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Computational and Numerical Analysis of Transient Problems in Acoustics, Elasticity, and Electromagnetism

Space–time Trefftz discontinuous Galerkin methods for wave problems

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Trefftz methods

Consider a PDE $\mathcal{L}u = 0$ that is: (i) linear, (ii) homogeneous (RHS=0), (iii) with piecewise constant coefficients.

Trefftz methods are finite element schemes such that test and trial functions are solutions of the PDE in each element K of the mesh $\mathcal{T}_h$, i.e.:

$$V_p \subset T(\mathcal{T}_h) = \Big\{ v \in L^2(\Omega) : \mathcal{L}v = 0 \text{ in each } K \in \mathcal{T}_h \Big\}.$$

E.g.: piecewise harmonic polynomials if $\mathcal{L}u = \Delta u$.

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Our main interest is in wave propagation, in:

- Frequency domain, Helmholtz eq. $-\Delta u - \omega^2 u = 0$
  lot of work done, $h/p/hp$-theory, extended to other eq.s;
  (recent survey: Hiptmair, AM, Perugia, arXiv:1506.04521)
- Time domain, wave equation $-\Delta U + \frac{1}{c^2}\frac{\partial^2}{\partial t^2} U = 0$
  Trefftz methods are in space–time.

Trefftz methods for wave equation

Why Trefftz methods? Comparing with standard DG,

- better accuracy per DOFs and higher convergence orders;
- PDE properties "known" by discrete space, e.g. dispersion;
- lower dimensional quadrature needed;
- simpler and more flexible;
- adapted bases and (one day) adaptivity. . .

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Existing works on Trefftz for time-domain wave equation:

- MACIĄG, SOKALA 2005–2011, Trefftz on a single element;
- PETERSEN, FARHAT, TEZAUR, WANG 2009&2014, DG with Lagrange multipliers;
- EGGER, KRETZSCHMAR, SCHNEPP, TSUKERMAN, WEILAND $3\times$2014–2015, Maxwell equations; KRETZSCHMAR, MOIOLA, PERUGIA, SCHNEPP $2\times$2015, analysis;
- BANJAY, GEORGOULIS, LIJOKA, interior penalty-DG.

Simplest basis: Trefftz polynomials

Consider wave equation $-\Delta U + \frac{1}{c^2} U'' = 0$ in $K \subset \mathbb{R}^{n+1}$ (c const.).

For $\mathbf{d} \in \mathbb{R}^n$, $|\mathbf{d}| = 1$, $f : \mathbb{R} \to \mathbb{R}$ smooth, $f(\mathbf{d} \cdot \mathbf{x} - ct)$ is solution.

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Choose Trefftz space of polynomials of deg. $\leq p$ on element K:

$$\mathbb{T}^p(K) := \left\{ v \in \mathbb{P}^p(K),\ -\Delta v + c^{-2} v'' = 0 \right\}$$
$$= \operatorname{span}\left\{ (\mathbf{d}_{j,\ell} \cdot \mathbf{x} - ct)^j,\ {}_{1 \leq \ell \leq L(j,n)}^{0 \leq j \leq p,} \right\}, \quad \text{with dimension}$$

$$\dim\left(\mathbb{T}^p(K)\right) = \binom{p+n-1}{n} \frac{2p+n}{p} = \mathcal{O}_{p\to\infty}(p^n) \ll \dim\left(\mathbb{P}^p(K)\right) = \binom{p+n+1}{n+1} = \mathcal{O}_{p\to\infty}(p^{n+1})$$

Taylor polynomial of (smooth) U belongs to $\mathbb{T}^p(K)$.

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Choice of directions $\mathbf{d}_{j,\ell}$: (corresponding to homog. polyn. deg. j)

- $n = 1$, left/right directions $\mathbf{d}_{j,1} = 1$, $\mathbf{d}_{j,2} = -1$, $\mathbb{T}^p(K) = \operatorname{span}\{(x \pm ct)^j\}$;
- $n = 2$, any distinct $\{\mathbf{d}_{j,\ell}\}_{\ell=1,\dots,2j+1}$ give a basis;
- $n = 3$, $(\mathbf{d}_{j,\ell} \cdot \mathbf{x} - ct)^j$ linearly indep. $\iff$ $[Y_N^m(\mathbf{d}_{j,\ell})]_{N \leq j, m; \ell}$ full rank.

