Mirror symmetry for elementary birational cobordisms

Gabriel Kerr

Homological Mirror Geometry, BIRS
March 10, 2016

Outline

Motivation

B-model

A-model

Sketch of the proof

Outline

Motivation

B-model

A-model

Sketch of the proof

Birational approach to HMS

Let X be a Fano DM stack and $W : X^{mir} \to \mathbb{C}$ its mirror potential.

Birational approach to HMS

Let X be a Fano DM stack and $W : X^{mir} \to \mathbb{C}$ its mirror potential.

$\frac{\text{HMS Conjecture}}{2}$: There is an equivalence

$$D(X) \cong FS(W).$$

Birational approach to HMS

Let X be a Fano DM stack and $W : X^{mir} \to \mathbb{C}$ its mirror potential.

$\frac{\text{HMS Conjecture}}{2}$: There is an equivalence

$$D(X) \cong FS(W).$$

Question 1 : What are natural decompositions for either side of this equivalence?

Birational approach to HMS

Let X be a Fano DM stack and $W : X^{mir} \to \mathbb{C}$ its mirror potential.

$\frac{\text{HMS Conjecture}}{2}$: There is an equivalence

$$D(X) \cong FS(W).$$

Question 1 : What are natural decompositions for either side of this equivalence?

Question 2 : Does HMS respect these decompositions?

Birational Approach

Question 1 : What are natural decompositions for either side of this equivalence?

Birational Approach

Question 1 : What are natural decompositions for either side of this equivalence?

B-model answer : If X admits a minimal model or VGIT sequence

$$X \overset{f_1}{\dashrightarrow} X_1 \overset{f_2}{\dashrightarrow} \cdots \overset{f_r}{\dashrightarrow} X_r$$

then

$$D(X) = \left\langle \mathcal{T}^B_{f_1}, \ldots, \mathcal{T}^B_{f_r}, D(X_r) \right\rangle .$$

If $X_r = \emptyset$, then this fully decomposes $D(X)$ in terms of wall contribution categories.

Birational Approach

Question 1 : What are natural decompositions for either side of this equivalence?

A-model answer : Take the following steps

Birational Approach

Question 1 : What are natural decompositions for either side of this equivalence?

A-model answer : Take the following steps

Step 1 : Replace $W : X^{mir} \to \mathbb{C}$ with $\mathbf{w}$ where

$$\begin{array}{ccc} X^{mir} & \longrightarrow & \mathcal{U} \\ \downarrow W & & \downarrow \pi \\ \mathbb{C} & \xrightarrow{\mathbf{w}} & \mathcal{M} \end{array}$$

and $\pi : \mathcal{U} \to \mathcal{M}$ is a moduli stack of hypersurfaces of X^{mir}.

Birational Approach

Question 1 : What are natural decompositions for either side of this equivalence?

A-model answer : Take the following steps

Step 2 : Compactify the range $\mathbb{C}$ of the superpotential W where

$$\begin{array}{ccc} \bar{X}^{mir} & \longrightarrow & \bar{\mathcal{U}} \\ {\scriptstyle W}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathbb{P}^1 & \xrightarrow{\mathbf{w}} & \bar{\mathcal{M}} \end{array}$$

and $\pi : \bar{\mathcal{U}} \to \bar{\mathcal{M}}$ is a compactified moduli stack of stable hypersurface degenerations of $\bar{X}^{mir}$.

Birational Approach

Question 1 : What are natural decompositions for either side of this equivalence?

A-model answer : Take the following steps

Step 3 : Consider a 1-parameter degeneration $\mathbf{w}_t$ of $\mathbf{w} = \mathbf{w}_1$

$$\begin{array}{ccc} \tilde{X}^{mir}_{\mathbf{w}_-} & \longrightarrow & \bar{\mathcal{U}} \\ {\scriptstyle W}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathbb{P}^1 \times D^* & \xrightarrow{\mathbf{w}_t} & \bar{\mathcal{M}} \end{array}$$

where D is the unit disc and $D^* = D\backslash\{0\}$.

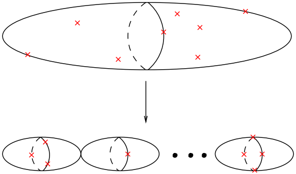
Birational Approach

Question 1 : What are natural decompositions for either side of this equivalence?

A-model answer : Take the following steps

Step 4 : Consider the degenerate potential $\mathbf{w}_0$ of $\mathbf{w}_t$ which is a map

$$\mathbf{w}_0 = \cup \mathbf{w}_0^i : \cup_1^r \mathbb{P}^1 \to \bar{\mathcal{M}}.$$



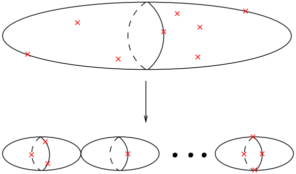
Birational Approach

Question 1 : What are natural decompositions for either side of this equivalence?

A-model answer : Take the following steps

Step 5 : Taking W_i to be the pullback of $\mathcal{U}$ along $\mathbf{w}_0^i$ and $\mathcal{T}_{\mathbf{w}^i}^A = FS(W_i)$, there is a semi-orthogonal decomposition

$$FS(W) = \left\langle \mathcal{T}_{\mathbf{w}^1}^A, \ldots, \mathcal{T}_{\mathbf{w}^r}^A \right\rangle .$$



Birational Approach

Question 2 : Does HMS respect these decompositions?

Birational Approach

Question 2 : Does HMS respect these decompositions?

Toric Setting : Diemer, Katzarkov, K. (DKK) give an explicit prescription for taking a VGIT sequence

$$X \overset{f_1}{\dashrightarrow} X_1 \overset{f_2}{\dashrightarrow} \cdots \overset{f_r}{\dashrightarrow} \emptyset$$

to a degeneration $\mathbf{w}_t$ of mirror potentials.

Birational Approach

Question 2 : Does HMS respect these decompositions?

Toric Setting The resulting decompositions

$$D(X) = \left\langle \mathcal{T}^B_{f_1}, \ldots, \mathcal{T}^B_{f_r} \right\rangle,$$

$$FS(W) = \left\langle \mathcal{T}^A_{\mathbf{w}^1}, \ldots, \mathcal{T}^A_{\mathbf{w}^r} \right\rangle$$

have the same number of components.

Theorem (DKK) : For mirror decompositions,

$$\mathrm{rk}(K_0(\mathcal{T}^B_{f_i})) = \mathrm{rk}(K_0(\mathcal{T}^A_{\mathbf{w}^i})).$$

Birational Approach

Question 2 : Does HMS respect these decompositions?

Toric Setting The resulting decompositions

$$D(X) = \left\langle \mathcal{T}^B_{f_1}, \ldots, \mathcal{T}^B_{f_r} \right\rangle,$$

$$FS(W) = \left\langle \mathcal{T}^A_{\mathbf{w}^1}, \ldots, \mathcal{T}^A_{\mathbf{w}^r} \right\rangle$$

have the same number of components.

