<u>Notation.</u> G complex reductive Lie group $\supset P$ parabolic subgroup.

$G^\vee$ dual group $\supset P^\vee$ parabolic with same nodes as P.

Partial flag variety $G^\vee/P^\vee$ — homogeneous, smooth, projective, Fano.

<u>Question:</u> Mirror symmetry for flag varieties?

The open Richardson $\mathring{G/P}$ is the complement in G/P of the Schubert and opposite Schubert divisors — smooth, affine Calabi-Yau, cluster variety.

<u>Conjecture</u> (Rietsch '08) $G^\vee/P^\vee$ is mirror to $(\mathring{G/P}, f)$.

Kim-Givental '95: complete flag varieties, i.e. $P, P^\vee =$ Borel.

The conjecture emerged in relation with works by Lusztig, Zelevinsky, Fomin and others on crystals, Peterson and others on quantum Schubert calculus, Witten, Vafa and others on Landau-Ginzburg models.

In work with T. Lam we approach the problem via <u>automorphic forms</u>:

1. Introduction 2. mirror theorem
3. Berenstein-Kazhdan crystals 4. quantum Chevalley and examples

$$Kl^{Std}_{GL(3)}(a) := \sum_{z_1, z_2 \in \mathbb{F}_p^\times} e^{\frac{2i\pi}{p}\left(z_1 + z_2 + \frac{a}{z_1 z_2}\right)}$$

hyper-Kloosterman

Deligne, SGA $4\frac{1}{2}$: pure lisse $\left|Kl^{Std}_{GL(3)}(a)\right| \leq 3p \quad \forall a \in \mathbb{F}_p^\times$

Katz book '96: rigid local system

Frenkel-Gross, Annals '09: construct rigid connection ∇^V_G on $\mathbb{P}^1$
for any representation (G, V)

Heinloth-Ngô-Yun, Annals '13: construct $Kl^V_G(a) = \sum_{z \in X(\mathbb{F}_p)} e^{\frac{2i\pi}{p} f_a(z)}$

Hecke eigensheaf

$$\begin{tikzcd} & Hk_V \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ Bun_{\check{G}} & & \mathbb{P}^1 \times Bun_{G^\vee} \end{tikzcd} \qquad p_{2!}\, p_1^*$$

Ramanujan bound over function field. Compare Ramanujan $\tau(p)$.

Lam-T '16: X is identified with $\mathring{G}/P$ and $f_a(z)$ is the potential function.

example above: $z \in X = \mathbb{G}_m \times \mathbb{G}_m = \mathring{\mathbb{P}}^2 = \mathring{GL(3)} \Big/ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}$

mirror symmetry

<u>in mathematics</u>: two varieties that exchange several invariants. Specifically the symplectic geometry of one is related to the complex geometry of the other.

Key: Deformations, families, holomorphic curves, periods.

<u>in physics</u>: two equivalent ways to describe the same theory.

<u>example</u>: Hodge diamond of Calabi-Yau manifolds.

$$h^{p,q}(X) = h^{n-p,q}(Y)$$

quintic 3-fold

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 1 & & 0 & \\
1 & & 101 & & 101 & & 1 \\
 & 0 & & 1 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

Dwork family

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 101 & & 0 & \\
1 & & 1 & & 1 & & 1 \\
 & 0 & & 101 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

Smooth projective Fano.		Landau-Ginzburg model = (quasi-projective Calabi-Yau, potential).
symplectic invariants: A-side		complex invariants: B-side
Fukaya category	$\overset{?}{\simeq}$ HMS	Matrix factorization category
quantum cohomology = enumerating rational curves		singularity theory.
Frobenius manifold		Saito mixed Hodge modules
isomonodromic deformations		miniversal deformations
small quantum differential equation (linear ODE)	$\overset{?}{\simeq}$ MIRROR	pushforward D-module

Reconstruction theorems: e.g. quantum product is associative (WDVV equation)

e.g. from small to big when cohomology is generated in degree 2.

early works that launched the program:

Givental ICM'94, Kontsevich ICM'94, Dubrovin ICM'98

<u>Example</u>: $\mathbb{C}\mathbb{P}^1 = GL(2) \Big/ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$