Initial–boundary value problem

First order initial–boundary value problem (Dirichlet): find $(v, \boldsymbol{\sigma})$

$$\begin{cases} \nabla v + \dfrac{\partial \boldsymbol{\sigma}}{\partial t} = \mathbf{0} & \text{in } Q = \Omega \times (0,T) \subset \mathbb{R}^{n+1},\ n \in \mathbb{N}, \\ \nabla \cdot \boldsymbol{\sigma} + \dfrac{1}{c^2}\dfrac{\partial v}{\partial t} = 0 & \text{in } Q, \\ v(\cdot, 0) = v_0, \quad \boldsymbol{\sigma}(\cdot, 0) = \boldsymbol{\sigma}_0 & \text{on } \Omega, \\ v(\mathbf{x}, \cdot) = g & \text{on } \partial\Omega \times (0,T). \end{cases}$$

Equivalent to $-\Delta U + c^{-2}\frac{\partial^2}{\partial t^2} U = 0$ setting $v = \frac{\partial U}{\partial t}$ and $\boldsymbol{\sigma} = -\nabla U$.

Velocity c piecewise constant. $\Omega \subset \mathbb{R}^n$ Lipschitz bounded.

Extensions:

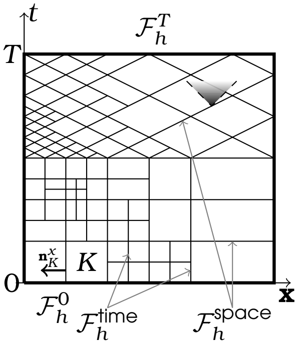
- Neumann $\boldsymbol{\sigma} \cdot \mathbf{n} = g$ & Robin $\frac{\vartheta}{c} v - \boldsymbol{\sigma} \cdot \mathbf{n} = g$ BCs (✓),
- Maxwell equations (✓),
- elasticity,
- 1^{st} order Friedrichs' systems,
- Maxwell equations in dispersive materials. . .

Space–time mesh and assumptions

Introduce space–time polytopic mesh $\mathcal{T}_h$ on Q.
Assume: $c = c(\mathbf{x})$ constant in elements.

Assume: each face $F = \partial K_1 \cap \partial K_2$ with normal $(\mathbf{n}_F^x, n_F^t)$ is either

- space-like: $c|\mathbf{n}_F^x| < n_F^t$, denote $F \subset \mathcal{F}_h^{\text{space}}$, or
- time-like: $n_F^t = 0$, denote $F \subset \mathcal{F}_h^{\text{time}}$.



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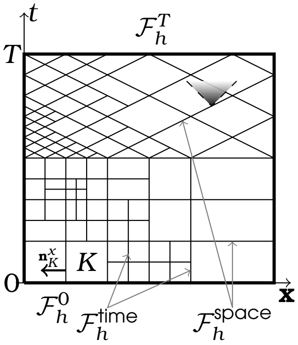

DG notation:

$$\{\!\{w\}\!\} := \frac{w_{|K_1} + w_{|K_2}}{2}, \qquad \{\!\{\boldsymbol{\tau}\}\!\} := \frac{\boldsymbol{\tau}_{|K_1} + \boldsymbol{\tau}_{|K_2}}{2},$$

$$[\![w]\!]_{\mathbf{N}} := w_{|K_1}\mathbf{n}_{K_1}^x + w_{|K_2}\mathbf{n}_{K_2}^x,$$

$$[\![\boldsymbol{\tau}]\!]_{\mathbf{N}} := \boldsymbol{\tau}_{|K_1} \cdot \mathbf{n}_{K_1}^x + \boldsymbol{\tau}_{|K_2} \cdot \mathbf{n}_{K_2}^x,$$

$$[\![w]\!]_t := w_{|K_1} n_{K_1}^t + w_{|K_2} n_{K_2}^t = (w^- - w^+) n_F^t,$$

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$$\mathcal{F}_h^0 := \Omega \times \{0\}, \qquad \mathcal{F}_h^T := \Omega \times \{T\},$$

$$\mathcal{F}_h^\partial := \partial\Omega \times [0, T].$$

DG elemental equation and numerical fluxes

Multiply PDEs with test $(w, \boldsymbol{\tau}) \in H^1(\mathcal{T}_h)^{1+n}$, integrate by parts in K:

$$-\iint_K \left(v \underbrace{\left(\nabla \cdot \boldsymbol{\tau} + \frac{1}{c^2} \frac{\partial w}{\partial t} \right)} + \boldsymbol{\sigma} \cdot \left(\underbrace{\frac{\partial \boldsymbol{\tau}}{\partial t} + \nabla w} \right) \right) \mathrm{d}\mathbf{x}\, \mathrm{d}t$$

$$+ \underbrace{\int_{\partial K} \left((\widehat{v}\, \boldsymbol{\tau} + \widehat{\boldsymbol{\sigma}}\, w) \cdot \mathbf{n}_K^x + \left(\widehat{\boldsymbol{\sigma}} \cdot \boldsymbol{\tau} + \frac{1}{c^2} \widehat{v}\, w \right) n_K^t \right) \mathrm{dS} = 0}\, .$$