Theorem (DKK) : For mirror decompositions,

$$\mathrm{rk}(K_0(\mathcal{T}^B_{f_i})) = \mathrm{rk}(K_0(\mathcal{T}^A_{\mathbf{w}^i})).$$

Birational Approach

Question 2 : Does HMS respect these decompositions?

Toric Setting The resulting decompositions

$$D(X) = \left\langle \mathcal{T}^B_{f_1}, \dots, \mathcal{T}^B_{f_r} \right\rangle,$$

$$FS(W) = \left\langle \mathcal{T}^A_{\mathbf{w}^1}, \dots, \mathcal{T}^A_{\mathbf{w}^r} \right\rangle$$

have the same number of components.

Theorem (DKK) : For mirror decompositions,

$$\mathrm{rk}(K_0(\mathcal{T}^B_{f_i})) = \mathrm{rk}(K_0(\mathcal{T}^A_{\mathbf{w}^i})).$$

Conjecture : There exists an equivalence of categories

$$\mathcal{T}^B_{f_i} \cong \mathcal{T}^A_{\mathbf{w}^i}.$$

Outline

Motivation

B-model

A-model

Sketch of the proof

Elementary birational cobordisms

- Let $(a_0, \ldots, a_d) \in \mathbb{Z}^{d+1}$ with $a_i \neq 0$ for all i.

Elementary birational cobordisms

- Let $(a_0, \ldots, a_d) \in \mathbb{Z}^{d+1}$ with $a_i \neq 0$ for all i.
- Take $X = \mathbb{C}^{d+1}$ and $\mathbb{C}^*$ acting via $\lambda \cdot (z_0, \ldots, z_d) = (\lambda^{a_0} z_0, \ldots, \lambda^{a_d} z_d)$.

Elementary birational cobordisms

- Let $(a_0, \ldots, a_d) \in \mathbb{Z}^{d+1}$ with $a_i \neq 0$ for all i.
- Take $X = \mathbb{C}^{d+1}$ and $\mathbb{C}^*$ acting via $\lambda \cdot (z_0, \ldots, z_d) = (\lambda^{a_0} z_0, \ldots, \lambda^{a_d} z_d)$.
- The unstable loci for the GIT quotients

$$B_- = \{(z_0, \ldots, z_d) : z_i = 0 \text{ for } a_i < 0\},$$
$$B_+ = \{(z_0, \ldots, z_d) : z_i = 0 \text{ for } a_i > 0\}.$$

Elementary birational cobordisms

- Let $(a_0, \ldots, a_d) \in \mathbb{Z}^{d+1}$ with $a_i \neq 0$ for all i.
- Take $X = \mathbb{C}^{d+1}$ and $\mathbb{C}^*$ acting via $\lambda \cdot (z_0, \ldots, z_d) = (\lambda^{a_0} z_0, \ldots, \lambda^{a_d} z_d)$.
- The unstable loci for the GIT quotients

$$\begin{aligned} B_- &= \{(z_0, \ldots, z_d) : z_i = 0 \text{ for } a_i < 0\}, \\ B_+ &= \{(z_0, \ldots, z_d) : z_i = 0 \text{ for } a_i > 0\}. \end{aligned}$$

- Let $X_\pm = X \backslash B_\pm$ and

$$X_+/\mathbb{C}^* \dashrightarrow X_-/\mathbb{C}^*$$

the associated VGIT.

Elementary birational cobordisms

- Let $a_{d+1} = -\sum_{i=0}^{d} a_i$ and $\mathbf{a} = (a_0, \ldots, a_{d+1}) \in \mathbb{Z}^{d+2}$.

Elementary birational cobordisms

- Let $a_{d+1} = -\sum_{i=0}^{d} a_i$ and $\mathbf{a} = (a_0, \ldots, a_{d+1}) \in \mathbb{Z}^{d+2}$.

Theorem (BFK, HL)

If $a_{d+1} < 0$ *then*

$$D(X_+/\mathbb{C}^*) \cong \left\langle \mathcal{T}_{\mathbf{a}}^{B}, D(X_-/\mathbb{C}^*) \right\rangle$$

where $\mathcal{T}_{\mathbf{a}}^{B}$ *admits a complete exceptional collection*

$$\left\langle E_0, \ldots, E_{-a_{d+1}-1} \right\rangle$$

Elementary birational cobordisms

- Let $a_{d+1} = -\sum_{i=0}^{d} a_i$ and $\mathbf{a} = (a_0, \ldots, a_{d+1}) \in \mathbb{Z}^{d+2}$.

Theorem (BFK, HL)

If $a_{d+1} < 0$ then

$$D(X_+/\mathbb{C}^*) \cong \left\langle \mathcal{T}_{\mathbf{a}}^{B}, D(X_-/\mathbb{C}^*) \right\rangle$$

where $\mathcal{T}_{\mathbf{a}}^{B}$ admits a complete exceptional collection

$$\left\langle E_0, \ldots, E_{-a_{d+1}-1} \right\rangle$$

- *Furthermore, there is a full and faithful embedding*

$$F : \mathcal{T}_{\mathbf{a}}^{B} \to D^{eq}(X)$$

for which $F(E_i) = \mathcal{O}_{B_-}(i)$.

Computing $\mathcal{T}_{\mathbf{a}}^B$

- Want to compute the dg-algebra

$$End^*_{Coh^{eq}(X)}(\oplus_{i=0}^{-a_{d+1}-1}\mathcal{O}_{B_-}(i))$$

in the dg category $Coh^{eq}(X)$.

Computing $\mathcal{T}_{\mathbf{a}}^B$

- Want to compute the dg-algebra

$$End^*_{Coh^{eq}(X)}(\oplus_{i=0}^{-a_{d+1}-1}\mathcal{O}_{B_-}(i))$$

in the dg category $Coh^{eq}(X)$.

- Given an equivariant sheaf $\mathcal{F}$,

$$\mathrm{Hom}^*_{Coh^{eq}(X)}(\mathcal{F}(i),\mathcal{F}(j)) \cong \mathrm{End}^*_{Coh(X)}(\mathcal{F})_{(j-i)}$$

and these isomorphisms are compatible with composition.

Computing $\mathcal{T}_{\mathbf{a}}^{B}$

- Want to compute the dg-algebra

$$End^{*}_{Coh^{eq}(X)}(\oplus_{i=0}^{-a_{d+1}-1}\mathcal{O}_{B_-}(i))$$

  in the dg category $Coh^{eq}(X)$.

- Given an equivariant sheaf $\mathcal{F}$,

$$\mathrm{Hom}^{*}_{Coh^{eq}(X)}(\mathcal{F}(i),\mathcal{F}(j)) \cong \mathrm{End}^{*}_{Coh(X)}(\mathcal{F})_{(j-i)}$$

  and these isomorphisms are compatible with composition.

