DO GROWING WHITTAKER FUNCTIONS *ACTUALLY* OCCUR IN AUTOMORPHIC FORMS?

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WHITTAKER FUNCTIONS IN MODULAR FORMS

Let F be a classical holomorphic modular form for SL(2,ℤ)

$$F\left(\frac{az+b}{cz+d}\right) = (cz+d)^k F(z)$$

Then since F(z+1) = F(z), it has a Fourier expansion in Re(z):

$$F(x+iy) = \sum_{n\in\mathbb{Z}} a_n(y)\, e^{2\pi i n x}$$

Since F is holomorphic, actually $a_n(y) = e^{-2\pi ny}$.

A similar argument for non-holomorphic Maass forms (using Laplacian Δ instead of Cauchy-Riemann operator $\overline{\partial}$) reduces $a_n(y)$ to the solution of a 2nd order linear ODE.

WHICH SOLUTION? Whittaker: there are 2, one grows & one rapidly decays.

$$I_\nu(x) = \sum_{n=0}^{\infty} \frac{(x/2)^{\nu+2n}}{n!\,\Gamma(n+\nu+1)} \qquad I_\nu(u) = \sqrt{\frac{1}{2\pi u}}\, e^u + O(u^{-3/2}e^u), \quad u\to\infty$$

$$K_\nu(x) = \frac{\pi}{2}\frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\pi\nu)}. \qquad K_\nu(u) = \sqrt{\frac{\pi}{2u}}\, e^{-u} + O(u^{-3/2}e^{-u}), \quad u\to\infty.$$

If growth condition: cannot grow, hence must rapidly decay.

GROWING WHITTAKER FUNCTIONS IN MODULAR FORMS

Growing functions **do** actually occur in some examples

- e.g., modular j-function $j(z) = e^{-2\pi i z} + 744 + 196884\, e^{2\pi i z} + \cdots$
- reciprocals of Ramanujan's Δ cusp form.
- weak Maass forms
- Poincare series formed from I-Bessel functions
- Automorphic Greens functions

These have many applications for SL(2).

Also in rank-one settings, e.g., SO(n,1) (Borcherds, Harvey-Moore,…).

Some recent movitation:

- [Viazovska; Cohn-Kumar-M-Radchenko-Viazovska] use growing modular forms of negative weight to resolve the sphere packing problem in dimensions 8 and 24.
- Thus harmonic analysis absolutely requires growing Whittaker functions, even though the automorphic representation viewpoint discourages them.

A RICH, BEAUTIFUL WHITTAKER THEORY IN HIGHER RANK

Standing assumptions for rest of talk:

- Stick to G=SL(3,$\mathbb{R}$), Γ=SL(3,$\mathbb{Z}$), K=SO(3)
- Spectral parameter λ is in general position (removable).

Whittaker functions are eigenfunctions of all invariant differential operators, and transform by a character under unit upper triangular matrices:

$$W\left(\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} gk\right) = e^{2\pi i(x+y)}\, W(g)$$

Studied by Kostant, Casselman-Zuckerman.

The decaying Whittaker function $W_\lambda(g)$ was earlier constructed by Jacquet, is most common.

There are 6 linearly independent solutions, of which Jacquet's $W_\lambda(g)$ is the only non-growing linear combination.

Analog of I-Bessel

$$\mathcal{M}_\lambda(a_{y_1,y_2}) = \frac{\pi^3}{\sin(\frac{\pi}{2}(\lambda_1-\lambda_2))\,\sin(\frac{\pi}{2}(\lambda_2-\lambda_3))\,\sin(\frac{\pi}{2}(\lambda_3-\lambda_1))}$$
$$\times\, |y_1y_2| \sum_{m=0}^{\infty} \frac{(\pi|y_1|)^{m-\lambda_3/2}(\pi|y_2|)^{m+\lambda_1/2}}{m!\,\Gamma(m+\frac{\lambda_1-\lambda_3}{2}+1)} I_{m+(\lambda_1-\lambda_2)/2}(2\pi|y_1|)I_{m+(\lambda_2-\lambda_3)/2}(2\pi|y_2|)$$

For generic λ, the 6-dimensional space is spanned by $\mathcal{M}_{w\lambda}(g)$ for w in the Weyl group

For special λ some coincide, and must replaced by appropriate limits.

