# A continuous matter distribution arising as an intrinsic flat limit of point particle configurations

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# Geometrostatic manifolds

- Addressed in great detail by Brill and Lindquist in their 1963 paper "Interaction energy in geometrostatics".
- Solutions of vacuum Einstein-Maxwell constraint equations

$$R[g] = 2|E|_g^2, \quad \mathrm{div}(E) = 0$$

on $\mathbb{R}^3 \setminus \{p_1, ..., p_n\}$ of the form

$$g = (\chi\psi)^2\delta, \quad E = \pm\mathrm{grad}_g\left(\ln(\chi/\psi)\right).$$

- "...static" refers to the existence of the electrostatic potential and vanishing second fundamental form; not suggesting initial data for static spacetime.

# Explicit Expressions

- The constraints reduce to $\Delta_\delta \chi = 0$, $\Delta_\delta \psi = 0$.
- Solutions compatible with asymptotically Euclidean behavior at infinity:

$$|\chi - 1|, |\psi - 1| = O(r^{-1}), \quad |d\chi|_\delta = |d\psi|_\delta = O(r^{-2}).$$

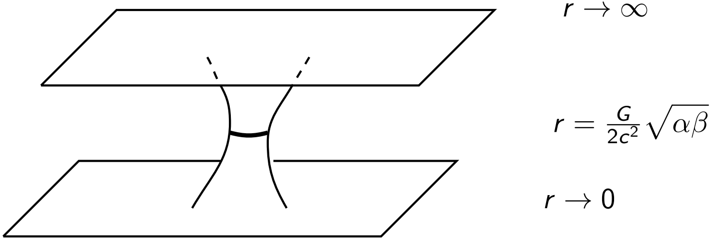
- Explicit solutions:

$$\chi(x) = 1 + \frac{G}{2c^2} \sum_{i=1}^{n} \frac{\alpha_i}{|x - p_i|}, \quad \psi(x) = 1 + \frac{G}{2c^2} \sum_{i=1}^{n} \frac{\beta_i}{|x - p_i|}.$$

- Impose $\alpha_i, \beta_i > 0$ for all $i$.

## Concrete example

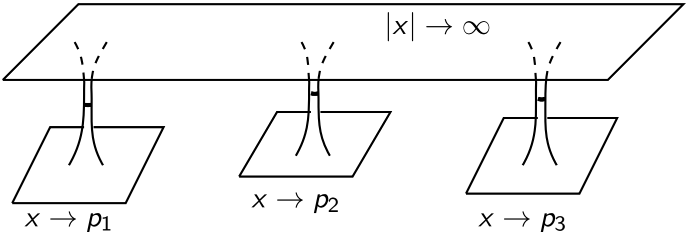
- $g = \left(1 + \frac{G}{2c^2} \cdot \frac{\alpha}{r}\right)^2 \left(1 + \frac{G}{2c^2} \cdot \frac{\beta}{r}\right)^2 \delta$ with $\alpha, \beta > 0$.
- Reissner-Nordström initial data; charged point particle.




- $m = (\alpha + \beta)/2$;
- If $\alpha\beta = 0$ we have an asymptotically cylindrical end instead.

# General picture: $n$ charged point particles

- Assuming $\alpha_i, \beta_i > 0$ for all $i$ we have $n+1$ asymptotically Euclidean ends.




- $|x| \to \infty$ asymptotic end has ADM mass of

$$m = \frac{1}{2}\sum(\alpha_i + \beta_i).$$

- Rough idea: If $m = \frac{1}{2}\sum(\alpha_i + \beta_i) \to 0$ then $\chi, \psi \to 1$ and $(M, g) \to (\mathbb{R}^3, \delta)$.