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Here $\widehat{v},\widehat{\boldsymbol{\sigma}}$ are numerical fluxes, approximations of traces of $(v,\boldsymbol{\sigma})$ on skeleton defined as:

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Here $\widehat{v},\widehat{\boldsymbol{\sigma}}$ are numerical fluxes, approximations of traces of $(v,\boldsymbol{\sigma})$ on skeleton defined as: $\alpha,\beta\in L^\infty(\mathcal{F}_h^{\text{time}}\cup\mathcal{F}_h^{\partial})$

$$\widehat{v}_{hp}:=\begin{cases} v_{hp}^- \\ v_{hp} \\ v_0 \\ \{\!\{v_{hp}\}\!\}+\beta[\![\boldsymbol{\sigma}_{hp}]\!]_{\mathbf{N}} \\ g \end{cases}\qquad \widehat{\boldsymbol{\sigma}}_{hp}:=\begin{cases} \boldsymbol{\sigma}_{hp}^- & \text{on } \mathcal{F}_h^{\text{space}}, \\ \boldsymbol{\sigma}_{hp} & \text{on } \mathcal{F}_h^{T}, \\ \boldsymbol{\sigma}_0 & \text{on } \mathcal{F}_h^{0}, \\ \{\!\{\boldsymbol{\sigma}_{hp}\}\!\}+\alpha[\![v_{hp}]\!]_{\mathbf{N}} & \text{on } \mathcal{F}_h^{\text{time}}, \\ \boldsymbol{\sigma}_{hp}-\alpha(v-g)\mathbf{n}_\Omega^x & \text{on } \mathcal{F}_h^{\partial}. \end{cases}$$

$\alpha=\beta=0\to$ Kretzschmar–S.–T.–W., $\quad\alpha\beta\geq\frac{1}{4}\to$ Monk–Richter.

TDG formulation

Trefftz space $\mathbf{T}(\mathcal{T}_h) := \Big\{(w,\boldsymbol{\tau}) \in L^2(Q), (w|_K, \boldsymbol{\tau}|_K) \in H^1(K)^{1+n},$
$\nabla w + \frac{\partial \boldsymbol{\tau}}{\partial t} = \mathbf{0}, \quad \nabla\cdot\boldsymbol{\tau} + c^{-2}\frac{\partial w}{\partial t} = 0 \ \forall K \in \mathcal{T}_h\Big\}.$

Choosing any $\mathbf{V}_p(\mathcal{T}_h) \subset \mathbf{T}(\mathcal{T}_h)$, summing over K, we write TDG as:

Seek $(v_{hp}, \boldsymbol{\sigma}_{hp}) \in \mathbf{V}_p(\mathcal{T}_h)$ s.t., $\quad \forall (w,\boldsymbol{\tau}) \in \mathbf{V}_p(\mathcal{T}_h)$,
$\mathcal{A}(v_{hp}, \boldsymbol{\sigma}_{hp}; w, \boldsymbol{\tau}) = \ell(w,\boldsymbol{\tau})$ where