- An elementary computation shows there are quasi-isomorphisms

$$\begin{aligned}\mathrm{End}^{*}_{Coh(X)}(\mathcal{O}_{B_-}) &\cong \mathrm{Ext}^{*}_{Coh(X)}(\mathcal{O}_{B_-},\mathcal{O}_{B_-}),\\ &\cong \Omega^{*}_{X/B_-},\\ &\cong \mathrm{Sym}^{*}(V_0 \oplus V_1).\end{aligned}$$

Computing $\mathcal{T}_{\mathbf{a}}^{B}$

$$\mathrm{End}^*_{Coh(X)}(\mathcal{O}_{B_-}) \cong \mathrm{Sym}^*(V_0 \oplus V_1).$$

- Here, we take

$$V_0 = \mathbb{C}\{z_i : a_i > 0\},$$
$$V_1 = \mathbb{C}\{dz_i : a_i < 0\}[1].$$

So that $R_{\mathbf{a}} := \mathrm{Sym}^*(V_0 \oplus V_1)$ is a weighted, super-symmetric algebra with $\mathrm{wt}(z_i) = a_i$ and $\mathrm{wt}(dz_i) = -a_i$.

Computing $\mathcal{T}_{\mathbf{a}}^{B}$

$$\mathrm{End}^*_{Coh(X)}(\mathcal{O}_{B_-}) \cong \mathrm{Sym}^*(V_0 \oplus V_1).$$

- Here, we take

$$V_0 = \mathbb{C}\{z_i : a_i > 0\},$$
$$V_1 = \mathbb{C}\{dz_i : a_i < 0\}[1].$$

  So that $R_{\mathbf{a}} := \mathrm{Sym}^*(V_0 \oplus V_1)$ is a weighted, super-symmetric algebra with $\mathrm{wt}(z_i) = a_i$ and $\mathrm{wt}(dz_i) = -a_i$.
- Due to formality, and working in the category $R_{\mathbf{a}}$-mod$^{\mathbb{Z}}$, the Yoneda functor yields an equivalence of categories

$$\mathcal{T}_{\mathbf{a}}^{B} \cong D\left(\mathrm{End}^*\left(\bigoplus_{k=0}^{-a_{d+1}-1} R_{\mathbf{a}}(k)\right)\text{-mod}\right)$$

Examples

$$R_{\mathbf{a}} = \mathrm{Sym}^*(V_0 \oplus V_1) \cong \mathrm{End}^*_{Coh(X)}(\mathcal{O}_{B_-}).$$

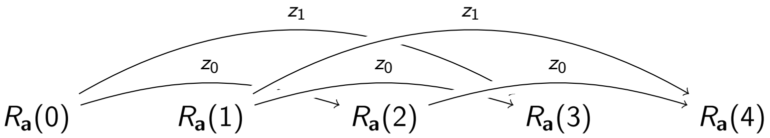
$\mathbf{a}$	$R_{\mathbf{a}}$	$X_+/\mathbb{C}^*$	$X_-/\mathbb{C}^*$
$(1, 1, -2)$	$\mathbb{C}[z_0, z_1, dz_2]$	$\mathbb{P}^1$	$\emptyset$

$$R_{\mathbf{a}}(0) \underset{z_0}{\overset{z_1}{\rightrightarrows}} R_{\mathbf{a}}(1)$$

Examples

$$R_{\mathbf{a}} = \mathrm{Sym}^*(V_0 \oplus V_1) \cong \mathrm{End}^*_{Coh(X)}(\mathcal{O}_{B_-}).$$

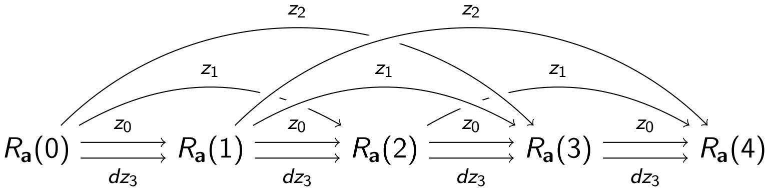
$\mathbf{a}$	$R_{\mathbf{a}}$	$X_+/\mathbb{C}^*$	$X_-/\mathbb{C}^*$
$(2, 3, -5)$	$\mathbb{C}[z_0, z_1, dz_2]$	$\mathbb{P}(2, 3)$	$\emptyset$



Examples

$$R_{\mathbf{a}} = \mathrm{Sym}^*(V_0 \oplus V_1) \cong \mathrm{End}^*_{Coh(X)}(\mathcal{O}_{B_-}).$$

$\mathbf{a}$	$R_{\mathbf{a}}$	$X_+/\mathbb{C}^*$	$X_-/\mathbb{C}^*$
$(1,2,3,-1,-5)$	$\mathbb{C}[z_0, z_1, z_2, dz_3, dz_4]$	$\mathcal{O}_{\mathbb{P}(1,2,3)}(-1)$	$\mathbb{C}^3$



Outline

Motivation

B-model

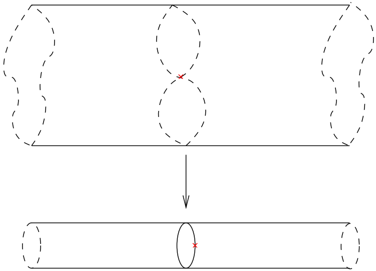
A-model

Sketch of the proof

Fukaya-Seidel for $W : Y \to \mathbb{C}^*$

Definition

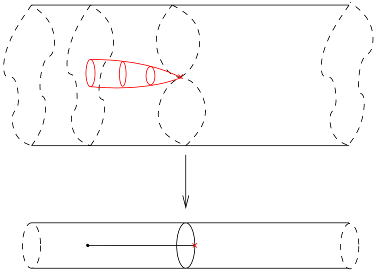
A symplectic Lefschetz fibration $W : Y \to \mathbb{C}^*$ is called atomic if it has a unique critical point p with critical value q.



Fukaya-Seidel for $W : Y \to \mathbb{C}^*$

Definition

A symplectic Lefschetz fibration $W : Y \to \mathbb{C}^*$ is called atomic if it has a unique critical point p with critical value q.

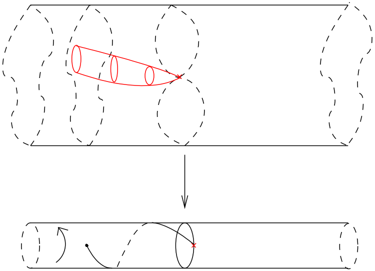


Choosing a basepoint $*$, the path δ_0 gives the vanishing thimble $T_0 \subset Y$ and the vanishing cycle $L_0 \subset W^{-1}(*)$.

Fukaya-Seidel for $W : Y \to \mathbb{C}^*$

Definition

A symplectic Lefschetz fibration $W : Y \to \mathbb{C}^*$ is called atomic if it has a unique critical point p with critical value q.

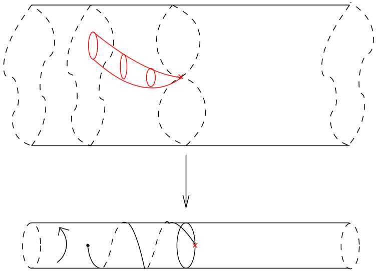


Applying monodromy gives paths $\delta_0, \delta_1, \ldots, \delta_{n-1}$ and a collection of vanishing cycles $L_0, L_1, \ldots, L_{n-1} \subset W^{-1}(*)$.

Fukaya-Seidel for $W : Y \to \mathbb{C}^*$

Definition

A symplectic Lefschetz fibration $W : Y \to \mathbb{C}^*$ is called atomic if it has a unique critical point p with critical value q.