# Almost rigidity of PMT in the geometrostatic context

## Theorem

*(Sormani - I.S, 2015/16) Let* $(M_k, g_k)$ *be a sequence of geometrostatic (Brill-Lindquist) manifolds with point particles* $P_k = \{p_1, p_2, ..., p_{n_k}\}$, *and let* $M'_k$ *denote the exterior portions*[1] *of* $M_k$. *Assume that* $0 \notin P_k$ *for all* $k$ *and that there is some* $R_0 > 0$ *such that* $P_k \subseteq B_\delta(0, R_0)$ *for all* $k$. *Let*

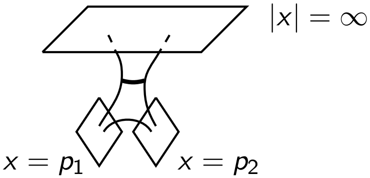
$$m_k = m_{\mathrm{ADM}}(M'_k, g_k), \quad \sigma_k = \min\{|p - p'|, |p| \;\mid p, p' \in P_k\}.$$

*If* $m_k \to 0$ *and* $m_k/\sigma_k \to 0$ *then for all* $R > R_0$ $B_{g_k}(0, R) \subseteq M'_k$ *converges to* $B_\delta(0, R) \subseteq \mathbb{R}^3$ *in the intrinsic flat sense.*

[1] Exterior portion refers to the portion of $M_k$ located outside of the outermost minimal surface(s).

# Comments on the proof

- Smallness of $m_k$ and $m_k/\sigma_k$ is used to *prevent* this scenario:




- We control the location of the outermost minimal surfaces by contrasting the Penrose Inequality with quadratic area growth along minimal surfaces.
- The rest of the proof is a consequence of the estimate of Lakzian and Sormani. (Similar to what will be shown later in the talk.)

# Simplification for the rest of today's talk

- $E = 0$, $\chi = \psi$, $R(g) = 0$ and
$$g = \left(1 + \frac{G}{2c^2}\sum_{i=1}^{n}\frac{a_i}{|x - p_i|}\right)^4 \delta, \quad a_i > 0.$$
- Distinguish "bare mass" $m_i$ from "effective mass" $a_i$.
- $x = p_i$ asymptotic end has ADM mass of
$$m_i = a_i + \frac{G}{2c^2}\sum_{j \neq i}\frac{a_i a_j}{|p_i - p_j|}.$$
- $m \neq \sum_i m_i$. Instead, we have interaction energy
$$mc^2 - \sum_i m_i c^2 \approx -G\sum_{i<j}\frac{m_i m_j}{|p_i - p_j|} + O\left(\frac{1}{|p_i - p_j|^2}\right).$$

# Discretizing a continuous distribution of "stuff"

- Consider a function $A(x)$ supported in a box $V$; model for a continuous distribution of "stuff".
- Subdivide the box $V$ into little boxes of size $\frac{1}{n}$; place an individual particle of appropriate "size" / "mass"

$$a_i = A(p_i)\frac{1}{n^3}$$

into the center $p_i$ of each subdivision.

- Consider the corresponding Brill-Lindquist (vacuum) metric:

$$\left(1 + \frac{G}{2c^2}\sum \frac{a_i}{|x - p_i|}\right)^4 \delta.$$

- Investigate what happens in the limit as $n \to \infty$. Dust?

# Dust?

- As $n \to \infty$:

$$1 + \frac{G}{2c^2} \sum \frac{a_i}{|x - p_i|} \to \underbrace{1 + \frac{G}{2c^2} \int_P \frac{A(p)}{|x - p|} \, \mathrm{dvol}_{\mathbb{R}^3}}_{\theta(x)}$$

- Naively:

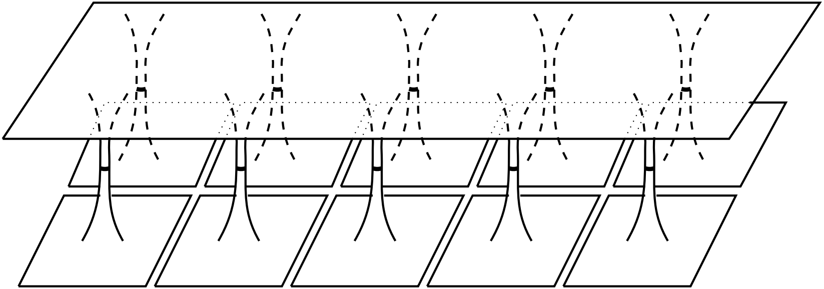
$$\left(1 + \frac{G}{2c^2} \sum \frac{a_i}{|x - p_i|}\right)^4 \delta \longrightarrow \theta(x)^4 \delta.$$