$$\mathcal{A}(v_{hp}, \boldsymbol{\sigma}_{hp}; w, \boldsymbol{\tau}) := \int_{\mathcal{F}_h^{\text{space}}} \Big(\frac{v_{hp}^- [\![w]\!]_t}{c^2} + \boldsymbol{\sigma}_{hp}^- \cdot [\![\boldsymbol{\tau}]\!]_t + v_{hp}^- [\![\boldsymbol{\tau}]\!]_{\mathbf{N}} + \boldsymbol{\sigma}_{hp}^- \cdot [\![w]\!]_{\mathbf{N}} \Big) \mathrm{d}S$$
$$+ \int_{\mathcal{F}_h^{\text{time}}} \Big(\{\!\{v_{hp}\}\!\} [\![\boldsymbol{\tau}]\!]_{\mathbf{N}} + \{\!\{\boldsymbol{\sigma}_{hp}\}\!\} \cdot [\![w]\!]_{\mathbf{N}} + \alpha [\![v_{hp}]\!]_{\mathbf{N}} \cdot [\![w]\!]_{\mathbf{N}} + \beta [\![\boldsymbol{\sigma}_{hp}]\!]_{\mathbf{N}} [\![\boldsymbol{\tau}]\!]_{\mathbf{N}} \Big) \mathrm{d}S$$
$$+ \int_{\mathcal{F}_h^T} (c^{-2} v_{hp} w + \boldsymbol{\sigma}_{hp} \cdot \boldsymbol{\tau}) \, \mathrm{d}S + \int_{\mathcal{F}_h^\partial} (\boldsymbol{\sigma}_{hp} \cdot \mathbf{n}_\Omega + \alpha v_{hp}) w \, \mathrm{d}S,$$
$$\ell(w,\boldsymbol{\tau}) := \int_{\mathcal{F}_h^0} (c^{-2} v_0 w + \boldsymbol{\sigma}_0 \cdot \boldsymbol{\tau}) \, \mathrm{d}S + \int_{\mathcal{F}_h^\partial} g(\alpha w - \boldsymbol{\tau}\cdot\mathbf{n}_\Omega) \, \mathrm{d}S.$$

Global, implicit and explicit schemes

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huge linear system!   Might be good for adaptivity.

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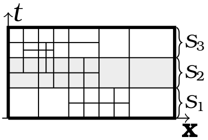

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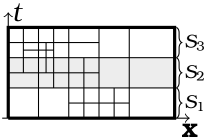
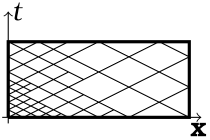


3 If mesh is suitably chosen, Trefftz-DG solution can be computed with a sequence of small local systems: (semi)-explicit method.

Smaller matrix blocks; allows parallelism!

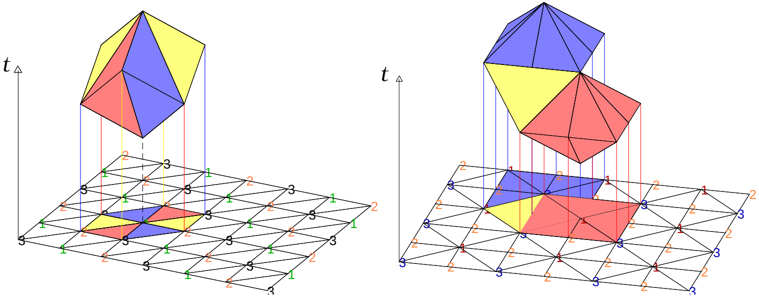


"Tent pitching algorithm" of Üngör–Sheffer, Falk–Richter, Monk–Richter, Gopalakrishnan–Monk–Sepúlveda. . .

Versions 1–2–3 are algebraically equivalent (on the same mesh).

Tent-pitched elements

Tent-pitched elements/patches obtained from regular space meshes in 2+1D give parallelepipeds or octahedra+tetrahedra:



Trefftz requires quadrature on faces only: element shapes do not matter much, simplices around a tent pole can be considered a single element.

Relation with UWVF and finite differences

With $\alpha c = \beta/c = \delta = 1/2$, TDG operator reads $(\mathrm{Id} - F^*\Pi)$,
F isometry, Π "trace-flipping", as in Cessenat–Despres' UWVF.
True in 1+1D; only formally because of a trace issue in n+1D. . .

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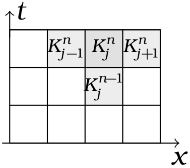
In 1+1D, without BCs, with piecewise constant basis, on Cartesian-product mesh, (implicit) TDG reads:

$$\frac{1}{c^2}\,\frac{v_j^n - v_j^{n-1}}{h_t} + \frac{\sigma_{j+1}^n - \sigma_{j-1}^n}{2h_x} = \alpha h_x\,\frac{v_{j-1}^n + v_{j+1}^n - 2v_j^n}{h_x^2},$$

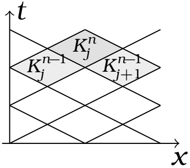
$$\frac{\sigma_j^n - \sigma_j^{n-1}}{h_t} + \frac{v_{j+1}^n - v_{j-1}^n}{2h_x} = \beta h_x\,\frac{\sigma_{j-1}^n + \sigma_{j+1}^n - 2\sigma_j^n}{h_x^2},$$



