Bounds for Poincaré constants on convex sets

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Banff, 11 July 2016

References

Some of the results here presented are contained in

- B. - Nitsch - Trombetti, **Comm. Contemp. Math.** (2015)
- B. - Santambrogio, **Springer Proc. Math. Stat.** (2016)

1. Poincaré constants

2. A sharp upper bound

3. A lower bound by Optimal Transport

4. Some generalizations

Poincaré inequalities

We take $1 < p < \infty$ and $\Omega \subset \mathbb{R}^N$ smooth and bounded

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(Ω need to be connected)

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The optimal $t_u \in \mathbb{R}$ is such that

$$\int_\Omega |u - t_u|^{p-2} (u - t_u) = 0$$

the inequality $(*)$ is equivalent to the one previously mentioned

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Discuss (sharp or not) geometric estimates on the **optimal Poincaré constant**

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Anticipating the conclusions

We will see that

$$\mu_p \simeq (\text{diameter})^{-p}$$

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In other words, they are **Neumann eigenfunctions** of the $p-$Laplacian

A minmax characterization of μ_p

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$\mu_p(\Omega)$ can be seen also as the **second critical value** of

$$u \mapsto \int_\Omega |\nabla u|^p \quad \text{on} \quad \mathcal{S}_p(\Omega) = \left\{ u \in W^{1,p}(\Omega) \,:\, \int_\Omega |u|^p = 1 \right\}$$

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Proposition

Consider the set of **continuous loops**

$$\Gamma_1 = \{\gamma : \mathbb{S}^1 \to \mathcal{S}_p(\Omega) \ : \ \textit{odd \& continuous}\}$$

then

$$\mu_p(\Omega) = \inf_{\gamma \in \Gamma_1} \max_{u \in \mathrm{Im}(\gamma)} \int_\Omega |\nabla u|^p$$

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Remark

This is the non-Hilbertian generalization of the minmax characterization of the **first nontrivial Neumann eigenvalue** of the Laplacian (*seen in Dorin's talk*)

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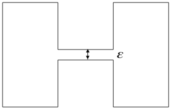

Figure : $\mu_p(\Omega_\varepsilon) \to 0$

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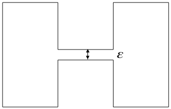

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YES if Ω **convex bounded** (Payne-Weinberger, Ferone-Nitsch-Trombetti)

$$\mu_p(\Omega) > \left(\frac{\pi_p}{\mathrm{diam}(\Omega)} \right)^p$$

<u>Estimate is sharp</u>

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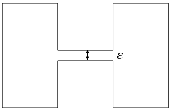

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Estimate is sharp for the sequence of collapsing rectangles

$$R_n = [0,1] \times \left[0, n^{-1}\right]$$

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Szegő-Weinberger

For a general open set

$$\mu_2(\Omega) \le \mu_2(\text{ball}) \left(\frac{|\text{ball}|}{|\Omega|}\right)^{\frac{2}{N}}$$

Known for $p = 2$ **only!** For $p \neq 2$ is **unknown**

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This is not always useful!

If $|\Omega| \ll 1$, the upper bound blows-up. But for the sequence of collapsing rectangles

$$R_n = [0, 1] \times \left[0, n^{-1}\right] \qquad \text{we have} \qquad \sup_{n \in \mathbb{N}} \mu_p(R_n) < +\infty$$

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Notation

For an open set $\Omega \subset \mathbb{R}^N$, we set

$$\lambda_p(\Omega) = \inf_{u \in W_0^{1,p}(\Omega)} \left\{ \int_\Omega |\nabla u|^p \, : \, \int_\Omega |u|^p = 1 \right\}$$

First **Dirichlet eigenvalue** of the $p-$Laplacian

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Theorem [B.-Nitsch-Trombetti]

Let $1 < p < \infty$, *for every* $\Omega \subset \mathbb{R}^N$ *convex we have*

$$\mu_p(\Omega) < \lambda_p(ball) \left(\frac{\mathrm{diam}(ball)}{\mathrm{diam}(\Omega)} \right)^p$$