- The metric $g_A(x) = \theta(x)^4 \delta$ satisfies

$$R(g_A) = \frac{16\pi G}{c^2} \underbrace{A\theta^{-5}}_{\varrho} \quad \ldots \quad \text{Dust?}$$

# Why naive?

$$g_n := \left(1 + \frac{G}{2c^2} \sum \frac{a_i}{|x - p_i|}\right)^4 \delta \text{ looks something like so:}$$

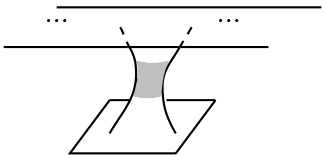


# Locating the "canonical" minimal surfaces

## Theorem

*(I.S, 2015/6) Let* $A(x) \geq 0$ *be a smooth function, compactly supported in a box* $V$*. There exists a constant* $C$ *and a natural number* $n_0$ *such that for all* $n \geq n_0$ *and all center points* $q$ *of a* $(1/n)$*-box in the subdivision of* $V$ *with* $A(q) \neq 0$ *the metric* $g_n$ *has a minimal surface in the region*

$$\left(\frac{G}{2c^2}\theta(q)^{-1} - \frac{C}{n}\right) \cdot \frac{A(q)}{n^3} \leq |x - q| \leq \left(\frac{G}{2c^2}\theta(q)^{-1} + \frac{C}{n}\right) \cdot \frac{A(q)}{n^3}.$$



$$r \approx \frac{G}{2c^2}\theta(q)^{-1}\frac{A(q)}{n^3}$$

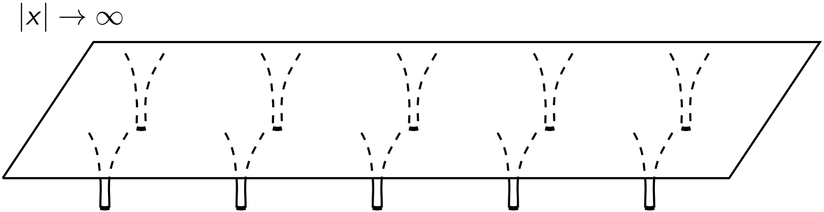
# Regarding the proof

- Zoom in / blow things up at $q$; do so at the rate of $A(q)/n^3$.
- If $n \gg 1$ (uniformly in $q$) the blown up metric is approximately equal to the Schwarzschild metric:

$$\left(\theta(q) + \frac{G}{2c^2} \cdot \frac{1}{|u|}\right)^4 \delta = \left(1 + \frac{G}{2c^2} \cdot \frac{\theta(q)^{-1}}{|u|}\right)^4 \theta(q)^4 \delta,$$

- Have precise information about fall-off rates.
- Do an Implicit-Function-Theorem-type-argument to see that the blown up metric has a minimal surface at about $|u| = \frac{G}{2c^2}\theta(q)^{-1}$; then rescale back.

# Cutting at "canonical" minimal surfaces: $(M_n, g_n)$




- The total Euclidean volume of "cut-outs" is on the order of $n^3 \cdot (\frac{1}{n^3})^3 \sim \frac{1}{n^6}$; serves as a hint that some kind of limit of $(M_n, g_n)$ as $n \to \infty$ is possible.
- Warning: the above need not be outermost minimal from the standpoint of $|x| \to \infty$.