On a uniform rhombic mesh, with piecewise constant basis, (explicit) TDG is Lax–Friedrichs:

$$v_j^n = \frac{v_j^{n-1} + v_{j+1}^{n-1}}{2} - c^2 h_t \frac{\sigma_{j+1}^{n-1} - \sigma_j^{n-1}}{2h_x},$$

$$\sigma_j^n = \frac{\sigma_j^{n-1} + \sigma_{j+1}^{n-1}}{2} - h_t \frac{v_{j+1}^{n-1} - v_j^{n-1}}{2h_x},$$



TDG a priori error analysis

Using jumps and averages, define 2 mesh- and flux-dependent seminorms $|||\cdot|||_{DG} \le |||\cdot|||_{DG^+}$ on $H^1(\mathcal{T}_h)^{1+n}$, norms on $\mathbf{T}(\mathcal{T}_h)$.

$$\forall(v,\boldsymbol{\sigma}),(w,\boldsymbol{\tau}) \in \mathbf{T}(\mathcal{T}_h): \qquad (\alpha,\beta>0)$$

$$\mathcal{A}(v,\boldsymbol{\sigma};v,\boldsymbol{\sigma}) \ge |||(v,\boldsymbol{\sigma})|||_{DG}^2 \qquad \text{coercivity,}$$

$$|\mathcal{A}(v,\boldsymbol{\sigma};w,\boldsymbol{\tau})| \le 2\,|||(v,\boldsymbol{\sigma})|||_{DG^+}\;|||(w,\boldsymbol{\tau})|||_{DG} \qquad \text{continuity,}$$

$$\Downarrow$$

Existence & uniqueness of discrete solution (only for Trefftz!)

Stability and quasi-optimality:

$$|||(v-v_{hp},\boldsymbol{\sigma}-\boldsymbol{\sigma}_{hp})|||_{DG} \le 3 \inf_{(w_{hp},\boldsymbol{\tau}_{hp})\in\mathbf{V}_p(\mathcal{T}_h)} |||(v-w_{hp},\boldsymbol{\sigma}-\boldsymbol{\tau}_{hp})|||_{DG^+}.$$

Energy dissipation: (if $g=0$)

$$\frac{1}{2}\int_{\Omega\times\{T\}}(c^{-2}v_{hp}^2+|\boldsymbol{\sigma}_{hp}|^2)\,\mathrm{d}\mathbf{x} \le \frac{1}{2}\int_{\Omega\times\{0\}}(c^{-2}v_0^2+|\boldsymbol{\sigma}_0|^2)\,\mathrm{d}\mathbf{x}.$$

Stability and error bound in $L^2(Q)$ norm

Error bound in space–time $L^2(Q)$ norm follows if we have

$$\left\| \frac{w}{c} \right\|_{L^2(Q)} + \|\boldsymbol{\tau}\|_{L^2(Q)^n} \le C_{(\mathcal{T}_h,\alpha,\beta)} |||(w,\boldsymbol{\tau})|||_{DG} \quad \forall (w,\boldsymbol{\tau}) \in \mathbf{T}(\mathcal{T}_h).$$

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Using Monk–Wang "duality" technique, this holds if,
for the auxiliary inhomogeneous IBVP

$$\begin{cases} \nabla z + \partial\boldsymbol{\zeta}/\partial t = \boldsymbol{\Phi} & \text{in } Q, \quad \boldsymbol{\Phi} \in L^2(Q)^n, \\ \nabla\cdot\boldsymbol{\zeta} + c^{-2}\,\partial z/\partial t = \psi & \text{in } Q, \quad \psi \in L^2(Q), \\ z(\cdot,0) = 0, \quad \boldsymbol{\zeta}(\cdot,0) = \mathbf{0} & \text{on } \Omega, \\ z(\mathbf{x},\cdot) = 0 & \text{on } \partial\Omega\times(0,T), \end{cases}$$

the following stability bound holds:

$$2\left\|n_t^{\frac{1}{2}}\frac{z}{c}\right\|^2_{L^2(\mathcal{F}_h^{\mathrm{sp}}\cup\mathcal{F}_h^T)} + 2\left\|n_t^{\frac{1}{2}}\boldsymbol{\zeta}\right\|^2_{L^2(\mathcal{F}_h^{\mathrm{sp}}\cup\mathcal{F}_h^T)^n} + \left\|\frac{z}{\beta^{\frac{1}{2}}}\right\|^2_{L^2(\mathcal{F}_h^{\mathrm{time}})} + \left\|\frac{\boldsymbol{\zeta}\cdot\mathbf{n}_K^x}{\alpha^{\frac{1}{2}}}\right\|^2_{L^2(\mathcal{F}_h^{\mathrm{time}}\cup\mathcal{F}_h^{\partial})}$$
$$\leq C^2_{(\mathcal{T}_h,\alpha,\beta)}\Big(\|\boldsymbol{\Phi}\|^2_{L^2(Q)^n} + \|c\psi\|^2_{L^2(Q)}\Big)^2 \qquad \forall(\boldsymbol{\Phi},\psi)\in L^2(Q)^{n+1}.$$

When does "adjoint stability" hold?

