crystal isomorphism between tensor products with permuted factors (L. and Lubovsky).

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** Braverman-Finkelberg, etc.: relation to q-Whittaker functions and Schubert calculus in quantum K-theory.

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Also recall the graded character of $V^+_{w_\circ}(\lambda)$ (global module for current algebra):

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Local and global Weyl modules for current algebras

Let $\lambda = \sum_{i \in I} m_i \omega_i$.

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Based on the corresponding combinatorial models, Naito-Sagaki gave a new, crystal-theoretic interpretation of the above relationship between local and global Weyl modules.

Braverman-Finkelberg q-Whittaker functions

$\Psi_\lambda(q)$: eigenfunctions of the quantum difference Toda integrable system (Etingof, Sevostyanov).

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Remark. As we have seen before, $\Psi_\lambda(q)$ and $\widehat{\Psi}_\lambda(q)$ are the characters of global and local Weyl modules for current algebras, respectively.

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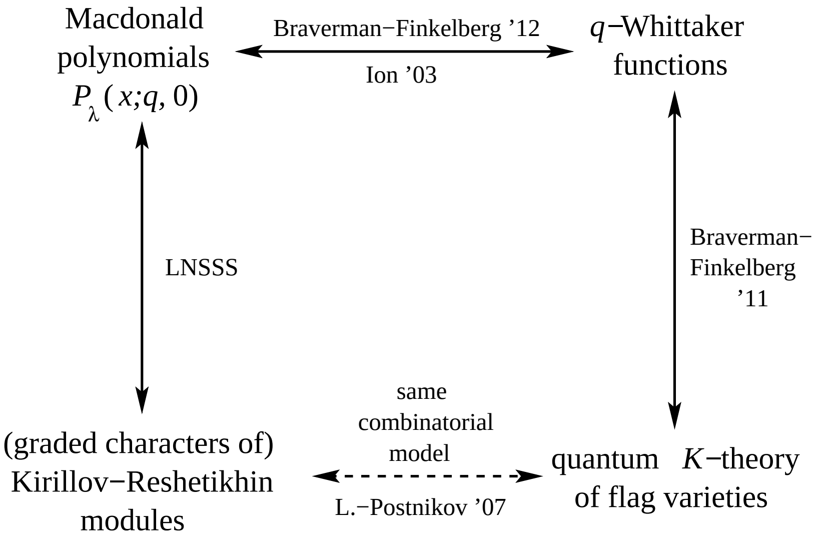
Thus, for the moment, our approach based on crystal combinatorics seems to be the only option.

Geometric connections



Schubert calculus on flag varieties

Flag variety G/B, Schubert variety $X_w = \overline{B^- wB/B}$, for $w \in W$.

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A k-point GW invariant (of degree d) counts curves of degree d passing through k given Schubert varieties.

Quantum K-theory

Givental and Lee defined more general, K-theoretic GW invariants.

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The structure constants for the quantum K-theory $QK(G/B)$ are defined based on the 2- and 3-point invariants (complex formula).

$K(G/B)$ and $QK(G/B)$: Chevalley formulas

Theorem. (L.-Postnikov, L.-Shimozono) *In $K(G/B)$ (finite-type or Kac-Moody), we have an explicit combinatorial formula (of Chevalley type) in terms of the alcove model for expanding:*

$$[\mathcal{O}_v] \cdot [\mathcal{O}_{s_k}] = \sum_{w \in W} c_w \, [\mathcal{O}_w] \, .$$

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Evidence. L.-Maeno. Also: computer experiments (A. Buch).

K-theoretic J-function

The K-theoretic J-function is the generating function of 1-point K-theoretic GW invariants.

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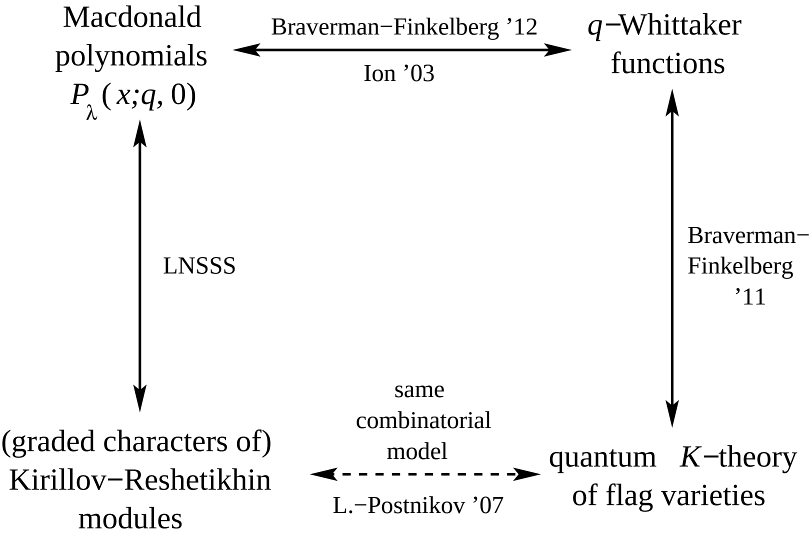
Theorem. (Braverman-Finkelberg) *In simply-laced types, the q-Whittaker function $\Psi_\lambda(q)$ (viewed as a function of λ) coincides with the K-theoretic J-function.*


Macdonald polynomials $P_\lambda(x;q,0)$
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Ion ’03
q−Whittaker functions
LNSSS
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same combinatorial model
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