ASYMPTOTICS OF WHITTAKER FUNCTIONS

Asymptotics are given by a conjecture of Zuckerman (proven by To, Templier): there is a basis $\phi_\lambda^{(m)}(g)$ with asymptotics

$$\log(\phi_\lambda^{(m)}(a_{y_1,y_2})) \sim 2\pi\left(p_3^{(m)}(y_1,y_2) - p_1^{(m)}(y_1,y_2)\right)$$

where

$$\begin{aligned}
p_3^{(1)}(y_1,y_2) - p_1^{(1)}(y_1,y_2) &= (y_1^{2/3} + y_2^{2/3})^{3/2},\\
p_3^{(2)}(y_1,y_2) - p_1^{(2)}(y_1,y_2) &= -(y_1^{2/3} + e^{-2\pi i/3} y_2^{2/3})^{3/2},\\
p_3^{(3)}(y_1,y_2) - p_1^{(3)}(y_1,y_2) &= -(y_1^{2/3} + e^{2\pi i/3} y_2^{2/3})^{3/2},\\
p_3^{(4)}(y_1,y_2) - p_1^{(4)}(y_1,y_2) &= (y_1^{2/3} + e^{-2\pi i/3} y_2^{2/3})^{3/2},\\
p_3^{(5)}(y_1,y_2) - p_1^{(5)}(y_1,y_2) &= (y_1^{2/3} + e^{2\pi i/3} y_2^{2/3})^{3/2},\\
p_3^{(6)}(y_1,y_2) - p_1^{(6)}(y_1,y_2) &= -(y_1^{2/3} + y_2^{2/3})^{3/2}.
\end{aligned}$$

$$\begin{pmatrix} p_1^{(m)}(y_1,y_2) & y_1^2 & 0 \\ -1 & p_2^{(m)}(y_1,y_2) & y_2^2 \\ 0 & -1 & p_3^{(m)}(y_1,y_2) \end{pmatrix} \text{ is a nilpotent matrix.}$$

This is a sharp form of Piatetski-Shapiro/Shalika “Multiplicity One Theorem”

Thus moderate growth implies only Jacquet’s decaying one (m=6) appears in a Fourier expansion.

$$W_\lambda(g) = \sum_{w\in\Omega} \mathcal{M}_{w\lambda}(g) = \phi^{(6)}(g)$$

WHY HAS NOBODY SEEN THEM IN AUTOMORPHIC FORMS?

Evidence for existence	Evidence against existence
SL(2) ones are not obvious to construct	None have been constructed
Exist for many real rank 1 groups, e.g, SO(n,1)	Koecher/Goetzky principle for Sp(2n) and Hilbert modular forms
Existing ones are fundamental, e.g., Greens functions, counting problems	Miatello-Wallach conjecture they don't exist; proved some cases over number fields
String theory, Ward identities for CFT on 3-torus	Will later show SL(3,$\mathbb{Z}$) doesn't have any modulo some natural assumptions

- Some subtleties:
  - In principle, can have only Jacquet's Whittaker function W_λ in the Fourier expansion…yet still violate moderate growth (!). This is tricky.
  - Do not assume any growth conditions on the automorphic form, so in principle rapidly growing objects such as exp(j(z)) could occur.

SO WHAT SHOULD A COUNTEREXAMPLE TO THE MIATELLO-WALLACH CONJECTURE LOOK LIKE?

Naively, we might look to rule out having some M's in the Fourier expansion instead of just W's.

We'd expect F to grow like some exponential in the cusp, like $j(z)$ does.

We'd expect to be able to manipulate Fourier series without convergence issues.

Upshot of talk: we rule out such an F under these assumptions, but have difficulty relaxing them completely.

[Miatello-Wallach] imposes no condition at all on the growth, so might have something like $e^{j(z)}$ which has an infinite number of M's in its expansion and doesn't have exponential growth.

FOURIER EXPANSIONS FOR SL(3,$\mathbb{Z}$)\SL(3,$\mathbb{R}$)/SO(3)

Standard reference is Bump's Springer Lecture Notes

Let F be an automorphic function and define

$$[P^{m,n}F](g) \quad := \quad \int_{(\mathbb{Z}\backslash\mathbb{R})^2} F\left(\begin{pmatrix} 1 & 0 & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} g\right) e^{-2\pi i(mz+ny)}\, dy\, dz$$

Thus $$F(g) \quad = \sum_{m,n\in\mathbb{Z}} [P^{m,n}F](g)$$

Key calculation (change variables in integral):

$$[P^{m,n}F]\left(\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} g\right) \quad = \quad [P^{ma+nc,mb+nd}F](g)\,, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})\,.$$