1. 1D, constant c, decomposing solution in left and right waves, $C \sim T(N_x + N_t)^{1/2}$ on a Cartesian-product $N_x \times N_t$ mesh.

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1. 1D, constant c, decomposing solution in left and right waves, $C \sim T(N_x + N_t)^{1/2}$ on a Cartesian-product $N_x \times N_t$ mesh.

2. 1D, general c, with Gronwall + energy + integration by parts +

$$\alpha_{|K_1 \cap K_2} = \frac{\mathtt{a} h^x}{\min\{c^2_{|K_1} h^x_{K_1}, c^2_{|K_2} h^x_{K_2}\}}, \quad \beta_{|K_1 \cap K_2} = \frac{\mathtt{b} h^x}{\min\{h^x_{K_1}, h^x_{K_2}\}}$$

$\Rightarrow C \sim (1/\max_{K \in \mathcal{T}_h}\{h^x_K\} + \mathrm{e}^T N^{\mathrm{space}}_{\mathrm{interfaces}})^{1/2}$, hp-type bound.

When does "adjoint stability" hold?

1. 1D, constant c, decomposing solution in left and right waves, $C \sim T(N_x + N_t)^{1/2}$ on a Cartesian-product $N_x \times N_t$ mesh.

2. 1D, general c, with Gronwall + energy + integration by parts +

$$\alpha_{|K_1 \cap K_2} = \frac{\mathtt{a}\, h^x}{\min\{c^2_{|K_1} h^x_{K_1}, c^2_{|K_2} h^x_{K_2}\}}, \quad \beta_{|K_1 \cap K_2} = \frac{\mathtt{b}\, h^x}{\min\{h^x_{K_1}, h^x_{K_2}\}}$$

$\Rightarrow C \sim (1/\max_{K \in \mathcal{T}_h}\{h^x_K\} + \mathrm{e}^T N^{\text{space}}_{\text{interfaces}})^{1/2}$, hp-type bound.

3. nD, no time-like faces ($\mathcal{F}_h^{\text{time}} = \emptyset$), impedance BCs only,
$\Rightarrow C \sim T\, h_t^{-1/2}$ on uniform meshes.

All bounding constants are explicit.

For general case, need bound on traces of z, $\zeta \cdot \mathbf{n}_x$ in $L^2(\mathcal{F}_h^{\text{time}})$.

hp convergence bounds in 1+1D (and h in n+1D)

We prove fully-explicit hp best-approximation bounds in 1+1D.

Combined with quasi-optimality, give convergence bounds:

$$|||(v - v_{hp}, \sigma - \sigma_{hp})|||_{DG}$$

$$\leq \frac{12}{\sqrt{c}} \sum_{K \in \mathcal{T}_h} \left(6\Big(c + \frac{h_K^x}{h_K^t}\Big) + 8c\xi_K\Big(1 + c\frac{h_K^t}{h_K^x}\Big) \right)^{1/2} (\mathrm{e}/2)^{\frac{s_K^2}{p_K}}$$

$$\cdot \frac{\left(h_K^x + ch_K^t\right)^{s_K + \frac{3}{2}}}{p_K^{s_K}} \left(|v/c|_{W_c^{s_K+1,\infty}(K)} + |\sigma|_{W_c^{s_K+1,\infty}(K)} \right).$$

$(\xi_K := \|\max\{\alpha c;\ 1/\alpha c;\ \beta/c;\ c/\beta\}\|_{L^\infty(\partial K)})$ $\qquad 1 \leq s_K \leq p_K$

- Exponential convergence for analytic solutions:
  $\sim \exp(-b\#\mathrm{DOFs})$ instead of $\exp(-b\sqrt{\#\mathrm{DOFs}})$.