Applying monodromy gives paths $\delta_0, \delta_1, \ldots, \delta_{n-1}$ and a collection of vanishing cycles $L_0, L_1, \ldots, L_{n-1} \subset W^{-1}(*)$.

Fukaya-Seidel for $W : Y \to \mathbb{C}^*$

Definition

Given an atomic Lefschetz fibration W, the n-unfolded category $\mathcal{A}^{1/n}$ of W is the directed A_∞-subcategory

$$\langle L_0, \dots, L_{n-1} \rangle$$

of the Fukaya category $\mathcal{F}(W^{-1}(*))$. The Fukaya-Seidel category $FS(W^{1/n})$ is the category of twisted complexes $\mathrm{Tw}(\mathcal{A}(W^{1/n}))$.

Fukaya-Seidel for $W : Y \to \mathbb{C}^*$

Definition

Given an atomic Lefschetz fibration W, the n-unfolded category $\mathcal{A}^{1/n}$ of W is the directed A_∞-subcategory

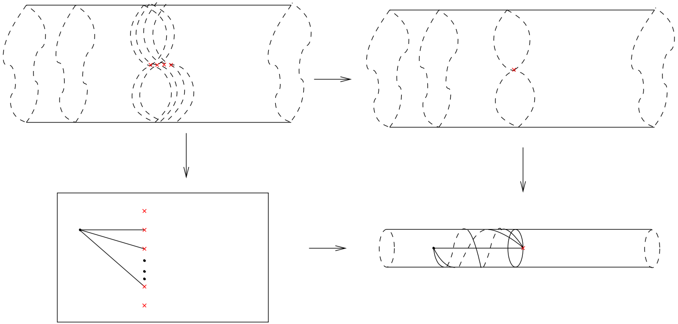
$$\langle L_0, \dots, L_{n-1} \rangle$$

of the Fukaya category $\mathcal{F}(W^{-1}(*))$. The Fukaya-Seidel category $FS(W^{1/n})$ is the category of twisted complexes $\mathrm{Tw}(\mathcal{A}(W^{1/n}))$.

Here

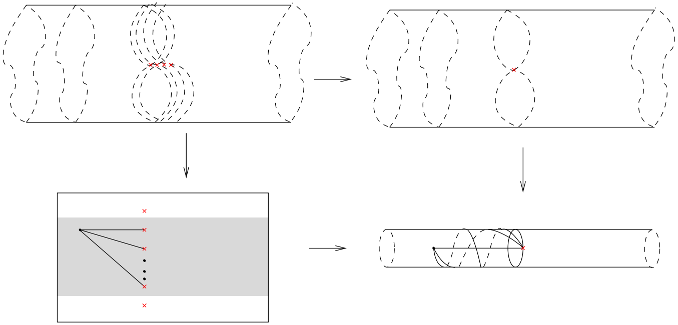
$$\mathrm{Hom}_{\mathcal{A}^{1/n}}(L_i, L_j) = \begin{cases} \mathrm{Hom}_{\mathcal{F}(W^{-1}(*))}(L_i, L_j) & \text{if } i < j, \\ 1_i & \text{if } i = j, \\ 0 & \text{if } i > j. \end{cases}$$

Fukaya-Seidel for $W : Y \to \mathbb{C}^*$



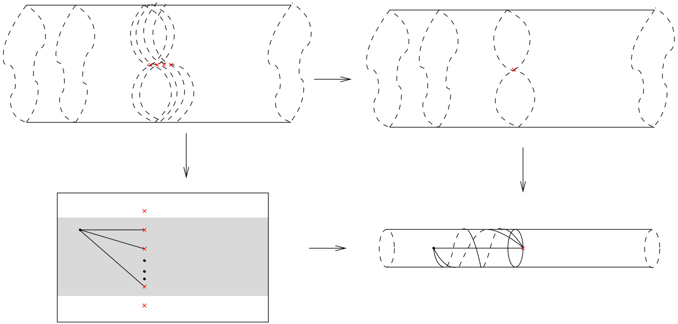
Alternatively, taking the pullback $\tilde{W} : \tilde{Y} \to \mathbb{C}$ of W along exp gives a periodic collection of critical values.

Fukaya-Seidel for $W : Y \to \mathbb{C}^*$



Restricting to a strip gives $\tilde{W}_S : \tilde{Y}_S \to S$.

Fukaya-Seidel for $W : Y \to \mathbb{C}^*$



Restricting to a strip gives $\tilde{W}_S : \tilde{Y}_S \to S$.

Theorem (Seidel)

There is an equivalence $FS(W^{1/n}) \cong FS(\tilde{W}_S)$.

Mirror potentials to elementary birational cobordisms

- Let

$$P_d = \left\{ [Z_0 : \cdots : Z_{d+1}] : \sum_{i=0}^{d+1} Z_i = 0, Z_i \neq 0 \right\} \subset \mathbb{P}^{d+1}$$

  be the d-dimensional pair of pants.

Mirror potentials to elementary birational cobordisms

- Let
$$P_d = \left\{ [Z_0 : \cdots : Z_{d+1}] : \sum_{i=0}^{d+1} Z_i = 0, Z_i \neq 0 \right\} \subset \mathbb{P}^{d+1}$$
be the d-dimensional pair of pants.
- Given $\mathbf{a} = (a_0, \ldots, a_{d+1}) \in \mathbb{Z}^{d+2}$ with $\sum_{i=0}^{d+1} a_i = 0$, consider the pencil $\psi_{\mathbf{a}} : \mathbb{P}^{d+1} \to \mathbb{P}^1$ defined by
$$\psi_{\mathbf{a}}([Z_0 : \cdots : Z_{d+1}]) := \left[\prod_{a_i > 0} Z_i^{a_i} : \prod_{a_i < 0} Z_i^{-a_i} \right].$$

Mirror potentials to elementary birational cobordisms

- Let

$$P_d = \left\{ [Z_0 : \cdots : Z_{d+1}] : \sum_{i=0}^{d+1} Z_i = 0, Z_i \neq 0 \right\} \subset \mathbb{P}^{d+1}$$

  be the d-dimensional pair of pants.
- Given $\mathbf{a} = (a_0, \ldots, a_{d+1}) \in \mathbb{Z}^{d+2}$ with $\sum_{i=0}^{d+1} a_i = 0$, consider the pencil $\psi_{\mathbf{a}} : \mathbb{P}^{d+1} \to \mathbb{P}^1$ defined by

$$\psi_{\mathbf{a}}([Z_0 : \cdots : Z_{d+1}]) := \left[\prod_{a_i > 0} Z_i^{a_i} : \prod_{a_i < 0} Z_i^{-a_i} \right].$$

Observation (GKZ, DKK)

The function $W_{\mathbf{a}} = \psi_{\mathbf{a}}|_{P_d}$ is an atomic Lefschetz fibration.

HMS for elementary birational cobordisms

The potential $W_{\mathbf{a}}$ appears in DKK as the equivariant quotient by $\mathbb{Z}/a_{d+1}$ of the homological mirror potential W to the VGIT defined by $(a_0, \ldots, a_d)$.