Gelfand - Tsetlin

quantum connection is

$$a \frac{d}{da} - \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}$$

$$\Big\downarrow$$

$$z \longrightarrow a$$

$$(*) \quad f_a(z) = \sum_{\text{arrows}} \frac{\text{head}}{\text{tail}} = z + \frac{a}{z}$$

$$\oint_{S^1} e^{z + \frac{a}{z}} \frac{dz}{2i\pi z} = \sum_{r=0}^{\infty} \frac{a^r}{(r!)^2} = I_0(2\sqrt{a})$$

$$\int_{-\infty}^{0} e^{z + \frac{a}{z}} \frac{dz}{z} = 2K_0(2\sqrt{a})$$

MIRROR $\Longrightarrow$ I_0, K_0 are in the kernel of the Bessel operator

$$\left(a \frac{d}{da}\right)^2 - a$$

Friedrich Wilhelm Bessel (1784-1846)

Fundamental example: projective space

$$\mathbb{P}^n \xrightarrow{\text{MIRROR}} (\mathbb{C}^\times)^n, \quad f_a(z) = z_1 + z_2 + \dots + z_n + \frac{a}{z_1 z_2 \cdots z_n}$$

ex (n=2) Kontsevich's formula for # of rational curves of deg d through 3d-1 generic points in the plane.

d	1	2	3	4	5	6
#	1	1	12	620	87 304	26 312 976

How to generalize it?

- $\mathbb{P}^n$ is an example of toric variety $\rightsquigarrow$ mirror symmetry for toric varieties (proved by Givental)

- $\mathbb{P}^n$ is an example of projective homogeneous variety $\longrightarrow$ this talk

$$\mathbb{P}^n = GL(n) \Big/ \begin{pmatrix} * & * \\ 0 - 0 & * \end{pmatrix}$$

- $\mathbb{P}^n$ is an example of Fano variety $\longrightarrow$ del Pezzo surfaces, Fano 3-folds, Fano 4-folds

Kim-Givental '95: complete flag variety G/B, dual $G^\vee/B^\vee$. B = Borel subgp.
Joe-Kim '03

$G^\vee/B^\vee$ —— MIRROR —— $(\overset{\circ}{G}/B, f)$

quantum connection — Toda lattice, Whittaker function — integral representation

Will be discussed in other talks in the conference this week!

<u>Non-exhaustive list</u> of <u>related works</u>: Jacquet integral '67; Kostant, Goodman-Wallach: rep theory; Jacquet-Piatetski-Shapiro-Shalika: Rankin-Selberg integrals; Casselman-Shalika-Shintani formula; Stade formula; Frenkel-Gaitsgory-Vilonen: geometric Langlands; Peterson, Kostant, Rietsch: Toda; Ginzburg-Jiang-Soudry: automorphic descent; Brubaker-Bump-Chinta-Friedberg: metaplectic; Borodin, Chhaibi, Corwin, O'Connell: probabilistic processes; Gerasimov-Lebedev-Oblezin: integrable systems; Braverman-Maulik-Okounkov: Springer resolution; Brumley-T'14: large values and singularities; Miller-Trinh '16: automorphic growth. Poincaré 1912, Bump-Friedberg-Goldfeld '88: Poincaré series.

<u>Zuckerman conjecture</u> (unpublished from '79) First appears in To's PhD thesis '95.

T. in progress: study it using ideas from mirror symmetry.

- Lefschetz thimbles of f in $\overset{\circ}{G}/B$.
- Dubrovin conjecture: exceptional collection on $G^\vee/B^\vee$.

(see also Gamma conjecture of Golyshev-Iritani-Galkin '13)

Theorem (Lam - T '16)

If $P^\vee$ is a minuscule parabolic, then there is an isomorphism

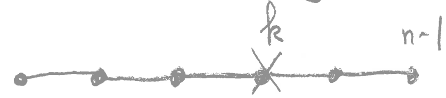
$$\begin{matrix} \text{quantum connection} \\ \text{for } G^\vee/P^\vee \end{matrix} \simeq \begin{matrix} \text{crystal } \mathcal{D}\text{-module} \\ \text{for } (\overset{\circ}{G/P}, f_a) \end{matrix}$$