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&\qquad \cdot \frac{\left(h_K^x + ch_K^t\right)^{s_K + \frac{3}{2}}}{p_K^{s_K}} \left(|v/c|_{W_c^{s_K+1,\infty}(K)} + |\sigma|_{W_c^{s_K+1,\infty}(K)} \right).
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- Exponential convergence for analytic solutions: $\sim \exp(-b\#\mathrm{DOFs})$ instead of $\exp(-b\sqrt{\#\mathrm{DOFs}})$.
- For $n > 1$, approximation in p is hard, in h follows from Taylor:

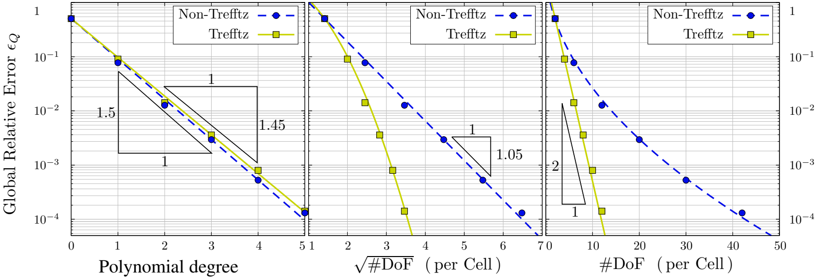
$-\Delta u + c^{-2}u'' = 0,\ u \in H^{p+1}(K),\ n = 2, 3 \Rightarrow\ \exists P \in \mathbb{T}^p(K)$ s.t. $\forall 0 \le j \le p$

$$|u - P|_{j,K} \le 4(1+j)^n \rho_0^{-2}\, h_K^{p+1-j}\, |u|_{p+1,K}$$

($K\star$-shaped wrt $B_{\rho_0 h_K}$).

Numerical example

Gaussian wave, uniform mesh of squares, p-convergence:



Very weak dependence on flux parameters, even for $\alpha, \beta = 0$.

Symmetric hyperbolic systems

As in MONK–RICHTER: piecewise-constant $\mathsf{A} > 0$, constant A_j

$$\begin{aligned} \mathsf{A}\mathbf{u}_t + \sum_j \mathsf{A}_j \mathbf{u}_{x_j} &= \mathbf{0} && \text{in } \Omega \times (0,T), \\ (\mathsf{D}-\mathsf{N})\mathbf{u} &= \mathbf{g} && \text{on } \partial\Omega \times (0,T), \\ \mathbf{u} &= \mathbf{u}_0 && \text{on } \Omega \times \{0\}, \end{aligned}$$

$$\mathsf{D}|_{\partial K} := \sum_j n_K^j \mathsf{A}_j,$$

+conditions on N.

Decomposition $\mathsf{M}|_{\partial K} := n_K^t \mathsf{A} + \sum_j n_K^j \mathsf{A}_j = \mathsf{M}_K^+ + \mathsf{M}_K^-$ such that $M^+ \geq 0$, $M^- \leq 0$, $\mathsf{M}_{K_1}^+ + \mathsf{M}_{K_2}^- = 0$ on $\partial K_1 \cap \partial K_2$, leads to

$$\mathcal{A}(\mathbf{u},\mathbf{w}) = \sum_{K_1,K_2} \int_{\partial K_1 \cap \partial K_2} \mathbf{u}_1 \cdot \mathsf{M}_{K_1}^+ (\mathbf{w}_1 - \mathbf{w}_2)\,\mathrm{d}S + \int_{\mathcal{F}_h^T} \mathbf{u} \cdot \mathsf{M}\mathbf{w}\,\mathrm{d}S + \frac{1}{2}\int_{\partial\Omega\times(0,T)} (\mathsf{D}+\mathsf{N})\mathbf{u}\cdot\mathbf{w}\,\mathrm{d}S,$$

$$\ell(\mathbf{w}) = -\int_{\mathcal{F}_h^0} \mathbf{u}_0 \cdot \mathsf{M}\mathbf{w}\,\mathrm{d}S - \frac{1}{2}\int_{\partial\Omega\times(0,T)} \mathbf{g}\cdot\mathbf{w}\,\mathrm{d}S.$$

$$|||\mathbf{u}|||_{DG}^2 := \mathcal{A}(\mathbf{u},\mathbf{u}) = \sum_{K_1,K_2} \int_{\partial K_1 \cap \partial K_2} (\mathbf{u}_1 - \mathbf{u}_2) \cdot \frac{\mathsf{M}^+ - \mathsf{M}^-}{2} (\mathbf{u}_1 - \mathbf{u}_2)\,\mathrm{d}S + \int_{\mathcal{F}_h^T \cup \mathcal{F}_h^0} \mathbf{u} \cdot \frac{\mathsf{M}^+ - \mathsf{M}^-}{2} \mathbf{u}\,\mathrm{d}S + \frac{1}{2}\int_{\partial\Omega\times(0,T)} \mathbf{u}\cdot\mathsf{N}\mathbf{u}\,\mathrm{d}S.$$

