Time-dependent Wave Splitting and Source Separation

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Content

Outline

Ambient medium
wave propagation speed
c_0 known

Unknown inclusion
wave propagation speed
$c(x) \geq c_0$ non constant

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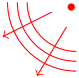
Incident wave u^I
sent in the medium



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Introduction and motivation

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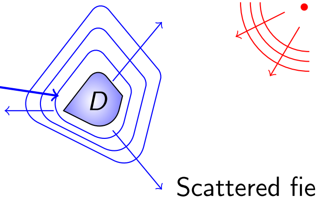
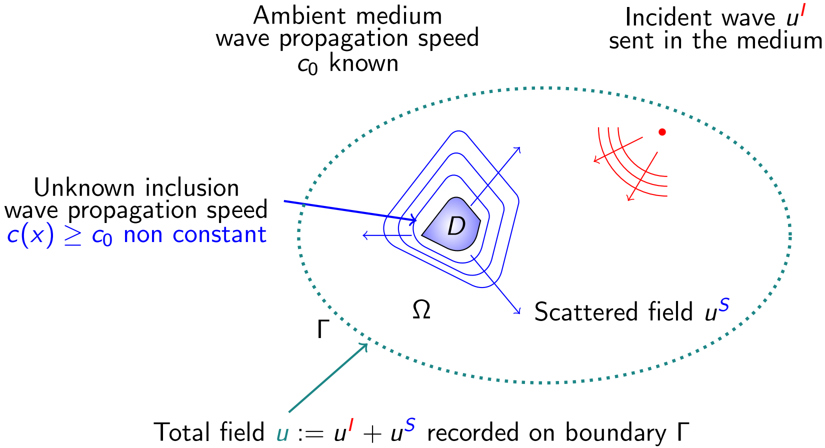


Scattered field u^S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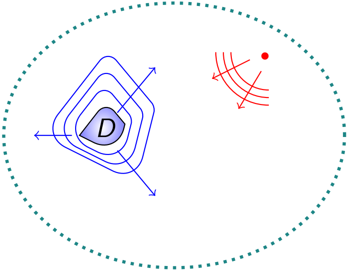
D
Scattered field u^S
Ω
Γ
Total field $u := u^I + u^S$ recorded on boundary Γ

Aim: solve a time-dependent inverse problem from measurements data in situations when the incident field is unknown

But!!! needed to solve inverse problems
$\Rightarrow$ computation of the forward problem
in the optimization process



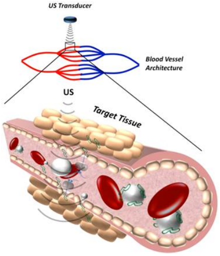
Assumptions about the incident field:

- location: approx. known
- time history: unknown

Examples of applications:

- in medical imaging
  e.g. Contrast-enhanced ultrasound: microbubbles as contrast agents

[1] M. Pernot, G. Montaldo, M. Tanter, and M. Fink. "Ultrasonic stars" for time reversal focusing using induced cavitation bubbles. *Appl. Phys. Lett.*, 88(3):034102, 2006.
[2] S. R. Sirsi, M. A. Borden. Advances in Ultrasound Mediated Gene Therapy Using Microbubble Contrast Agents, *Theranostics*, 2(12):1208-1222, 2012.



from [2]

- in geophysics
  e.g. Full Waveform Inversion or imaging

[3] N. Tu, A. Y. Aravkin, T. van Leeuwen, and F. J. Herrmann. Fast least-squares migration with multiples and source estimation, *EAGE* 2013.

Idea: split the measurements data into incident and scattered wave fields

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1. wave splitting
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   $\Rightarrow$ recover the unknown inclusion by PDE-constrained optimization

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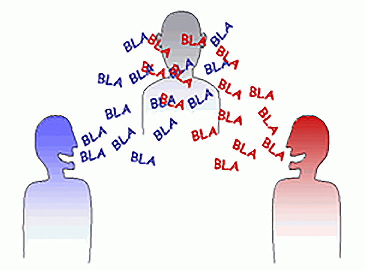
Focus on Wave Splitting...

Outline

Multiple scattering problem: $u = u_1 + u_2$, in $\Omega := \mathbb{R}^d \setminus (S_1 \cup S_2)$



u satifies:

$$\frac{\partial^2 u}{\partial t^2} - c_0^2 \Delta u = 0 \qquad \text{in } \Omega, \ t > 0.$$

Question: Given the measured total field u, can we recover u_1 and u_2 without knowing in advance either of them ?

