Effective Index Estimates via Euclidean isometric embeddings

Alessandro Carlotto

ETH

July 19th 2016

Overview

General goal

Given a compact Riemannian manifold (X^{n+1}, g) of dimension at least three, possibly with non-empty boundary, and considered the class

$$\Lambda := \{\text{compact minimal hypersurfaces in } (X^{n+1}, g), \text{ possibly with suitable boundary conditions}\}$$

we want to establish **universal comparison theorems**, namely algebraic inequalities relating $Ind(M^n)$ and the dimension of the real homology groups of M^n with coefficients that only depend on the ambient manifold (X^{n+1}, g).

An example

Theorem (A. Savo, 2010)

In the round three-sphere (S^3, g_0) for a non-equatorial minimal surface of genus γ

$$Ind(M^2) \geq \frac{\gamma}{2} + 4.$$

An example

Theorem (A. Savo, 2010)

In the round three-sphere (S^3, g_0) for a non-equatorial minimal surface of genus γ

$$Ind(M^2) \geq \frac{\gamma}{2} + 4.$$

Remarkable improvement of the well-known general bound by F. Urbano (1991) asserting that the index of non-equatorial minimal surfaces is at least five

$$Ind(M^2) \geq 5.$$

with equality attained only by Clifford tori.

Two settings

Joint works with **Lucas Ambrozio** and **Benjamin Sharp**.

(N^{n+1}, g) compact, $\partial N = \emptyset$	(Ω^{n+1}, g) compact, $\partial\Omega \neq \emptyset$
$\Lambda := \{$**closed** $\mathrm{min.\ hypers.}\}$	$\Lambda := \{$**free boundary** $\mathrm{min.\ hypers.}\}$
paper 1 - January 2016	paper 2 - May 2016
arXiv: 1601.08152	arXiv: 1605.09704

In this talk I will only focus on the **free boundary** case.

Free boundary minimal hypersurfaces

(Ω^{n+1}, g) compact Riemannian domain with $\partial\Omega \neq \emptyset$.

Definition

$M^n \subset \Omega^{n+1}$ properly embedded is called **free boundary minimal hypersurface** if it is a critical point for the $n-$dimensional area functional for the class of geometric variations given by vector fields that are tangent to $\partial\Omega$ when restricted to ∂M.

Free boundary minimal hypersurfaces

(Ω^{n+1}, g) compact Riemannian domain with $\partial\Omega \neq \emptyset$.

Definition

$M^n \subset \Omega^{n+1}$ properly embedded is called **free boundary minimal hypersurface** if it is a critical point for the $n-$dimensional area functional for the class of geometric variations given by vector fields that are tangent to $\partial\Omega$ when restricted to ∂M.

First variation

$M^n \subset \Omega^{n+1}$ properly embedded is a **free boundary minimal hypersurface** IFF the following two condtions hold:

1. vanishing mean curvature, namely $H \equiv 0$ along M;
2. orthogonal intersection at the boundary, namely $\nu \perp \partial\Omega$ along ∂M.

Examples and existence results

Trivial examples in the unit ball

- flat disks $D^2 \hookrightarrow D^3 \subset \mathbb{R}^3$;
- critical catenoids $\mathbf{Cat}_c \hookrightarrow D^3 \subset \mathbb{R}^3$.

Examples and existence results

Trivial examples in the unit ball

- flat disks $D^2 \hookrightarrow D^3 \subset \mathbb{R}^3$;
- critical catenoids $\mathbf{Cat}_c \hookrightarrow D^3 \subset \mathbb{R}^3$.

Recent examples in the unit ball

Topological type described by integers (γ, r)

- Fraser-Schoen (2013) proved existence for $\gamma = 0$ and any $r \geq 2$;
- Folha-Pacard-Zolotareva (2015) $\gamma = 0, 1$ and any $r \geq r_0$.

Lots of existence results with very **diverse** methods: Courant (1940), Hildebrandt-Nitsche (1979), Meeks-Yau (1980), Struwe (1984), Jost and Grüter-Jost (1986), Li (2015), Maximo-Nunes-Smith (2016)...