Important special case $$[P^{m,n}F](g) \quad = \quad [P^{0,\ell}F]\left(\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} g\right)$$

$$\ell = \gcd(m,n),\; c = \tfrac{m}{\gcd(m,n)},\; d = \tfrac{n}{\gcd(m,n)}$$

FOURIER EXPANSIONS FOR SL(3,$\mathbb{Z}$)\SL(3,$\mathbb{R}$)/SO(3)

Recap:

$$F(g) = \sum_{m,n\in\mathbb{Z}} [P^{m,n}F](g) \qquad [P^{m,n}F](g) = [P^{0,\ell}F]\left(\left(\begin{smallmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) g\right)$$

$$\ell = \gcd(m,n),\ c = \tfrac{m}{\gcd(m,n)},\ d = \tfrac{n}{\gcd(m,n)}$$

Take another Fourier series:

$$[P^{0,\ell}F](g) = \sum_{k\in\mathbb{Z}} [P^{k,0,\ell}F](g)$$

where

$$[P^{k,0,\ell}F](g) := \int_{(\mathbb{Z}\backslash\mathbb{R})^3} F\left(\left(\begin{smallmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{smallmatrix}\right) g\right) e^{-2\pi i(kx+\ell y)}\, dx\, dy\, dz$$

Get Piatetski-Shapiro/Shalika Fourier Expansion

$$\begin{aligned} F(g) &= \sum_{k\in\mathbb{Z}} [P^{k,0,0}F](g) + \sum_{\ell=1}^{\infty} \sum_{\gamma\in\Gamma_\infty^{(2)}\backslash\Gamma^{(2)}} \sum_{k\in\mathbb{Z}} [P^{k,0,\ell}F]\left(\left(\begin{smallmatrix} \gamma & \\ & 1 \end{smallmatrix}\right) g\right) \\ &= \sum_{\ell\in\mathbb{Z}} [P^{0,0,\ell}F](g) + \sum_{k=1}^{\infty} \sum_{\gamma\in\Gamma_\infty^{(2)}\backslash\Gamma^{(2)}} \sum_{\ell\in\mathbb{Z}} [P^{k,0,\ell}F]\left(\left(\begin{smallmatrix} 1 & \\ & \gamma \end{smallmatrix}\right) g\right) \end{aligned}$$

with $\Gamma^{(2)} = SL(2,\mathbb{Z})$, $\Gamma_\infty^{(2)}$ is its subgroup of unit upper triangular matrices.

MIATELLO-WALLACH CONJECTURE FOR SL(3,$\mathbb{Z}$)

Let F be an automorphic function on SL(3,$\mathbb{Z}$)\SL(3,$\mathbb{R}$)/SO(3)

Assume F is an eigenfunction of all SL(3,$\mathbb{R}$)-invariant differential operators

Conjecture: ϕ has moderate growth, i.e., $F(g) \ll \|g\|^A$ for some $A > 0.$

(this is an assumption in the definition of automorphic form)

In terms of torus variables $$a_{y_1,y_2} = \begin{pmatrix} y_1^{2/3}y_2^{1/3} & 0 & 0 \\ 0 & y_1^{-1/3}y_2^{1/3} & 0 \\ 0 & 0 & y_1^{-1/3}y_2^{-2/3} \end{pmatrix}$$

The conjecture is that $F(a_{y_1,y_2}) \ll (y_1y_2)^A$ for $y_1, y_2 \geq \sqrt{3/4}.$

Some natural assumptions:

- (ExpGro) Instead, $F(a_{y_1,y_2}) \ll e^{K(y_1+y_2)}$ for $y_1, y_2 \geq \sqrt{3/4}$ and some $K > 0.$
- (AbsCon) The Fourier series are absolutely convergent for fixed g (implies terms are bounded) .

Main Theorem (M-Trinh):

- any counterexample to the Miatello-Wallach Conjecture must violate violate (AbsCon).
- (ExpGro) will be violated unless non-decaying Whittaker functions appear in the Fourier expansion.

Example: no analog of j-function for SL(3,Z).

WHY CAN'T GROWING WHITTAKER FUNCTIONS OCCUR?