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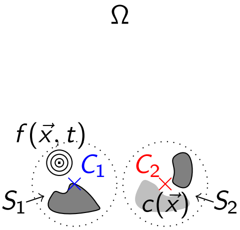

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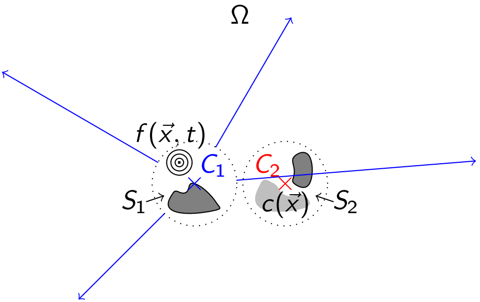

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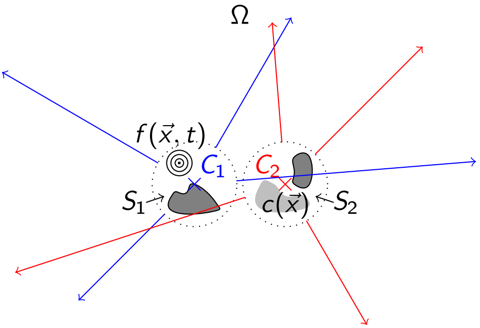

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Question: Given the measured total field u, can we recover u_1 and u_2 without knowing in advance either of them ?

Other works on Wave splitting:

- in the frequency domain
  - F. Ben Hassen, J. Liu, and R. Potthast. (2007)

    On source analysis by wave splitting with applications in inverse scattering of multiple obstacles. *J. Comput. Math*, **25**(3):266–281.
  - R. Griesmaier, M. Hanke, and J. Sylvester. (2014)

    Far field splitting for the Helmholtz equation. *SIAM J. Numer. Anal.*, **52**(1):343–362.
  - H. Wang and J. Liu. (2013)

    On decomposition method for acoustic wave scattering by multiple obstacles. Acta Mathematica Scientia, 33B(1):1–22.
- in the time-dependent domain
  - R. Potthast, F. M. Fazi, and P. A. Nelson. (2010)

    Source splitting via the point source method. *Inv. Problems*, **626**(4):045002.

Our method is local in space and time, deterministic, and also avoids a priori assumptions on the frequency spectrum of the signal.

Outside S_1 and S_2, u satisfies:

$$\frac{\partial^2 u}{\partial t^2} - c_0^2 \Delta u = 0 \qquad \text{in } \Omega,\ t > 0,$$

$c_0 > 0$ constant.

At $t = 0$, no signal in Ω, then uniqueness of splitting[1]

$$u = u_1 + u_2 \qquad \text{in } \Omega,\ t > 0$$

and u_k outgoing (3D):

$$u_k(t, r_k, \theta_k, \varphi_k) = \frac{1}{r_k} \sum_{i \geq 0} \frac{f_{k,i}(r_k - c_0 t, \theta_k, \varphi_k)}{(r_k)^i}$$

$(r_k, \theta_k, \varphi_k)$ spherical coordinates centered at C_k.

[1] M. J. Grote and C. Kirsch. Nonreflecting boundary condition for time-dependent multiple scattering. J. Comput. Phys., 221(1):41–67, 2007.

Since

$$u_k(t, r_k, \theta_k, \varphi_k) \;=\; \frac{1}{r_k} \sum_{i \geq 0} \frac{f_{k,i}(r_k - c_0 t, \theta_k, \varphi_k)}{(r_k)^i}$$

$(r_k, \theta_k, \varphi_k)$ spherical coordinates centered at C_k,

m^{th}-order absorbing boundary condition[2] on any Γ in Ω

$$B_k[u_k] = \; O\left(\frac{1}{r_k^{2m+1}}\right), \qquad k = 1, 2$$

[2] A. Bayliss and E. Turkel. Radiation boundary conditions for wave-like equations. Comm. Pure Appl. Math., 33(6):707–725, 1980.