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Inequality is **strict**, *but the estimate is sharp.*

Indeed, there exist $\{\mathcal{D}_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ open convex sets such that

1. $\mathrm{diam}(\mathcal{D}_n) = 2$
2. $\mathcal{D}_n$ *collapse to a segment*
3. $\mu_p(\mathcal{D}_n) \to \lambda_p(B_1)$ *(B_1 is the ball of radius 1)*

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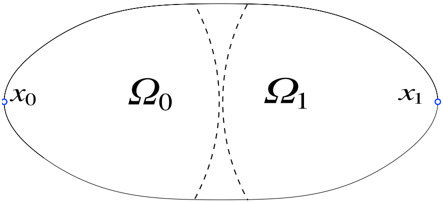

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- more precisely, we take the test function

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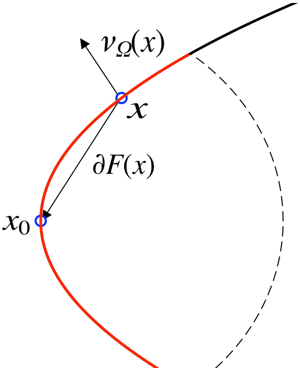
$$= \int_{\Omega_0\cap\partial\Omega} F\,|\nabla F|^{p-2}\,\frac{\partial F}{\partial \nu_\Omega} + \lambda_p(B_1)\int_{\Omega_0} |F|^p$$



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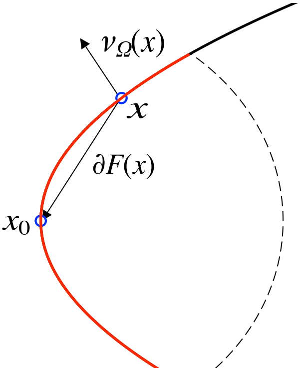
$$\overset{(\textit{see picture})}{\leq} \qquad \lambda_p(B_1)\int_{\Omega_0} |F|^p$$



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- first inequality is strict, since the **test function can not be an eigenfunction** (by Harnack's inequality) $\square$

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   To be sharp, one should make Ω "collapse"

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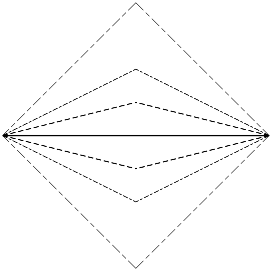

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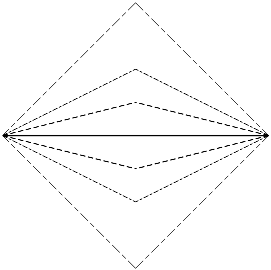

- **vague idea:** the variational characterization of $\mu_p(\mathcal{D}_n)$ converges to the minimization of a $1D$ weighted Rayleigh quotient, which is the same defining the first Dirichlet eigenfunction on the ball (which is radial, i.e. $1D$) $\square$

A shape optimization problem (without solution)

Corollary

The shape optimization problem

$$\sup\{\mu_p(\Omega)\,:\,\Omega \textit{ convex},\quad \mathrm{diam}(\Omega)=c\}$$

does not admit a solution. *A maximizing sequence is given by the "shrinking kites"* $\{\mathcal{D}_n\}_{n\in\mathbb{N}}$

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Proof.

From the previous estimate, we have

$$\mu_p(\Omega) < \lambda_p(\text{ball of radius } 1)\left(\frac{2}{c}\right)^{\frac{p}{N}}$$

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Proof.