Recall Fourier expansion
$$F(g) = \sum_{k\in\mathbb{Z}}[P^{k,0,0}F](g) + \sum_{\ell=1}^{\infty}\sum_{\gamma\in\Gamma_\infty^{(2)}\backslash\Gamma^{(2)}}\sum_{k\in\mathbb{Z}}[P^{k,0,\ell}F]\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix} g\right)$$
$$= \sum_{\ell\in\mathbb{Z}}[P^{0,0,\ell}F](g) + \sum_{k=1}^{\infty}\sum_{\gamma\in\Gamma_\infty^{(2)}\backslash\Gamma^{(2)}}\sum_{\ell\in\mathbb{Z}}[P^{k,0,\ell}F]\left(\begin{pmatrix}1 & \\ & \gamma\end{pmatrix} g\right)$$

where
$$[P^{k,0,\ell}F](g) = \sum_{m=1}^{6} c(k,\ell,m)\,\phi_\lambda^{(m)}(a_{k,\ell}\,g)$$

- KEY IDEA: the same Whittaker functions are summed over and over again using γ. Aside from Jacquet's decaying Whittaker function, all grow in the direction that γ pushes towards.

- Recall: **Multiplicity 1 theorem.** [Piatetski-Shapiro, Shalika] Among the 6 dimensional span of Whittaker functions, only Jacquet's $\phi_\lambda^{(6)}(a_{y_1,y_2})$ decays rapidly for large y_1,y_2.

- **Coroot Multiplicity 1** [Trinh] Assume $[P^{k,0,\ell}F](a_{t-2,t})\,,\ [P^{k,0,\ell}F](a_{t,t-2}) = O(1)\ ,\quad t\to\infty.$
Then $[P^{k,0,l}F](g)$ is a scalar multiple of $\phi_\lambda^{(6)}(g)$ = Jacquet's $W_\lambda(g)$.

- **Consequently** if F's Fourier expansion converges absolutely, it only includes W_λ's.

TRINH'S PROOF OF COROOT MULTIPLICITY 1

Starts with Ishii-Stade representation of Whittaker function:

$$\mathcal{M}_\lambda(a_{kt^{-2},\ell t}) = \frac{\pi^3\,|k|^{m+1-\lambda_3/2}\,|\ell|^{m+1+\lambda_1/2}}{\sin(\frac{\pi}{2}(\lambda_1-\lambda_2))\,\sin(\frac{\pi}{2}(\lambda_2-\lambda_3))\,\sin(\frac{\pi}{2}(\lambda_3-\lambda_1))}$$
$$\times \sum_{m=0}^{\infty} \frac{\pi^{2m+\lambda_1+\lambda_2/2}t^{-m-1+\lambda_3+\lambda_1/2}}{m!\,\Gamma(m+\frac{\lambda_1-\lambda_3}{2}+1)} I_{m+(\lambda_1-\lambda_2)/2}(2\pi|k|t^{-2})I_{m+(\lambda_2-\lambda_3)/2}(2\pi|\ell|t)$$

Consider the integral representation

$$I_\mu(x)\,I_\nu(x) = \frac{2}{\pi}\int_0^{\pi/2} I_{\mu+\nu}(2\,x\,\cos\theta)\,\cos((\mu-\nu)\theta)\,d\theta\ ,\quad \mathrm{Re}\,\mu+\nu > -1$$

Let $\mu = \sigma + it$ and $\nu = \bar{\mu} = \sigma - it$, where $\sigma, t \geq 0$.

Differentiating under the integral sign in σ and t shows that I_μ decreases in its real part, and increases in its imaginary part.

Ishii-Stade formula then makes the asymptotics in coroot direction manifest.

MODERATE GROWTH?

We just saw

Theorem [M-Trinh] if the Fourier series for F converges absolutely, the Fourier expansion is built out of W's (no M's). Thus

But this does not show F has moderate growth

- Only have reverse implication: moderate growth implies no M's.

There is no *a priori* control on the Fourier coefficients c(k,l) from $[P^{k,0,\ell}F](g) \;=\; c(k,\ell)\,W_\lambda(a_{k,\ell}\,g)$

We show subexponential estimate $c(k,\ell) = O_\varepsilon(e^{\varepsilon \max(|k|,|\ell|)^{1/3}\min(|k|,|\ell|)^{2/3}})$

by using estimates on $W_\lambda(a_{k,l}\gamma g)$ for varying γ.

In turn, the subexponential estimate on c(k,l) implies moderate growth. Thus

Theorem [M-Trinh] if the Fourier series for F converges absolutely, then F has moderate growth.

EXPONENTIAL GROWTH?

The previous result required the (AbsCon) assumption that the Fourier expansion converges absolutely (can get by with simply boundedness of terms).

A separate assumption is (ExpGro): $F(a_{y_1,y_2}) \ll e^{K(y_1+y_2)}$ for $y_1, y_2 \geq \sqrt{3/4}$ and some $K > 0$.

All automorphic eigenfunctions studied in the literature (to my knowledge) obey such an assumption.