# Our theorem

## Theorem

*(C. Sormani - I.S, 2016) If R is sufficiently large then* $M_{1,n} = B_\delta(p_0, R) \cap M_n$ *equipped with the metric*

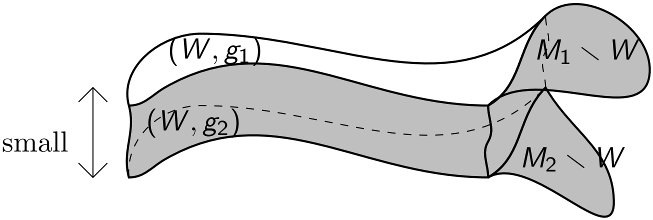
$$g_n = \left(1 + \frac{G}{2c^2} \sum \frac{a_i}{|x - p_i|}\right)^4 \delta$$

*converges in the intrinsic flat sense to* $M_2 = B_\delta(p_0, R)$ *equipped with*

$$g_A = \left(1 + \frac{G}{2c^2} \int_p \frac{A(p)}{|x - p|}\right)^4 \delta.$$

# Intrinsic flat distance: brief reminder

$(M_1, g_1)$ and $(M_2, g_2)$ with $g_1 \approx g_2$ over a subset $W \subset M_1 \cap M_2$.

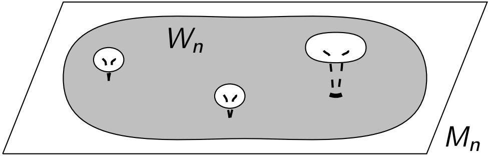



Lakzian-Sormani estimate on $d_{\mathcal{F}}(M_1, M_2)$:

$$\mathrm{Vol}_{g_1}(M_1 \setminus W) + \mathrm{Vol}_{g_2}(M_2 \setminus W)$$
$$+\text{“small”}\left(\mathrm{Vol}_{g_1}(W) + \mathrm{Vol}_{g_2}(W) + \mathrm{Vol}_{g_1}(\partial W) + \mathrm{Vol}_{g_2}(\partial W)\right).$$

# About the proof; part 1

- To apply Lakzian-Sormani we need $W_n$ on which $g_n \approx g_A$;




- $W_n = B_\delta(0, R) \setminus \left(\cup_i B_\delta(p_i, A(p_i)\frac{1}{n^2})\right)$;
- Recall that the "canonical" minimal surface is located more-or-less on the order of $A(p_i)\frac{1}{n^3}$;
- $\|g_n - g_A\|_{L^\infty(W_n)} = O(\frac{1}{n})$.

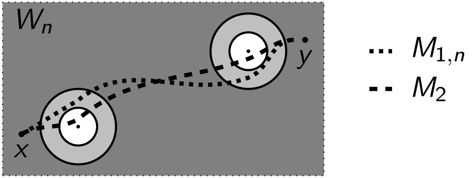
# About the proof; part 2

- $g_n$ and $g_A$ are bounded by a uniform multiple of $\delta$ on $M_{1,n} \subseteq M_2$;
- By Lakzian-Sormani $d_{\mathcal{F}}(M_{1,n}, M_2)$ is controlled by

$$\mathrm{Vol}_\delta(M_2 \setminus W_n) + \underbrace{(\sqrt{\lambda_n R} + \tfrac{R}{\sqrt{n}})}_{\text{"small"}} \Big(\mathrm{Vol}_\delta(W_n) + \mathrm{Vol}_\delta(\partial W_n)\Big)$$

- Need:
  - Smallness of $\mathrm{Vol}_\delta(M_2 \setminus W_n)$;
  - Smallness of $\lambda_n$;
  - Boundedness of $\mathrm{Vol}_\delta(W_n) + \mathrm{Vol}_\delta(\partial W_n)$.

# About the proof; part 3




- $\lambda_n = \sup_{x,y \in W_n} |d_{M_{1,n}}(x,y) - d_{M_2}(x,y)|$
- Estimates:
  - $\mathrm{Vol}_\delta(M_2 \setminus W_n) = O(n^3(\frac{1}{n^2})^3) = O(\frac{1}{n^3})$;
  - $\lambda_n = O(\frac{R}{n})$ by a direct brute force computation;
  - $\mathrm{Vol}_\delta(W_n) = O(R^3)$;
  - $\mathrm{Vol}_\delta(\partial W_n) = O(R^2) + O(n^3(\frac{1}{n^2})^2) = O(R^2)$.