From the previous estimate, we have

$$\mu_p(\Omega) < \lambda_p(\text{ball of radius } 1)\left(\frac{2}{c}\right)^{\frac{p}{N}}$$

The upper bound on the right is asymptotically attained by the sequence $\{\mathcal{D}_n\}_{n\in\mathbb{N}}$ $\square$

Summary

- Both shape optimization problems

$$\sup\{\mu_p(\Omega)\,:\,\Omega \text{ convex}, \quad \mathrm{diam}(\Omega) = c\}$$

and

$$\inf\{\mu_p(\Omega)\,:\,\Omega \text{ convex}, \quad \mathrm{diam}(\Omega) = c\}$$

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Summary

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do not admit solution

• In both cases, optimizing sequences undergo a **concentration phenomenon** and collapse to a segment

Comparison of constants

Corollary (*weak Szegő-Weinberger*)

For $1 < p < \infty$ and $\Omega \subset \mathbb{R}^N$ convex, we have

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In the quadratic case $p = 2$, the previous is a consequence of

$$\mu_2(\Omega) \le \mu_2(B) \left(\frac{|B|}{|\Omega|} \right)^{\frac{2}{N}} \qquad \text{(Szegő-Weinberger)}$$

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A clue of a potentially exhisting Szegő-Weinberger for $p \neq 2$

1. Poincaré constants

2. A sharp upper bound

3. A lower bound by Optimal Transport

4. Some generalizations

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$$\left(\frac{\pi_p}{\mathrm{diam}(\Omega)}\right)^p < \mu_p(\Omega)$$

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The proof uses Optimal Transport tools, so let us recall...

...some facts from Optimal Transport

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Definition (Wasserstein distance)

If ρ_0, ρ_1 are probabilities on Ω, we set

$$\Pi(\rho_0, \rho_1) = \Big\{ \gamma \text{ probability on } \Omega \times \Omega \text{ with marginals } \rho_0 \text{ and } \rho_1 \Big\}$$

Then for $1 < \alpha < \infty$ we define the $\alpha-$*Wasserstein distance*

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$$\mathbb{W}_\alpha(\Omega) = \begin{array}{c}\text{“space of probabilities on } \Omega \\ \text{endowed with the } \alpha-\text{Wasserstein distance”}\end{array}$$

(This is a complete and separable metric space)

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Remark

The curve μ_t is a geodesic in $\mathbb{W}_\alpha(\Omega)$, driven by the velocity field $\mathbf{v}_t$

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Let $1 < p < \infty$ *and* $1 < q < p$. *Let* $\Omega \subset \mathbb{R}^N$ *be an open convex set. Let* ϕ *smooth and let* ρ_0, ρ_1 *probabilities. Then*

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◇ Use Holder inequality and <u>geodesic convexity</u> of $t \mapsto \|\mu_t\|_{L^{q'}}$ $\square$

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  and optimally transport ρ_0 on ρ_1
- i.e. use the **expedient estimate** with ρ_0 and ρ_1, observe that
$$\int \phi\,(\rho_0 - \rho_1) = 2\,\frac{\displaystyle\int |\phi|^q}{\displaystyle\int |\phi|^{q-1}}$$

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Remark

Taking $q \nearrow p$ implies that we use the expedient estimate with

$$W_\infty(\rho_0, \rho_1)$$

i.e. we use the $\infty-$Wasserstein distance to prove the estimate

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Remark

The **lower bound on** μ_p and the **Nash-type inequality** are consequences of this general result

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$$\mu_{p,q}(\Omega) := \inf_{u \in W^{1,p}(\Omega)} \left\{ \int_\Omega |\nabla u|^p \, dx \ : \ \int_\Omega |u|^q = 1, \ \int_\Omega |u|^{q-2} \, u = 0 \right\}$$

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For every sequence of convex sets $\{\Omega_n\}_{n \in \mathbb{N}}$ with $|\Omega_n| \to 0$ and $\mathrm{diam}(\Omega_n) \geq c > 0$

$$\lim_{n \to \infty} \mu_{p,q}(\Omega_n) = \begin{cases} 0, & \text{if } q > p \\ +\infty, & \text{if } q < p \end{cases}$$

Upper bound for $q > p$

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Theorem [B.-Nitsch-Trombetti]

For $q > p$, *the shape optimization problem*

$$\sup\{\mu_{p,q}(\Omega) \, : \, \Omega \text{ convex}, \quad \mathrm{diam}(\Omega) = c\}$$

now **has a solution**

Many thanks for your kind attention

"*Discipline is never an end in itself, only a means to an end*" (`R. Fripp`)