Morse index- I

Second variation

If M^n is two-sided, then can reduce to scalar variations and

$$Q^M(\phi,\phi) = \int_M |\nabla^M \phi|^2 - \int_M (Ric(N,N) + |A|^2)\phi^2 - \int_{\partial M} II^{\partial\Omega}(N,N)\phi^2$$

so integration by parts gives

$$Q^M(\phi,\phi) = -\int_M \phi\mathcal{L}\phi + \int_{\partial M} \phi\left(\frac{\partial\phi}{\partial\nu} - II^{\partial\Omega}(N,N)\phi\right).$$

Morse index- I

Second variation

If M^n is two-sided, then can reduce to scalar variations and

$$Q^M(\phi,\phi) = \int_M |\nabla^M \phi|^2 - \int_M (Ric(N,N) + |A|^2)\phi^2 - \int_{\partial M} II^{\partial\Omega}(N,N)\phi^2$$

so integration by parts gives

$$Q^M(\phi,\phi) = -\int_M \phi\mathcal{L}\phi + \int_{\partial M} \phi\left(\frac{\partial\phi}{\partial\nu} - II^{\partial\Omega}(N,N)\phi\right).$$

Eigenvalue problem

$$\begin{cases} \mathcal{L}\phi = -\lambda\phi \\ \frac{\partial\phi}{\partial\nu} = II^{\partial\Omega}(N,N)\phi \end{cases}$$

Morse index - II

Discrete spctrum

Have a complete basis of $L^2(M^n)$ say $\{\phi_j\}$ and associated discrete class $\{\lambda_j\}$ with $\lambda_j \uparrow +\infty$ such that

$$\begin{cases} \mathcal{L}\phi_j = -\lambda_j \phi_j \\ \frac{\partial \phi_j}{\partial \nu} = II^{\partial\Omega}(N,N)\phi_j \end{cases}$$

Morse index

$$Ind(M) = \#\left\{\lambda_j \text{ eigenvalue} \mid \lambda_j < 0\right\}.$$

Some motivations

From the **constructive** viewpoint, there seem to be no method will allow to control both the topology and the Morse index of the free boundary surface. This poses the need of classification and comparison criteria.

Some motivations

From the **constructive** viewpoint, there seem to be no method will allow to control both the topology and the Morse index of the free boundary surface. This poses the need of classification and comparison criteria.

- Fraser-Schoen (2016) proved that if $M^2 \subset B^3_1$ not flat (equiv. not a disk by Nitsche (1985)) then $Ind(M^2) \geq 4$. Questions:
  1. improve by bringing topology into play;
  2. flexible counterparts (theory for non symmetric domains and in presence of ambient curvature).
- Cheng-Fraser-Pang (2015) proved that if $Scal_\Omega \geq 0$ and $H_{\partial\Omega} > 0$ then the only free-boundary stable minimal surfaces are *either* disks *or* totally geodesic annuli. Questions:
  1. general classification theory for $Ind(M^2) = k$;
  2. asymptotic analysis.

Main result - Euclidean setting

Theorem (-, Ambrozio, Sharp)

Let Ω^{n+1} be a smooth, compact domain of the $(n+1)$-dimensional Euclidean space, $n \geq 2$ and let M^n be a compact, orientable, properly embedded free boundary minimal hypersurface in Ω^{n+1}.

1. If Ω^{n+1} is strictly mean convex, then

$$index(M^n) \geq \frac{2}{n(n+1)} dim H_1(M^n, \partial M^n; \mathbb{R}).$$

2. If Ω^{n+1} is strictly two-convex, then

$$index(M^n) \geq \frac{2}{n(n+1)} \max \dim \{H_1(M^n, \partial M^n; \mathbb{R}); H_{n-1}(M^n, \partial M^n; \mathbb{R})\}.$$

The case of surfaces - I

Corollary 1

Let Ω be a strictly mean convex domain of the three-dimensional Euclidean space. Let M^2 be a compact, orientable, properly embedded free boundary minimal surface in Ω with genus γ and $r \geq 1$ boundary components. Then

$$index(M) \geq \frac{1}{3}(2\gamma + r - 1).$$

The case of surfaces - I

Corollary 1

Let Ω be a strictly mean convex domain of the three-dimensional Euclidean space. Let M^2 be a compact, orientable, properly embedded free boundary minimal surface in Ω with genus γ and $r \geq 1$ boundary components. Then

$$index(M) \geq \frac{1}{3}(2\gamma + r - 1).$$

Corollary 2

In the Euclidean unit ball there exist free boundary minimal surfaces of arbitrarily large Morse index. The examples constructed by Fraser and Schoen and the examples constructed by Folha, Pacard and Zolotareva have their Morse indices growing linearly with the number of boundary components.