HMS for elementary birational cobordisms

The potential $W_{\mathbf{a}}$ appears in DKK as the equivariant quotient by $\mathbb{Z}/a_{d+1}$ of the homological mirror potential W to the VGIT defined by $(a_0, \ldots, a_d)$.

Theorem (K.)

For any $0 \leq n \leq \sum_{a_i > 0} a_i$ *there is a strict, fully faithful functor*

$$\Phi : \mathcal{A}^{1/n} \to R_{\mathbf{a}}\text{-mod}^{\mathbb{Z}}$$

for which

$$\Phi(L_k) = R_{\mathbf{a}}(k).$$

Two corollaries

Letting $\mathcal{T}_{\mathbf{a}}^{A} = FS(W^{1/-a_{d+1}})$ the theorem implies,

Corollary

There is an equivalence $\mathcal{T}_{\mathbf{a}}^{B} \cong \mathcal{T}_{\mathbf{a}}^{A}$.

Two corollaries

Letting $\mathcal{T}_{\mathbf{a}}^{A} = FS(W^{1/-a_{d+1}})$ the theorem implies,

Corollary

There is an equivalence $\mathcal{T}_{\mathbf{a}}^{B} \cong \mathcal{T}_{\mathbf{a}}^{A}$.

As a special case when $a_i > 0$ for all $i \neq d+1$,

Corollary

HMS holds for weighted projective spaces

$$D(\mathbb{P}(a_0, \ldots, a_d)) \cong FS(W_{\mathbf{a}}^{1/-a_{d+1}}) = FS(W_{\mathbf{a}}^{HV}).$$

Outline

Motivation

B-model

A-model

Sketch of the proof

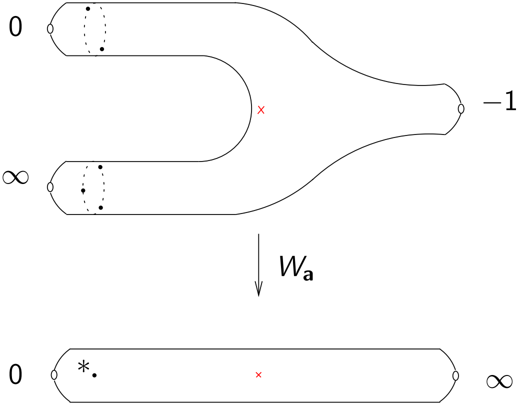
Base case $d = 1$

Assume $\mathbf{a} = (a_0, a_1, a_2) \in \mathbb{Z}^3$ satisfies $a_0, a_1 > 0$. Parameterize $P_1 \cong \mathbb{P}^1 \backslash \{0, -1, \infty\}$ with $[Z_0 : Z_1]$. Then

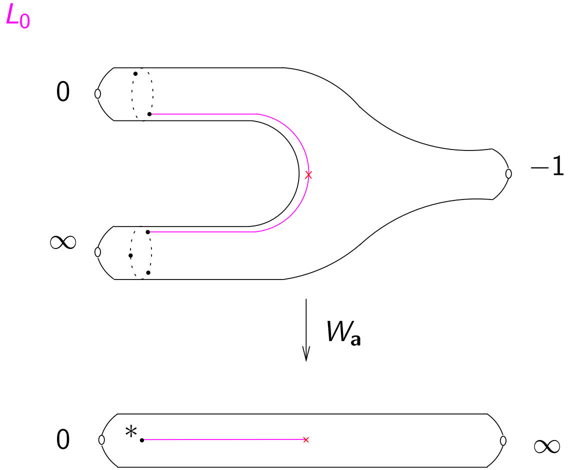
$$W_{\mathbf{a}}([Z_0 : Z_1]) = \frac{Z_0^{a_0} Z_1^{a_1}}{(-Z_0 - Z_1)^{a_0 + a_1}}$$

is an $(a_0 + a_1)$-fold branched covering with ramification degree a_0, a_1 and a_2 at $0, \infty$ and -1, respectively, and a single critical point at $[a_0 : a_1] \in P_1$. The admissible path δ_0 from $W_{\mathbf{a}}([a_0 : a_1])$ to zero has vanishing thimble equal to the component of $W^{-1}(\delta_0)$ containing $[a_0, a_1]$.