Neglecting the higher order error term:

$$B_j[u_k] \ = B_j[u_k + u_j] = \ B_j[u], \qquad j = 1, 2, \quad k \neq j$$

Recover u_1 and u_2 by solving:

$$\begin{cases} B_2[u_1] = B_2[u] & (1) \\ B_1[u_2] = B_1[u] & (2) \end{cases}$$

where u is known (measurements on Γ)

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where u is known (measurements on Γ)

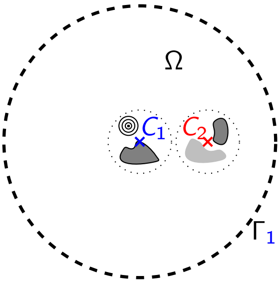
Difficulty: integration of partial differential equation (1)-(2) on the submanifold Γ

- Find adequate initial and boundary conditions
- Change of coordinates from $(r_k, \theta_k, \varphi_k)$ to $(r_j, \theta_j, \varphi_j)$
- Remove normal/radial derivatives (equation on Γ involving only (t, θ_j, φ_j))

In 2D, Bayliss-Turkel first order absorbing boundary condition

$$B_j[u] \; = \; \frac{1}{c_0}\frac{\partial u}{\partial t} + \frac{\partial u}{\partial r_j} + \frac{u}{2r_j}$$

For simplicity, let $\Gamma := \Gamma_1$ be a circle centered at C_1.



E.g. to recover u_1 on Γ_1

$$\begin{aligned} B_2[u_1] &= B_2[u] \\ \frac{1}{c_0}\frac{\partial u_1}{\partial t} + \frac{\partial u_1}{\partial r_2} + \frac{u_1}{2r_2} &= \frac{1}{c_0}\frac{\partial u}{\partial t} + \frac{\partial u}{\partial r_2} + \frac{u}{2r_2} \end{aligned}$$

How to solve this PDE for u_1?

- need initial and boundary conditions
- remove the radial derivative! we solve on Γ_1
- derivatives in (r_2, θ_2), when domain in (r_1, θ_1)

$\Longrightarrow$ rewrite the PDE using only $\dfrac{\partial}{\partial t}$, $\dfrac{\partial}{\partial \theta_1}$ and 0^{th}-order term

PDE reads:

$$\left(\frac{1}{c_0}\frac{\partial}{\partial t} + \frac{\partial}{\partial r_2} + \frac{1}{2r_2}\right) u_1 \;=\; B_2[u]$$

First step:
Change of coordinate system from (r_2, θ_2) to (r_1, θ_1)

$$\frac{\partial}{\partial r_2} \;=\; K(r_1,\theta_1)\frac{\partial}{\partial r_1} \;+\; M(r_1,\theta_1)\frac{\partial}{\partial \theta_1}$$

where K, M only depend on the change of coordinates, hence

$$\left(\frac{1}{c_0}\frac{\partial}{\partial t} + K(\theta_1)\,\frac{\partial}{\partial r_1} + M(\theta_1)\,\frac{\partial}{\partial \theta_1} + \frac{1}{2r_2}\right) u_1 \;=\; B_2[u]\,, \quad \text{on } \Gamma_1,\; t>0$$

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!!! on Γ_1, solution only depends on t and θ_1 since r_1 constant

Second step:
Assume from the progressive wave expansion

$$u_1(t, r_1, \theta_1) \simeq \frac{1}{\sqrt{r_1}} f_1(r_1 - c_0 t, \theta_1)$$

Then f_1 satisfies:

$$\frac{\partial f_1}{\partial r_1} = -\frac{1}{c_0} \frac{\partial f_1}{\partial t}$$

by replacing in the PDE

$$\left(\frac{1}{c_0} \frac{\partial}{\partial t} + K(\theta_1) \frac{\partial}{\partial r_1} + M(\theta_1) \frac{\partial}{\partial \theta_1} + \frac{1}{2r_2} \right) \left(\frac{1}{\sqrt{r_1}} f_1 \right) = B_2[u]$$

$$\left(\frac{1}{c_0 \sqrt{r_1}} \frac{\partial}{\partial t} + \frac{K(\theta_1)}{\sqrt{r_1}} \left(\frac{\partial}{\partial r_1} - \frac{1}{2r_1} \right) + \frac{M(\theta_1)}{\sqrt{r_1}} \frac{\partial}{\partial \theta_1} + \frac{1}{2r_2 \sqrt{r_1}} \right) f_1 = B_2[u]$$

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$$= -\frac{1}{c_0}\frac{\partial}{\partial t}$$