# Is "every" dust a limit of Brill-Lindquist data? (Pt 1)

- Is every compactly supported, conformally flat, asymptotically Euclidean, time-symmetric, dust initial data a limit of Brill-Lindquist data?
- Yes: Suppose $g = \theta^4\delta$ with compactly supported $R(g) = \frac{16\pi G}{c^2}\varrho \geq 0$. Our construction for

$$A = \varrho\theta^5$$

recovers this particular $g$.

  - The relationship $A = \rho\theta^5$ comes from combining $\theta = 1 + \frac{G}{2c^2}\int_y \frac{A(y)}{|x-y|}\,\mathrm{dvol}_\delta$ and $R(g) = -8\theta^{-5}\Delta_\delta\theta = \dfrac{16\pi G}{c^2}\varrho$.

# Interaction

- In some sense both $A\,\mathrm{dvol}_\delta$ and $\varrho\,\mathrm{dvol}_g$ communicate density.
- Expression $\theta = 1 + \frac{G}{2c^2}\int_y \frac{A(y)}{|x-y|}\,\mathrm{dvol}_\delta$ suggests that $A$ is density with respect to the Euclidean metric. Naively one might expect $A = \varrho\theta^6$, and not $A = \varrho\theta^5$. Discrepancy is due to interaction energy:
  - Here $A\,\mathrm{dvol}_\delta = \varrho\theta^{-1}\mathrm{dvol}_g$ corresponds to "effective mass density".
  - This is to be distinguished from $\varrho\,\mathrm{dvol}_g$ which corresponds to "bare mass density".
  - The expression $\int \varrho\theta^{-1}\mathrm{dvol}_g - \int \varrho\,\mathrm{dvol}_g$ is the continuous version of Brill-Lindquist formula for interaction energy.

# Is "every" dust a limit of Brill-Lindquist data? (Pt 2)

- What if somebody just prescribes a compactly supported continuous distribution of "dust particles" on $\mathbb{R}^3$? Is it realizable (as a limit of Brill-Lindquist data)?
- Not a well phrased question: everything depends on whether you are prescribing dust using metric-dependent or metric-independent quantities.
  - Metric-dependent approach: Prescribe a scalar (density) function $\varrho$; the constraint equation states $R(g) = \frac{16\pi G}{c^2}\varrho$.
  - Metric-independent approach: Prescribe a 3-form $\omega$. The constraint equation states $R(g)\,\mathrm{dvol}_g = \frac{16\pi G}{c^2}\omega$.

# Is "every" dust a limit of Brill-Lindquist data? (Pt 3)

These questions reduce to the questions of solvability of ....

- Metric-dependent approach: $\Delta_\delta \theta = -4\pi \frac{G}{2c^2} \varrho \theta^5$ with $\theta \to 1$. This problem does not have solutions when $\varrho$ is large enough. (E.g. when $\frac{G}{2c^2} \int_y \frac{\varrho(y)}{|x-y|} \mathrm{dvol}_\delta \geq 1$.)
- Metric-independent approach: $\Delta_\delta \theta = -4\pi \frac{G}{2c^2} \omega_0 \theta^{-1}$ with $\theta \to 1$ and $\omega = \omega_0 \mathrm{dvol}_\delta$. Here the exponent of $-1$ works in our favor!

# Is "every" dust a limit of Brill-Lindquist data? (Pt 4)

## Corollary

*Let $\omega$ be a compactly supported 3-form on $\mathrm{R}^3$. Then there is a unique conformally flat, asymptotically Euclidean, time-symmetric initial data $g_\omega$ with $R(g_\omega)\,\mathrm{dvol}_{g_\omega} = \frac{16\pi G}{c^2}\omega$. Furthermore, $g_\omega$ arises as a pointed intrinsic flat limit of Brill-Lindquist data.*

So roughly speaking:

- One can prescribe as much "stuff" on $\mathbb{R}^3$ as one might like. However, the conformal factor will spread things apart, increase volume and make the density of "stuff" relatively low.
- Thanks to David Maxwell for pointing us in the direction of using $\omega$ instead of $\varrho$.

# Thank you for your attention!