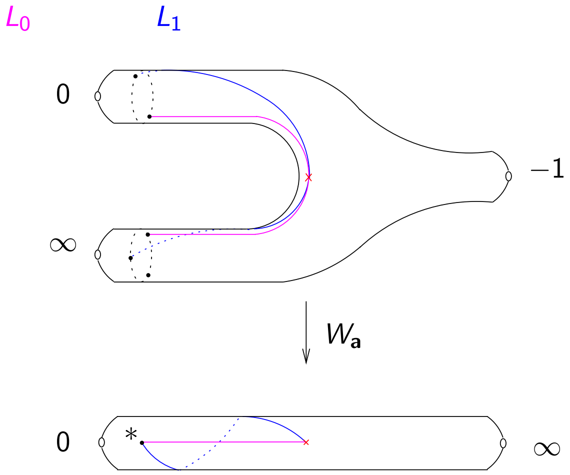
Example $\mathbf{a} = (2, 3, -5)$



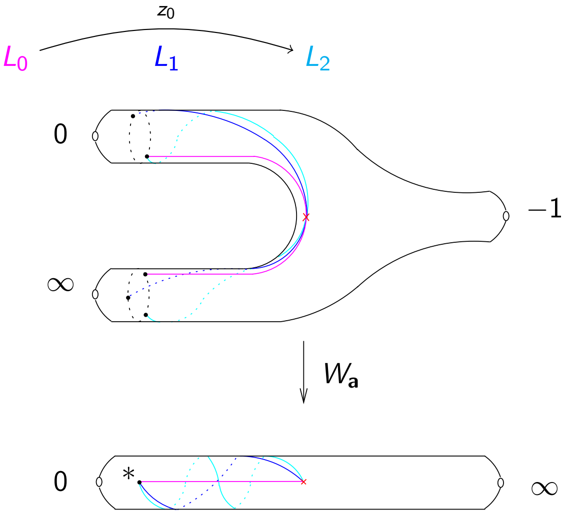
Example $\mathbf{a} = (2, 3, -5)$



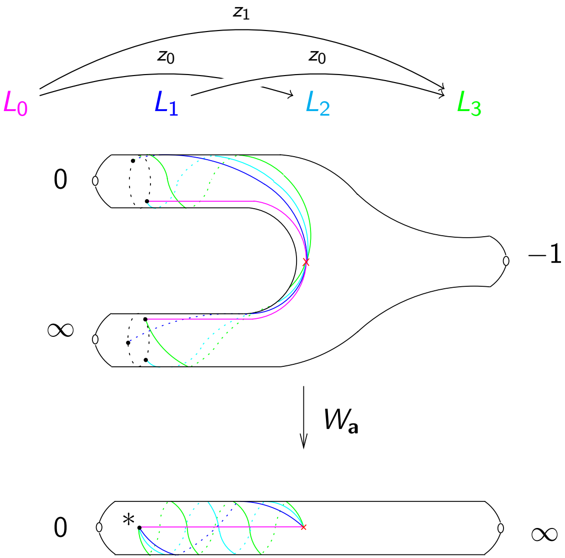
Example $\mathbf{a} = (2, 3, -5)$



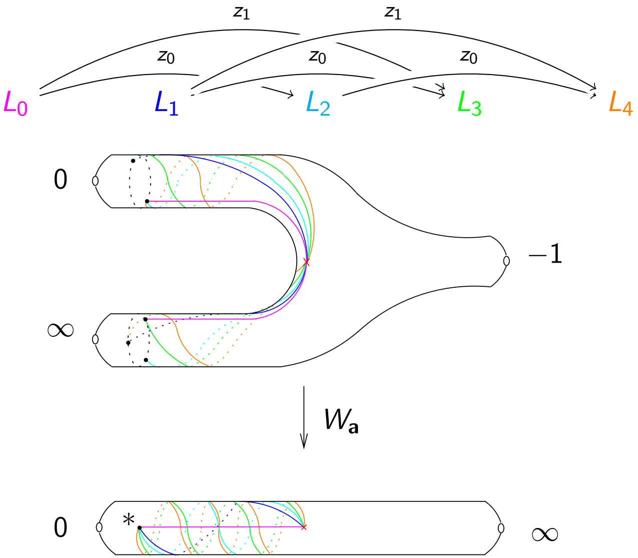
Example $\mathbf{a} = (2, 3, -5)$



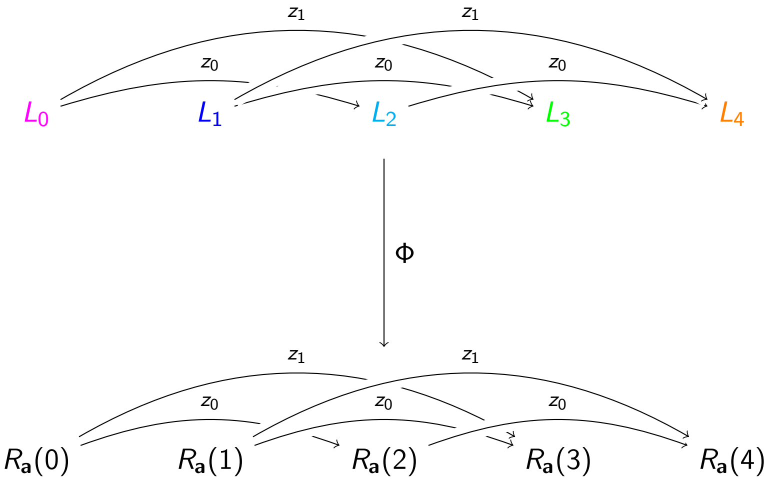
Example $\mathbf{a} = (2, 3, -5)$



Example $\mathbf{a} = (2, 3, -5)$

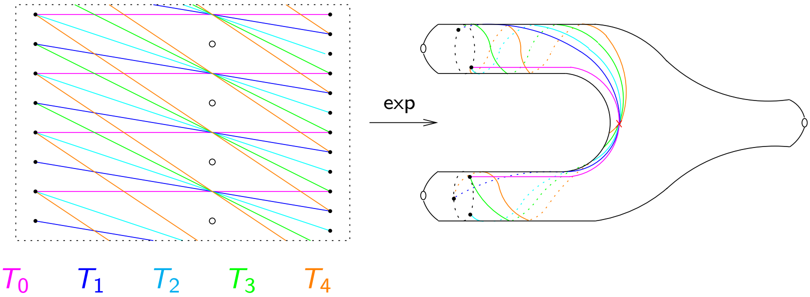


Example $\mathbf{a} = (2, 3, -5)$



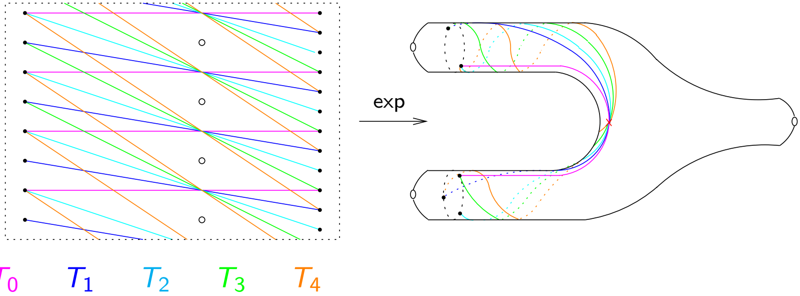
The logarithmic picture

To better understand the Floer theory of the vanishing thimbles T_i, consider the logarithm $\log : P_1 \to \mathbb{C} \backslash (\pi i + 2\pi i \mathbb{Z})$.



The logarithmic picture

To better understand the Floer theory of the vanishing thimbles T_i, consider the logarithm $\log : P_1 \to \mathbb{C} \backslash (\pi i + 2\pi i \mathbb{Z})$.



As the slope of T_i is decreasing relative to i, any holomorphic polygon with counter-clockwise boundary on the thimbles $\{T_{i_1}, \ldots, T_{i_m}\}$ with $i_j < i_{j+1}$ must be a triangle.

Induction Step

- Assume the theorem holds for $\dim < d$ and let $\mathbf{a} = (a_0, \ldots, a_{d+1})$. We may assume that $a_0, a_1 > 0$ (or apply Koszul duality).

Induction Step

- Assume the theorem holds for dim $< d$ and let $\mathbf{a} = (a_0, \ldots, a_{d+1})$. We may assume that $a_0, a_1 > 0$ (or apply Koszul duality).
- In order to understand the Floer theory for $\mathrm{Hom}(L_i, L_j)$, we must understand the geometry of $W_{\mathbf{a}}^{-1}(*)$.

Induction Step

- Assume the theorem holds for dim $< d$ and let $\mathbf{a} = (a_0, \ldots, a_{d+1})$. We may assume that $a_0, a_1 > 0$ (or apply Koszul duality).
- In order to understand the Floer theory for $\mathrm{Hom}(L_i, L_j)$, we must understand the geometry of $W_{\mathbf{a}}^{-1}(*)$.
- We accomplish this by choosing an auxiliary Lefschetz fibration

$$f([Z_0 : \cdots : Z_{d+1}]) = [Z_0 : Z_1 : -Z_0 - Z_1]$$

Induction Step

- Assume the theorem holds for dim $< d$ and let $\mathbf{a} = (a_0, \ldots, a_{d+1})$. We may assume that $a_0, a_1 > 0$ (or apply Koszul duality).
- In order to understand the Floer theory for $\mathrm{Hom}(L_i, L_j)$, we must understand the geometry of $W_{\mathbf{a}}^{-1}(*)$.
- We accomplish this by choosing an auxiliary Lefschetz fibration

$$f([Z_0 : \cdots : Z_{d+1}]) = [Z_0 : Z_1 : -Z_0 - Z_1]$$

- Taking $D = \{Z_0 + Z_1 = 0\} \subset P_d$ and $F_t = W_{\mathbf{a}}^{-1}(t) \backslash D$, f restricts to a Lefschetz fibration

$$f : F_t \to P_1$$

for all $W_{\mathbf{a}}$ regular values t.