The case of surfaces - II

Corollary 3

Let Ω be a compact domain in $\mathbb{R}^3$ whose boundary is strictly convex. Then any sequence $\{M_i^2\}$ of compact, properly embedded free boundary minimal surfaces in Ω that has uniformly bounded index has a subsequence converging smoothly and graphically to a compact properly embedded free boundary minimal surface M^2 in Ω.

This follows from the compactness theory by Fraser-Li, which in turn is the free boundary version of the classical results by Choi-Schoen.

Main result - general setting

Given (Ω^{n+1}, g) a Riemannian manifold with boundary, consider an isometric embedding

$$\Omega^{n+1} \hookrightarrow (\mathbb{R}^d, \delta)$$

with 2nd fundamental form II^{Ω}. Moral statement: the conclusions above are true provided a certain **curvature positivity condition** is satisfied: we require curvature to be large with respect to II^{Ω} in L^2-sense.

In practice, this poses a problem of finding some **optimal isometric embedding** with respect to the above criterion.

Main result - general setting

Theorem (-, Ambrozio, Sharp)

Given $M^n \subset (\Omega^{n+1}, g)$ suppose that for any non-zero vector field on M^n

$$\int_M \left[tr_M(Rm^\Omega(\cdot, X, \cdot, X)) + Ric^\Omega(N,N)|X|^2 \right] > \int_M \left[(|II^\Omega(\cdot, X)|^2 - |II^\Omega(X,N)|^2) + (|II^\Omega(\cdot, N)|^2 - |II^\Omega(N,N)^2|)|X|^2 \right].$$

If Ω^{n+1} is strictly mean convex (i.e. $H^{\partial\Omega} > 0$) then

$$index(M^n) \geq \frac{2}{d(d-1)} dim H_1(M^n, \partial M^n; \mathbb{R}).$$

If Ω^{n+1} is strictly two-convex (i.e. $(\lambda_1 + \lambda_2)^{\partial\Omega} > 0$) then

$$index(M^n) \geq \frac{2}{d(d-1)} \max \dim \left\{ H_1(M^n, \partial M^n; \mathbb{R}); H_{n-1}(M^n, \partial M^n; \mathbb{R}) \right\}.$$

Examples of applicability

1. mean convex (resp. two-convex) domains in all rank 1 symmetric spaces:
   - $\mathbb{S}^n, \ n \geq 3$;
   - $\mathbb{RP}^n, \ n \geq 3$;
   - $\mathbb{CP}^n, \ n \geq 2$;
   - $\mathbb{HP}^n, \ n \geq 1$;
   - $Ca\mathbb{P}^2$.
2. product of spheres $\mathbb{S}^p \times \mathbb{S}^q$ for $(p, q) \neq (2, 2)$;
3. suitably pinched 3-manifolds (pinching only involving $Scal_\Omega$).

Spaces of forms and BC

Let d be the exterior differential on M^n and let $d^* : \Omega^p(M) \to \Omega^{p-1}(M)$ denote the codifferential defined in terms of the Hodge star operator on (M^n, g) (so that, as a result, $d^* = (-1)^{n(p+1)+1} * d *$). We define the sets

$$\mathcal{H}^p_N(M,g) = \{\omega \in \Omega^p(M);\ d\omega = 0,\ d^*\omega = 0 \text{ on } M^n \text{ and } i_\nu\omega = 0 \text{ on } \partial M\}$$

and

$$\mathcal{H}^p_T(M,g) = \{\omega \in \Omega^p(M);\ d\omega = 0,\ d^*\omega = 0 \text{ on } M^n \text{ and } \nu \wedge \omega = 0 \text{ on } \partial M\}.$$

We use the expression "harmonic form" to call any differential form that is simultaneously closed and co-closed.

Hodge theory for manifolds with boundary

In the setting above, the Hodge-de Rham theorem can be stated as follows.