Finally: PDE to recover $f_1 = \sqrt{r_1} u_1$ on Γ_1, $t > 0$

$$\left(\alpha_1(\theta_1) \frac{\partial}{\partial t} + \beta_1(\theta_1) \frac{\partial}{\partial \theta_1} + \gamma_1(\theta_1) \right) f_1 = \left(\frac{1}{c_0} \frac{\partial}{\partial t} + \frac{\partial}{\partial r_2} + \frac{1}{2 r_2} \right) u,$$

with

$$\begin{aligned} \alpha_1(\theta_1) &= \frac{\sqrt{r_1^2 + \ell^2 - 2 r_1 \ell \cos(\theta_1)} - r_1 + \ell \cos(\theta_1)}{c_0 \sqrt{r_1} \sqrt{r_1^2 + \ell^2 - 2 r_1 \ell \cos(\theta_1)}}, \\ \beta_1(\theta_1) &= \frac{\ell \sin(\theta_1)}{r_1 \sqrt{r_1} \sqrt{r_1^2 + \ell^2 - 2 r_1 \ell \cos(\theta_1)}}, \\ \gamma_1(\theta_1) &= \frac{\ell \cos(\theta_1)}{2 r_1 \sqrt{r_1} \sqrt{r_1^2 + \ell^2 - 2 r_1 \ell \cos(\theta_1)}}, \end{aligned}$$

and ℓ the signed distance between C_1 and C_2.

We want to recover $f_1 = \sqrt{r_1} u_1$ which satisfies on Γ, $t > 0$

$$\left(\alpha_1(\theta_1)\frac{\partial}{\partial t} + \beta_1(\theta_1)\frac{\partial}{\partial \theta_1} + \gamma_1(\theta_1)\right) f_1 = \left(\frac{1}{c_0}\frac{\partial}{\partial t} + \frac{\partial}{\partial r_2} + \frac{1}{2r_2}\right) u,$$

Initial condition?

At $t = 0$, no signal in Ω: all sources in $S_1 \cup S_2$

$\Longrightarrow$ f_1 and f_2 vanish in Ω, thus on $\Gamma_1 \cup \Gamma_2$

the initial condition is:

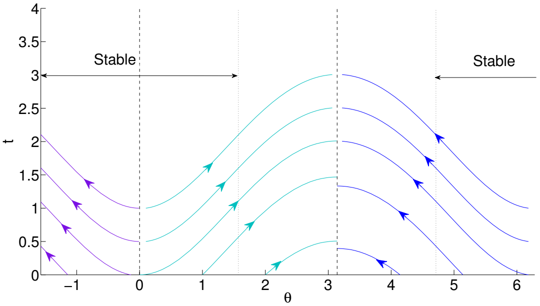
$$f_1 = 0, \qquad \text{on } \Gamma_1, \text{ at } t = 0.$$

Hyperbolic PDE

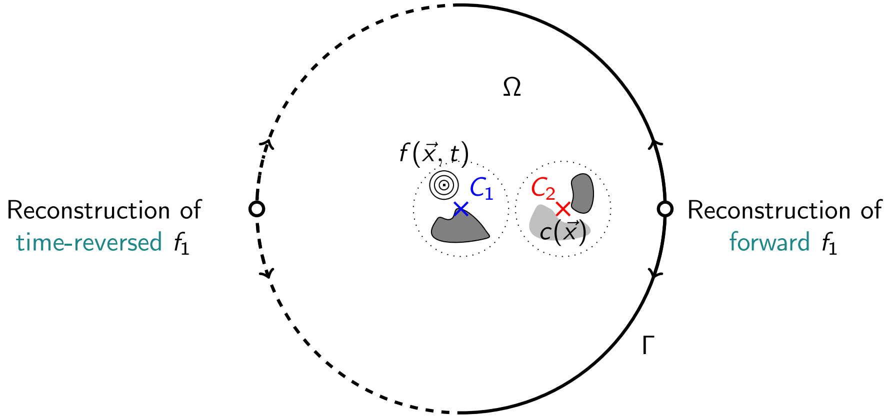
$$\left(\alpha_1(\theta_1)\frac{\partial}{\partial t} + \beta_1(\theta_1)\frac{\partial}{\partial \theta_1} + \gamma_1(\theta_1)\right) f_1 = \left(\frac{1}{c_0}\frac{\partial}{\partial t} + \frac{\partial}{\partial r_2} + \frac{1}{2r_2}\right) u$$

trivial at $\theta_1 = 0$ or π modulo 2π, since $\alpha_1(\theta_1) = 0, \beta_1(\theta_1) = 0$