Induction Step

Lemma (K.)

Letting

$$\mathbf{b} = (a_0 + a_1, a_2, \ldots, a_{d+1}) \in \mathbb{Z}^{d+1},$$
$$\mathbf{c} = (a_0, a_1, -a_0 - a_1) \in \mathbb{Z}^3,$$

F_t *is the pullback*

$$\begin{array}{ccc} F_t & \longrightarrow & P_{d-1} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle W_{\mathbf{b}}} \\ P_1 & \xrightarrow{tW_{-\mathbf{c}}} & \mathbb{C}^* \end{array}$$

Induction Step

Lemma (K.)

Letting

$$\mathbf{b} = (a_0 + a_1, a_2, \ldots, a_{d+1}) \in \mathbb{Z}^{d+1},$$
$$\mathbf{c} = (a_0, a_1, -a_0 - a_1) \in \mathbb{Z}^3,$$

F_t is the pullback

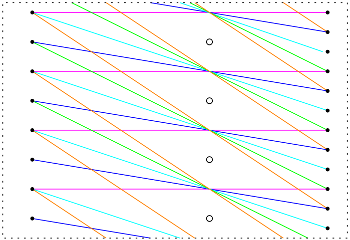
$$\begin{array}{ccc} F_t & \longrightarrow & P_{d-1} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle W_{\mathbf{b}}} \\ P_1 & \xrightarrow{tW_{-\mathbf{c}}} & \mathbb{C}^* \end{array}$$

Corollary

The vanishing cycles L_i of $W_{\mathbf{a}}$ are f-matching cycles over the thimbles T_i for $W_{\mathbf{c}}$. Furthermore, L_i is the pullback along $tW_{-\mathbf{c}}$ of a vanishing thimble of $W_{\mathbf{b}}$.

Example $\mathbf{a} = (2, 3, 1, -2, -4)$

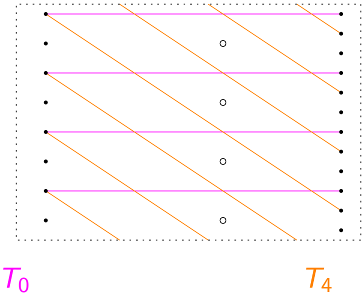
Here $\mathbf{b} = (5, 1, -2, -4)$ and $\mathbf{c} = (2, 3, -5)$.



T_0 T_1 T_2 T_3 T_4

Example $\mathbf{a} = (2, 3, 1, -2, -4)$

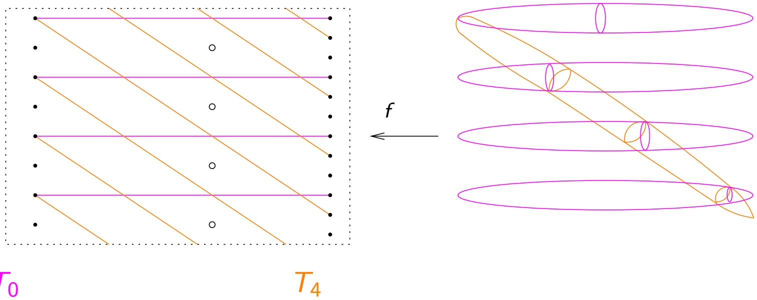
Here $\mathbf{b} = (5, 1, -2, -4)$ and $\mathbf{c} = (2, 3, -5)$.



Consider just two of these thimbles.

Example $\mathbf{a} = (2, 3, 1, -2, -4)$

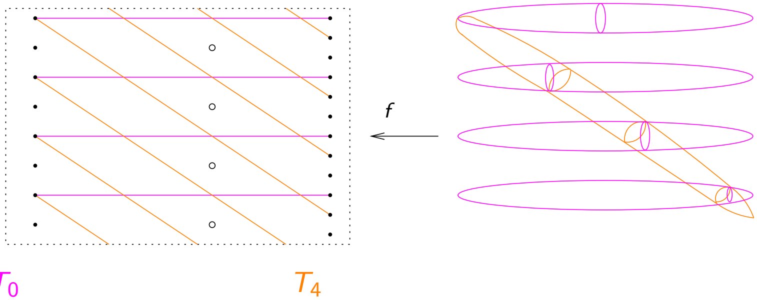
Here $\mathbf{b} = (5, 1, -2, -4)$ and $\mathbf{c} = (2, 3, -5)$.



The corollary asserts that the $W_{\mathbf{a}}$ vanishing cycles $L_0^{\mathbf{a}}, L_4^{\mathbf{a}}$ are fibered over T_0, T_4 via f, collapsing at its endpoints.

Example $\mathbf{a} = (2, 3, 1, -2, -4)$

Here $\mathbf{b} = (5, 1, -2, -4)$ and $\mathbf{c} = (2, 3, -5)$.

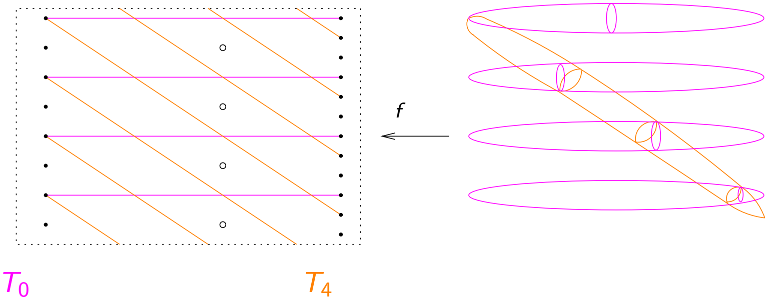


This gives a decomposition

$$\mathrm{Hom}_{FS(W_{\mathbf{a}})}(L_0^{\mathbf{a}}, L_4^{\mathbf{a}}) = CF^*(L_0^{\mathbf{a}}, L_4^{\mathbf{a}}) = \oplus_{y \in T_0 \cap T_4} CF^*(L_{0,y}^{\mathbf{b}}, L_{4,y}^{\mathbf{b}})$$

Example $\mathbf{a} = (2, 3, 1, -2, -4)$

Here $\mathbf{b} = (5, 1, -2, -4)$ and $\mathbf{c} = (2, 3, -5)$.