- But exp(j(z)) does not.
- There may be a reduction to such an exponential bound using Hecke operators.

Theorem [M-Trinh] If F obeys (ExpGro), then it does not have growing Whittaker functions in its Fourier expansion (only Jacquet's decaying W's).

However, we cannot show F grows moderately.

Nevertheless, enough to rule out the naïve picture that a growing automorphic form looks like j(z), in that it grows exponentially and has M's in its Fourier expansion.

PROOF (ASSUMING AN EXPONENTIAL BOUND)

Fourier coefficients inherit bounds on F, so

$$[P^{0,\ell}F]\left(\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^{-2} \end{pmatrix}\right) = [P^{m,n}F]\begin{pmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^{-2} \end{pmatrix} \ll e^{Kt^3},\ t>1,$$

SL(2)-Iwasawa: $\operatorname{Im}\frac{ai+b}{ci+d} = \frac{1}{c^2+d^2}$, thus

$$[P^{0,\ell}F]\left(\begin{pmatrix} (c^2+d^2)^{-1/2} & \theta_\gamma(c^2+d^2)^{1/2} & 0 \\ 0 & (c^2+d^2)^{1/2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^{-2} \end{pmatrix}\right) =$$

$$\sum_{k\in\mathbb{Z}}[P^{k,0,\ell}F]\left(\begin{pmatrix} (c^2+d^2)^{-1/2} & \theta_\gamma(c^2+d^2)^{1/2} & 0 \\ 0 & (c^2+d^2)^{1/2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^{-2} \end{pmatrix}\right) \ll e^{Kt^3},\ t>1$$

Now the growing coroot goes from t^3 to $(c^2+d^2)^{1/2}\,t^3$, which can be arbitrarily large.

APPLY ZUCKERMAN'S CONJECTURE

(which here is a Theorem due to To and Templier)

$$\log(\phi_\lambda^{(m)}(a_{y_1,y_2})) \sim 2\pi\left(p_3^{(m)}(y_1,y_2) - p_1^{(m)}(y_1,y_2)\right)$$

$$\begin{aligned}
p_3^{(1)}(y_1,y_2) - p_1^{(1)}(y_1,y_2) &= \quad (y_1^{2/3} + y_2^{2/3})^{3/2}, \\
p_3^{(2)}(y_1,y_2) - p_1^{(2)}(y_1,y_2) &= -(y_1^{2/3} + e^{-2\pi i/3} y_2^{2/3})^{3/2}, \\
p_3^{(3)}(y_1,y_2) - p_1^{(3)}(y_1,y_2) &= -(y_1^{2/3} + e^{2\pi i/3} y_2^{2/3})^{3/2}, \\
p_3^{(4)}(y_1,y_2) - p_1^{(4)}(y_1,y_2) &= \quad (y_1^{2/3} + e^{-2\pi i/3} y_2^{2/3})^{3/2}, \\
p_3^{(5)}(y_1,y_2) - p_1^{(5)}(y_1,y_2) &= \quad (y_1^{2/3} + e^{2\pi i/3} y_2^{2/3})^{3/2}, \\
p_3^{(6)}(y_1,y_2) - p_1^{(6)}(y_1,y_2) &= -(y_1^{2/3} + y_2^{2/3})^{3/2}.
\end{aligned}$$

Gives very precise behavior of Whittaker functions in y_2.

For $(c^2+d^2)^{1/2}$ sufficiently large, 3 of these violate the bound on the previous slide.

Use other parabolic to rule out 2 more.

Conclude that only $\phi_\lambda^{(6)}(g)$ = Jacquet's W_λ occurs.

CONCLUSIONS AND SPECULATIONS

[Miatello-Wallach] made a brave conjecture:

the moderate growth condition in the definition of automorphic forms is automatic in higher rank.

It's generically correct for $SL(3,\mathbb{Z})\backslash SL(3,\mathbb{R})/SO(3)$ with some natural assumptions

Absolute convergence is a big subtlety, as are bounds on Fourier coefficients for such hypothetical forms.

What does this mean?

- We can't expect to see automorphic forms on higher rank with Fourier coefficients that grow faster than polynomials
  - Somewhat problematic for certain string theory expectations
- Greens function Fourier expansion will not generalize easily from $SL(2,\mathbb{Z})$
- Langlands-style automorphic representations are enough to capture harmonic analysis in higher rank
  - leaves rank 1 settings separate from the rest of automorphic harmonic analysis.
- Koecher principle is not purely a Hartog's phenomenon.

Thank you for your time and for inviting me