Maxwell's equations

$$\nabla \times \mathbf{E} + \frac{\partial(\mu \mathbf{H})}{\partial t} = \mathbf{0}, \qquad \nabla \times \mathbf{H} - \frac{\partial(\epsilon \mathbf{E})}{\partial t} = \mathbf{0} \quad \text{in } Q \subset \mathbb{R}^{3+1},$$

$$\mathbf{n}_\Omega^{\mathbf{x}} \times \mathbf{E} = \mathbf{n}_\Omega^{\mathbf{x}} \times \mathbf{g}(\mathbf{x}, t) \qquad \text{Dirichlet/PEC BCs},$$

$$\begin{cases} [\![\mathbf{v}]\!]_t := (\mathbf{v}^- - \mathbf{v}^+) \\ [\![\mathbf{v}]\!]_{\mathbf{T}} := \mathbf{n}_{K_1}^{\mathbf{x}} \times \mathbf{v}_{|K_1} + \mathbf{n}_{K_2}^{\mathbf{x}} \times \mathbf{v}_{|K_2} \end{cases} \qquad \text{(tangential) jumps.}$$

Trefftz-DG formulation defined by:

$$\begin{aligned}
\mathcal{A}_{\mathcal{M}}(\mathbf{E}_{hp}, \mathbf{H}_{hp}; \mathbf{v}, \mathbf{w}) = &\int_{\mathcal{F}_h^{\text{space}}} \Big(\epsilon \mathbf{E}_{hp}^- \cdot [\![\mathbf{v}]\!]_t + \mu \mathbf{H}_{hp}^- \cdot [\![\mathbf{w}]\!]_t - \mathbf{E}_{hp}^- \cdot [\![\mathbf{w}]\!]_{\mathbf{T}} + \mathbf{H}_{hp}^- \cdot [\![\mathbf{v}]\!]_{\mathbf{T}} \Big) \, \mathrm{d}S \\
&+ \int_{\mathcal{F}_h^T} (\epsilon \mathbf{E}_{hp} \cdot \mathbf{v} + \mu \mathbf{H}_{hp} \cdot \mathbf{w}) \, \mathrm{d}S + \int_{\mathcal{F}_h^\partial} \big(\mathbf{H}_{hp} + \alpha (\mathbf{n}_\Omega^{\mathbf{x}} \times \mathbf{E}_{hp}) \big) \cdot (\mathbf{n}_\Omega^{\mathbf{x}} \times \mathbf{v}) \, \mathrm{d}S \\
&+ \int_{\mathcal{F}_h^{\text{time}}} \Big(- \{\!\{\mathbf{E}_{hp}\}\!\} \cdot [\![\mathbf{w}]\!]_{\mathbf{T}} + \{\!\{\mathbf{H}_{hp}\}\!\} \cdot [\![\mathbf{v}]\!]_{\mathbf{T}} + \alpha [\![\mathbf{E}_{hp}]\!]_{\mathbf{T}} \cdot [\![\mathbf{v}]\!]_{\mathbf{T}} + \beta [\![\mathbf{H}_{hp}]\!]_{\mathbf{T}} \cdot [\![\mathbf{w}]\!]_{\mathbf{T}} \Big) \, \mathrm{d}S,
\end{aligned}$$

$$\ell_{\mathcal{M}}(\mathbf{v}, \mathbf{w}) = \int_{\mathcal{F}_h^0} (\epsilon \mathbf{E}_0 \cdot \mathbf{v} + \mu \mathbf{H}_0 \cdot \mathbf{w}) \, \mathrm{d}S + \int_{\mathcal{F}_h^\partial} (\mathbf{n}_\Omega^{\mathbf{x}} \times \mathbf{g}) \cdot \big(- \mathbf{w} + \alpha (\mathbf{n}_\Omega^{\mathbf{x}} \times \mathbf{v}) \big) \, \mathrm{d}S.$$

Well-posedness and stability identical to wave equation.
Explicit approximation bounds in h. Impedance BCs also fine.

Extensions and open problems

- More general space–time meshes (not aligned to t);
- non/less dissipative methods (is our dissipation too much?);
- analysis of non-penalised methods ($\alpha = \beta = 0$);
- L^2 stability in more general cases;
- Maxwell, elasticity, first-order hyperbolic systems, dispersive/Drude-type models for plasmas, . . . ;
- Trefftz hp-approximation theory in dimensions > 1;
- other bases: non-polynomial, trigonometric, directional. . . ;
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Thank you!