$\Longrightarrow$ **Dirichlet boundary condition:** $f_1 = \dfrac{B_2[u]}{\gamma_1(0)}$ at $\theta_1 = 0$



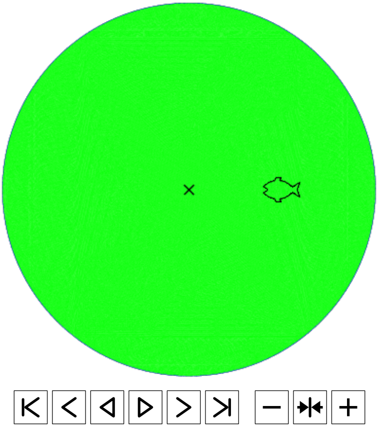
... and by using time reversal[1]



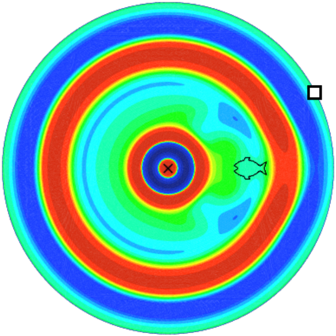
A similar equation can be derived for f_2 on the same boundary $\Gamma = \Gamma_1$.

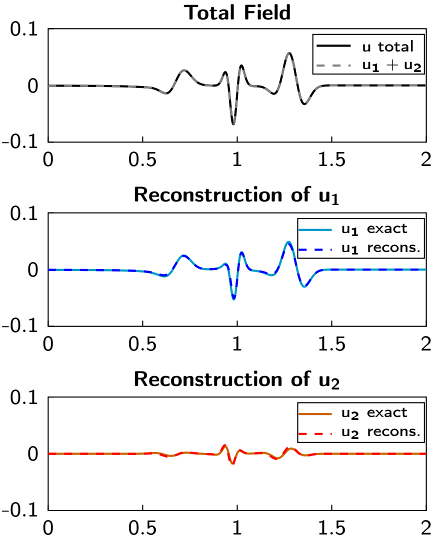
[1] M.J. Grote, M. Kray, F. Nataf and F. Assous. Time-dependent wave splitting and source separation. (2016)

- **Incident wave field from a point source**
- **Scattered wave field from a penetrable fish-shaped inclusion**



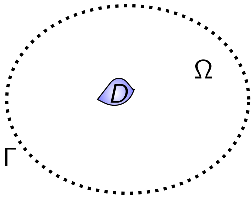
Time history of wave fields at one location: incident wave impinges on a penetrable inclusion





Outline

Aim: Reconstruct the outgoing wave field u in $\Omega \setminus D$ from measurements on Γ reversed in time.



The wave equation is time reversible. The time reversed field $u_R(t,\cdot) := u(T-t,\cdot)$ is solution of a wave equation as well:

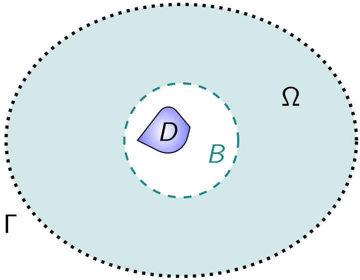
$$\begin{cases} \dfrac{\partial^2 u_R}{\partial t^2} - c_0^2 \Delta u_R = 0 & \text{in } (0,T)\times(\Omega\setminus D), \\ u_R(t,\cdot) = u(T-t,\cdot) & \text{on } (0,T)\times\Gamma, \\ u_R = ? & \text{on } (0,T)\times\partial D, \end{cases}$$

with homogeneous initial conditions.

This problem is undetermined because D is unknown!