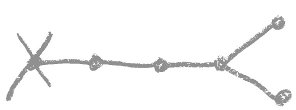
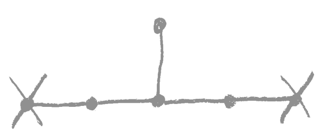
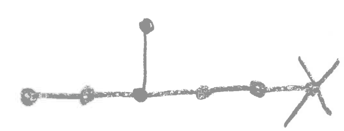
$$a\frac{d}{da} - \sigma *_a \qquad\qquad \int_{\overset{\circ}{G/P}} e^{f_a}$$

connection 1-form = quantum multiplication by σ, where $\mathrm{Pic}(G^\vee/P^\vee) = \mathbb{Z}\sigma$

pushforward $\mathcal{D}$-module $\mathcal{D} = \mathbb{C}[a, a^{-1}]\langle a\frac{d}{da}\rangle$

List of minuscule flag varieties (also compact Hermitian symmetric)

- $\mathbb{P}^n$ and Grassmannian $Gr(k,n)$ 

- even-dimensional quadric 
- Spinor variety = orthogonal Grassmannian $OG(n,2n)$ 
- Cayley plane = projective Octonions (dim = 16) 
- Freudenthal variety (dim = 27) 

<u>idea of proof</u>: via automorphic forms

quantum $G^\vee/P^\vee$ —— MIRROR THM —— $(\mathring{G}/P, f_a)$

Langlands reciprocity

Kloosterman sheaves as Hecke eigenvalues

rigid automorphic form A_G on $\mathbb{P}^1$

we identify the crystal $\mathcal{D}$-module as the automorphic side. (technically our main result).

Heinloth-Ngô-Yun '13: construct A_G
- tame unipotent at $\{0\}$
- <u>wild simple supercuspidal</u> at $\{\infty\}$

Zhu '16: quantization Hitchin system in the <u>ramified</u> case.

<u>remark</u>. Witten "gauge theory and wild ramification" '07 relates Langlands reciprocity and T-duality of the Hitchin systems for G and $G^\vee$.
See also Hausel-Thaddeus, Kapustin-Witten, Gukov-Witten, Boalch, ...

Berenstein-Kazhdan crystal $\xrightarrow{\text{tropicalize}}$ Lusztig-Kashiwara combinatorial crystal

$$\begin{tikzcd} \lambda_1 \arrow[d] & \\ z \arrow[r] & \lambda_2 \end{tikzcd} \qquad f_\lambda(z) = \frac{z}{\lambda_1} + \frac{\lambda_2}{z}$$

$$\begin{array}{c} \lambda_1 \\ \rotatebox{90}{$\leqslant$} \\ z \geqslant \lambda_2 \end{array} \qquad \max(z-\lambda_1, \lambda_2 - z) \leqslant 0$$

Gelfand-Tsetlin pattern $\overset{\longleftrightarrow}{1:1}$ Semi-std Young tableaux of shape λ.

<u>Kloosterman sum</u>: $\sum_{z \in \mathbb{F}_p^\times} \chi(z) e^{\frac{2i\pi}{p} f(z)}$ $\quad \chi : \mathbb{F}_p^\times \to S^1$

<u>Schur polynomial</u> $s_\lambda(x) = \sum_{\lambda_2 \leqslant z \leqslant \lambda_1} x^z$

<u>Bessel-Whittaker function</u>: $\int \chi(z) e^{f(z)} \frac{dz}{z}$

<u>crystal $\mathcal{D}$-module</u>: $\int \mathcal{L}_\chi \otimes f_\lambda^* \mathrm{Exp}$

$\mathcal{D} = \mathbb{C}[z]\langle d \rangle$

$$\mathcal{L}_\chi := \mathcal{D}/\mathcal{D}(d - \chi z) \qquad \mathrm{Exp} := \mathcal{D}/\mathcal{D}(d-1)$$

$$f^* \mathrm{Exp} = \mathcal{D}/\mathcal{D}(d - f')$$

W_P: Weyl group of the Levi subgroup L_P of P.

W^P: minimal representatives for W/W_P in Bruhat order. w_P^{-1}: longest element of W^P.

B_-: opposite Borel. $\psi : U \longrightarrow \mathbb{A}^1$ non degenerate additive character.

<u>Berenstein-Kazhdan geometric crystal</u>: $U\,Z(L_P)\,w_P\,U \cap B_-$

$a \in Z(L_P)$ $\quad f_a(u_1 w_P u_2) := \psi(u_1) + \psi(u_2)$ potential.