Hodge and BC

Let (M^n, g) be a compact orientable manifold with non-empty boundary. For every $p = 0, \ldots, n$, the set of harmonic p-forms on M^n that are tangential at ∂M is isomorphic to the p-th cohomology group of M^n with real coefficients, i.e.

$$\mathcal{H}^p_N(M, g) \simeq H^p(M; \mathbb{R}).$$

Observe that the Hodge star operator of (M^n, g) gives an isomorphism between $\mathcal{H}^p_N(M, g)$ and $\mathcal{H}^{n-p}_T(M, g)$. Hence, we have the isomorphisms

$$\mathcal{H}^p_T(M, g) \simeq \mathcal{H}^{n-p}_N(M, g) \simeq H^{n-p}(M; \mathbb{R}) \simeq H_p(M, \partial M; \mathbb{R}).$$

the last following by Poincaré-Lefschetz duality.

The test functions obtained from harmonic one-forms

Let Ω be a domain in $\mathbb{R}^{n+1}$ with smooth boundary $\partial\Omega$.

Denote by $\{\theta_1, \ldots, \theta_{n+1}\}$ a fixed orthonormal basis of $\mathbb{R}^{n+1}$. Given a compact, free boundary minimal hypersurface M^n in Ω, we want to compute the index form on the functions

$$u_{ij} := \langle N \wedge \omega^{\sharp}, \theta_i \wedge \theta_j \rangle \quad \text{where } 1 \leq i < j \leq n+1$$

for ω a one-form (in fact, we will apply this for **harmonic** one forms).

The form of the identities in the Euclidean setting

Flat case

Let Ω be a domain in $\mathbb{R}^{n+1}$ whose boundary has mean curvature $H^{\partial\Omega}$ and second fundamental form $II^{\partial\Omega}$ with respect to the outward normal ν. Let M^n be a compact, orientable free boundary minimal hypersurface in Ω.

1. Given a harmonic one-form ω on M^n that is normal at the boundary ∂M then

$$\sum_{1\leq i<j\leq n+1} Q(u_{ij}, u_{ij}) = -\int_{\partial M} H^{\partial\Omega}|\omega|^2 d\sigma.$$

2. Given a harmonic one-form ω on M^n that is tangential at the boundary ∂M then

$$\sum_{1\leq i<j\leq n+1} Q(u_{ij}, u_{ij}) = -\int_{\partial M} (II^{\partial\Omega}(N,N)|\omega|^2 + II^{\partial\Omega}(\omega^\sharp, \omega^\sharp))d\sigma.$$

The core argument given the identities - I

Let us assume that M^n has index k, and denote by $\{\phi_q\}_{q=1}^{\infty}$ an $L^2(M, d\mu)$ orthonormal basis of eigenfunctions of the Jacobi operator of M^n satisfying the Robin boundary conditions. Let Φ denote the linear map defined by

$$\begin{array}{rccc} \Phi: & \mathcal{H}^1_T(M,g) & \to & \mathbb{R}^{n(n+1)k/2} \\ & \omega & \mapsto & \left[\int_M \langle N \wedge \omega^\sharp, \theta_i \wedge \theta_j \rangle \phi_q d\mu\right], \end{array}$$

where, $1 \leq i < j \leq n+1$ and q varies from 1 to k. Clearly,

$$dim\mathcal{H}^1_T(M,g) \leq dimKer(\Phi) + \frac{n(n+1)}{2}k$$

Since $\mathcal{H}^1_T(M,g) \simeq H_1(M, \partial M; \mathbb{R})$ the result will follow once we analyse the dimension of the kernel of Φ (and prove that such map is **injective**).

The core argument given the identities - II

Let ω be an element of the kernel of the map Φ. This means that all functions $u_{ij} = \langle N \wedge \omega^\sharp, \theta_i \wedge \theta_j \rangle$ are orthogonal to the first k eigenfunctions, namely $\phi_1, \ldots, \phi_k$. Since $index(M) = k$, we must have

$$Q(u_{ij}, u_{ij}) \geq \lambda_{k+1} \int_M u_{ij}^2 \, d\mu \geq 0 \quad \text{for all} \quad 1 \leq i < j \leq n+1,$$

by the variational characterization of the eigenvalues. In particular, by the first identity above, we have

$$0 \leq \sum_{1 \leq i < j \leq n+1} Q(u_{ij}, u_{ij}) = -\int_{\partial M} H^{\partial\Omega} |\omega|^2 d\sigma.$$

Since $H^{\partial\Omega} > 0$, the above inequality happens only if $|\omega|$ vanishes identically on ∂M. But then $\omega = 0$ on M. Hence, if the domain is strictly mean convex, Φ has trivial kernel, and the conclusion follows.

Conclusion

Thank you very much for your attention!