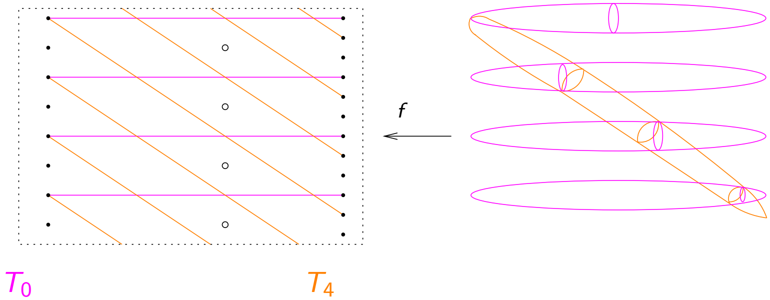


Implying $\mathrm{Hom}_{FS(W_{\mathbf{a}})}(L_0^{\mathbf{a}}, L_4^{\mathbf{a}})$ is isomorphic to

$$\mathrm{Hom}_{FS(W_{\mathbf{b}})}(L_0^{\mathbf{b}}, L_0^{\mathbf{b}}) \oplus \mathrm{Hom}_{FS(W_{\mathbf{b}})}(L_0^{\mathbf{b}}, L_2^{\mathbf{b}}) \oplus \cdots$$
$$\cdots \oplus \mathrm{Hom}_{FS(W_{\mathbf{b}})}(L_0^{\mathbf{b}}, L_4^{\mathbf{b}}) \oplus \mathrm{Hom}_{FS(W_{\mathbf{b}})}(L_0^{\mathbf{b}}, L_1^{\mathbf{b}}).$$

Example $\mathbf{a} = (2, 3, 1, -2, -4)$

Here $\mathbf{b} = (5, 1, -2, -4)$ and $\mathbf{c} = (2, 3, -5)$.

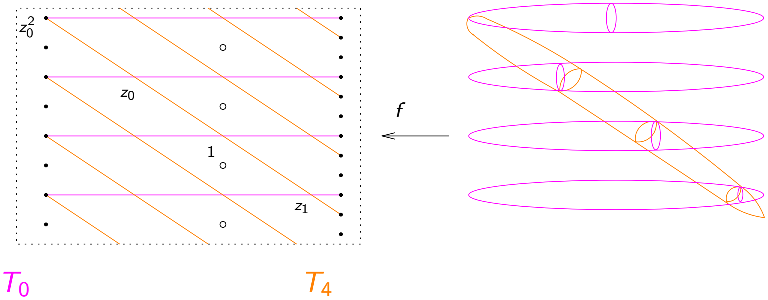


By induction, we have $\text{Hom}_{FS(W_{\mathbf{a}})}(L_0^{\mathbf{a}}, L_4^{\mathbf{a}})$ is isomorphic to

$$\mathbb{C} \cdot \{1\} \oplus \mathbb{C} \cdot \{d\tilde{z}_2\} \oplus \mathbb{C} \cdot \{d\tilde{z}_3\} \oplus \mathbb{C} \cdot \{\tilde{z}_1\} \subset R_{\mathbf{b}}.$$

Example $\mathbf{a} = (2, 3, 1, -2, -4)$

Here $\mathbf{b} = (5, 1, -2, -4)$ and $\mathbf{c} = (2, 3, -5)$.



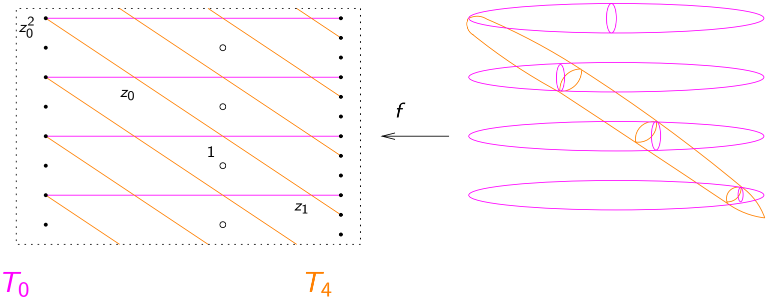
Taking $\tilde{z}_j, d\tilde{z}_i \in R_{\mathbf{b}}$ to $z_{j+1}, dz_{i+1} \in R_{\mathbf{a}}$, and multiplying by a power of z_0 or z_1 (depending on the summand), we obtain

$$\mathbb{C} \cdot \{1\} \oplus \mathbb{C} \cdot \{d\tilde{z}_2\} \oplus \mathbb{C} \cdot \{d\tilde{z}_3\} \oplus \mathbb{C} \cdot \{\tilde{z}_1\} \subset R_{\mathbf{b}},$$

$$\mathbb{C} \cdot \{z_0^2\} \oplus \mathbb{C} \cdot \{z_0 dz_3\} \oplus \mathbb{C} \cdot \{dz_4\} \oplus \mathbb{C} \cdot \{z_1 z_2\} = R_{\mathbf{a}}(4)$$

Example $\mathbf{a} = (2, 3, 1, -2, -4)$

Here $\mathbf{b} = (5, 1, -2, -4)$ and $\mathbf{c} = (2, 3, -5)$.



This yields the isomorphism of vector spaces

$$\Phi : \mathrm{Hom}_{FS(W_{\mathbf{a}}^{1/n})}(L_0^{\mathbf{a}}, L_4^{\mathbf{a}}) \xrightarrow{\cong} R_{\mathbf{a}}(4).$$

Induction Step

This decomposition is compatible with the Floer product, which defines the functor Φ on morphisms.

Induction Step

This decomposition is compatible with the Floer product, which defines the functor Φ on morphisms.
Utilizing the observation that only holomorphic triangles exist bounding the thimbles T_i, one obtains a formality result on the n-th unfolded category $\mathcal{A}^{1/n}$. This gives that Φ is an equivalence of categories.

Future directions

Recall that the original conjecture was an equivalence of decompositions

$$D(X) = \left\langle \mathcal{T}^{B}_{f_1}, \dots, \mathcal{T}^{B}_{f_r} \right\rangle,$$
$$FS(W) = \left\langle \mathcal{T}^{A}_{\mathbf{w}^1}, \dots, \mathcal{T}^{A}_{\mathbf{w}^r} \right\rangle.$$

Future directions

Recall that the original conjecture was an equivalence of decompositions

$$D(X) = \left\langle \mathcal{T}^B_{f_1}, \dots, \mathcal{T}^B_{f_r} \right\rangle,$$
$$FS(W) = \left\langle \mathcal{T}^A_{\mathbf{w}^1}, \dots, \mathcal{T}^A_{\mathbf{w}^r} \right\rangle.$$

- To check this holds, we must identify $(\mathcal{T}^B_{f_i}, \mathcal{T}^B_{f_{i+1}})$ and $(\mathcal{T}^A_{\mathbf{w}^i}, \mathcal{T}^A_{\mathbf{w}^{i+1}})$ bimodules, which glue the pieces together, and prove their equivalence.

Future directions

Recall that the original conjecture was an equivalence of decompositions

$$D(X) = \left\langle \mathcal{T}^B_{f_1}, \dots, \mathcal{T}^B_{f_r} \right\rangle,$$
$$FS(W) = \left\langle \mathcal{T}^A_{\mathbf{w}^1}, \dots, \mathcal{T}^A_{\mathbf{w}^r} \right\rangle.$$

- To check this holds, we must identify $(\mathcal{T}^B_{f_i}, \mathcal{T}^B_{f_{i+1}})$ and $(\mathcal{T}^A_{\mathbf{w}^i}, \mathcal{T}^A_{\mathbf{w}^{i+1}})$ bimodules, which glue the pieces together, and prove their equivalence.
- At a more elementary level, the equivalence between $\mathcal{T}^B_{\mathbf{a}}$ and $\mathcal{T}^A_{\mathbf{a}}$ must be shown in the case when some $a_i = 0$ (e.g. blowing up subvarieties of positive dimension).