Time Reversed Absorbing Condition (*TRAC*) method:

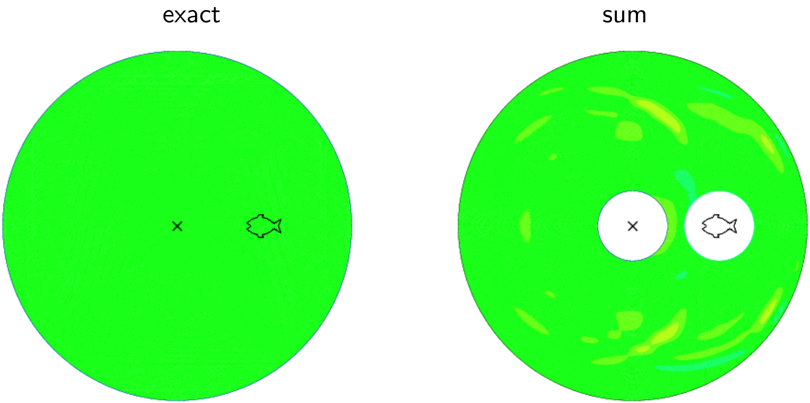
Introduce a subdomain B enclosing the inclusion D.



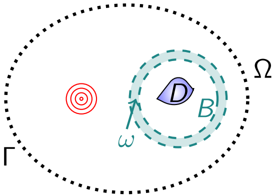
Reconstruct the time-reversed wave field in $\Omega \setminus B$
by imposing a relevant boundary condition on ∂B.

$\Longrightarrow$ *TRAC*

Reconstruction of the total wave field



Aim: recover the location, shape and properties of inclusion D from the reconstructed data on a reduced computational domain



To solve the inverse problem, we minimize the functional:

$$J(p) = \frac{1}{2}\int_0^T \int_\omega |u(p) - u^{obs}|^2 \, dx \, dt + \frac{\alpha}{2}\int_B |\nabla p|^2 \, dx,$$

with p the parameter to reconstruct, such that: $c^2(x) = c_0^2 + p(x)\chi_B(x)$ using

- *optimize-then-discretize* reduced-space approach
- BFGS algorithm
- finite elements method

Adaptive process[3]:

From an initial guess $p^{(0)}$, look for parameter p in the space spanned by the K first eigenfunctions of the elliptic operator:

$$p(x) = \sum_{i=1}^{K} p_i \phi_i(x), \quad \text{with} \quad \begin{cases} -\nabla \cdot (A(x) \nabla \phi_i) = \lambda_i \phi_i & \text{in } B, \\ \phi_i = 0 & \text{on } \partial B. \end{cases}$$

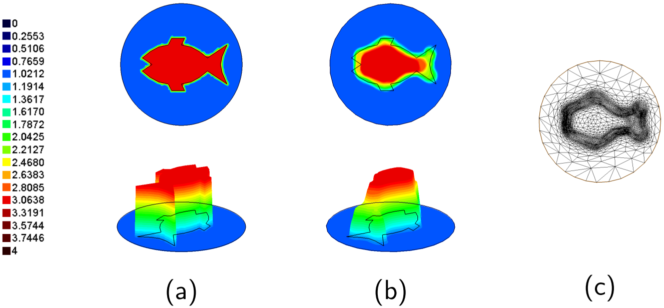
Matrix A is chosen with respect to the result obtained from the previous iteration:

$$A(x) = \frac{1}{|\nabla p^{(0)}(x)|^q} Id.$$

+ Mesh adaptation

[3] M. de Buhan, M. Kray. A new approach to solve the inverse scattering problem for waves: combining the TRAC and the Adaptive Inversion methods. Inverse Problems, 29(8), 2013.

Reconstruction of a fish: (from [de Buhan, K. 2013])



(a) Exact propagation speed in B

(b) Reconstruction with AI from data on ω

(c) Final mesh through adaptative process

Outline

Time-dependent Wave Splitting and Source Separation

New partial differential equation

- on a submanifold Γ
- in the time-dependent domain
- local in space and time
- independent on the frequency spectrum

Method extendable to:

- 2 or more scatterers
- vector-valued wave equations from electromagnetics and elasticity
- improved accuracy with higher order absorbing boundary condition (more terms in the progressive wave expansion)

Conclusion

Wave Splitting and adaptive eigenspace inversion
for time-dependent inverse problems

Procedure in 3 steps:

1. split the total wave field to recover the incident wave field, necessary for the optimization process
2. incident and scattered wave fields reconstructed from split data by using the *TRAC* method
3. adaptive eigenspace inversion to solve the inverse problem from the reconstructed data (in progress)

References

- **Wave Splitting**

[1] **M.J. Grote, M. Kray, F. Nataf and F. Assous.** Wave splitting for time-dependent scattered field separation. *C. R. Acad. Sci., Serie I*, 353(6) (2015)

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