$=$ sum of ratios of generalized minors on G.

$$a\,\eta(u)\,w_P\,u^{-1} \longleftarrow\!\shortmid\; u$$

$$\begin{tikzcd}
X_a & B_- w_P B_- \cap U \arrow[l, "\sim"', "\text{Fomin-Zelevinsky twist map } \eta"] \arrow[r, "\sim"] & \mathcal{R}^{w_0}_{w_P w_0} \arrow[d, "\wr"] \arrow[r, phantom, "\subset"] & G/B \arrow[d] \\
 & \text{Richardson variety (cluster, affine Calabi-Yau)} : & \overset{\circ}{G}/P \arrow[r, phantom, "\subset"] & G/P
\end{tikzcd}$$

$$\begin{tikzcd}
 & U\,Z(L_P)\,w_P\,U \cap B_- \arrow[dl, "f"'] \arrow[dr, "\pi"] & \\
\mathbb{A}^1 & & Z(L_P)
\end{tikzcd}$$

$\pi_! f^* \mathcal{D}/\mathcal{D}(\partial - 1)$

crystal $\mathcal{D}$-module

cell in Beilinson-Drinfeld affine Grassmannian.

$p_{2!}\, p_1^* =$ Hecke correspondence

Heinloth-Ngô-Yun rigid automorphic form A_G is a D-module on Bun_G

$$\begin{tikzcd}
 & & \mathrm{Gr}^\circ \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
 & \mathrm{Bun}_G & & \mathbb{P}^1 - \{0,\infty\} \times \mathrm{Bun}_G \\
\mathbb{A}^1 & U/[U,U] \times \mathbb{A}^1 \arrow[l, "\psi"'] \arrow[u, phantom, "\cup" description] & & \{pt\} \arrow[u, phantom, "\cup" description]
\end{tikzcd}$$

restrict this diagram to this point and compare with previous page $\Longrightarrow$

Thm (Lam-T '16)
If $P^\vee$ is minuscule, then $Kl_{G^\vee}^\vee$ coincides with $\int_{\mathring{G}/P} e^{f_a}$

$$p_{2!}\, p_1^*(A_G) = Kl_{G^\vee}^\vee \boxtimes A_G$$

generalized Kloosterman D-module on $\mathbb{P}^1 - \{0,\infty\}$ as Hecke eigenvalue of A_G.

(compare $T_a(f) = \lambda(a) f$ for a classical modular form f on $SL_2\mathbb{Z}$)

Consequences of the mirror theorem:

- we establish the Peterson isomorphism (announced '97) for minuscule flag varieties $G^\vee/P^\vee$. This is the semi-classical limit ($\hbar \to 0$) of the mirror theorem

$$QH^*(G^\vee/P^\vee) \simeq \mathcal{O}(Y_P)$$

small quantum cohomology ring — ring of regular functions on the Peterson variety Y_P.

$$Y := \{ g \in G/B \,,\; Ad(g^{-1}) f \in [u,u]^\perp \}$$

$$Y_P := Y \cap B_- w_P B$$

principal nilpotent in $\mathfrak{b}_-$

- the conjecture of Batyrev–Ciocan-Fontanine–Kim–Van Straten (Acta Math '00) for Grassmannians $Gr(k,n)$ using Gelfand-Tsetlin coordinates as a cluster chart.

- a conjecture of Marsh-Rietsch '13 for $Gr(k,n)$ and Pech-Rietsch-Williams '15 for quadrics

Euler-Poincaré characteristic calculation + purity.

quantum Chevalley formula (Fulton - Woodward '04 Witten '91)

$$H^*(G/P) = \bigoplus_{w \in W/W_P} \mathbb{C}\,\sigma_w$$ Schubert basis.

$$W^P \xrightarrow{\sim} W/W_P \qquad \pi_P : W \longrightarrow W/W_P$$

minimal representatives in Bruhat order

Th: If P is minuscule, $\exists !$ root γ such that $\forall w \in W^P$

$$\sigma_1 * \sigma_w = \sum_{\substack{\beta \in R^+ \setminus R_P^+ \\ W^P \ni w s_\beta > w}} \langle \beta^\vee, \omega_1 \rangle \, \sigma_{w s_\beta} + a \, \langle \gamma^\vee, \omega_1 \rangle \, \sigma_{\pi_P(w s_\gamma)}$$

if $\ell(\pi_P(w s_\gamma)) = \ell(w) + 1 - \langle \gamma^\vee, 2(\rho - \rho_P) \rangle$

Example $Gr(2,4)$ $\quad W = \mathfrak{S}_4 \quad W_P = \mathfrak{S}_2 \times \mathfrak{S}_2$

W^P: ϕ $\quad \sigma_\phi = 1$, σ_1, σ_2, σ_{11}, σ_{21}, σ_{22}

$$\sigma_1 * \sigma_{21} = \sigma_{22} + a$$: add a box $\quad \sigma_{21} \longrightarrow \sigma_{22}$

$$\sigma_1 * \sigma_{22} = a\,\sigma_1$$: remove a rim $\quad \sigma_{22} \longrightarrow \sigma_1 \qquad \sigma_{21} \longrightarrow \phi \quad \sigma_\phi = 1$

Example: $Gr(2,4) = GL(4) \Big/ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} = 4$-dimensional quadric

$\bullet \!-\!\!-\!\!-\!\! \times \!\!-\!\!-\!\!-\! \bullet \quad A_3$ Dynkin

quantum connection is

$$a \frac{d}{da} - \begin{pmatrix} 0 & 0 & 0 & 0 & a & 0 \\ 1 & 0 & 0 & 0 & 0 & a \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Gelfand-Tsetlin coordinates

$$\begin{tikzcd} {} \arrow[d] & & \\ z_1 \arrow[r] \arrow[d] & z_3 \arrow[d] & \\ z_2 \arrow[r] & z_4 \arrow[r] & a \end{tikzcd}$$

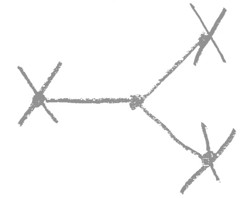
$$f_a(z) = \sum_{\text{arrows}} \frac{\text{head}}{\text{tail}} = z_1 + \frac{z_2}{z_1} + \frac{z_3}{z_2} + \frac{z_4}{z_3} + \frac{z_4}{z_2} + \frac{a}{z_4}$$

Corollary $\Longrightarrow$ $\oint e^{f_a(z)} \frac{dz}{z}$ is in the kernel of the connection.

Which can be verified directly: $\displaystyle\sum_{r=0}^{\infty} \frac{(2r)!}{r!^6} a^r$

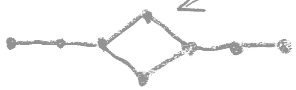
is in the kernel of $\partial^5 - 2a(2\partial + 1)$ $\qquad \partial = a\dfrac{d}{da}$

Example: 6-dimensional quadric $= SO(8)/P$



D_4-triality

Hasse diagram



middle cohomology is 2-dim.

quantum connection is

$$a\frac{d}{da} - \begin{pmatrix} 0 & \cdots & \cdots & \cdots & \cdots & \cdots & a & 0 \\ 1 & & & & & & 0 & a \\ & 1 & & & 0 & & & \vdots \\ & & 1 & & & & & \vdots \\ & & 1 & & & & & \vdots \\ & & & 1 & 1 & & & \vdots \\ 0 & & & & & 1 & & \\ & & & & & & 1 & 0 \end{pmatrix}$$

quiver Pech-Rietsch-Williams '15

$$\begin{tikzcd} {} \arrow[d] & \\ z_1 \arrow[d] & \\ z_2 \arrow[d] & \\ z_3 \arrow[r] \arrow[d] & z_5 \arrow[d] \\ z_4 \arrow[r] & a \end{tikzcd}$$

$$f_a(z) = z_1 + \frac{z_2}{z_1} + \frac{z_3}{z_2} + \frac{z_4}{z_3} + \frac{z_5}{z_3} + \frac{a}{z_4} + \frac{a}{z_5}.$$

7-dim stable subspace generated by σ.

$= \mathcal{D}/\mathcal{D}(\partial^7 - 2a(2\partial+1))$ where $\mathcal{D} = \mathbb{C}[a,a^{-1}]\langle\partial\rangle$.

Thm (Katz, Frenkel-Gross) The monodromy group is G_2.

because of S_3-symmetry of D_4

because it is the (1,7)-hypergeometric ${}_1F_7\left(\begin{matrix} 1/2 \\ 1\,1\,1\,1\,1\,1\,1 \end{matrix}; a\right)$

thm 4.1.5 in "Exponential sums and diff. equations", Annals of Math Studies.