Time approximation of transient boundary integral equations

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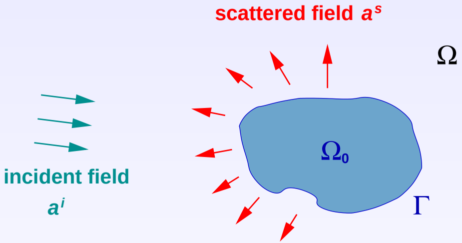
Motivation: acoustic scattering

Problem: $a^i(x,t)$ is incident on Γ for $t > 0$ – find the scattered field $a^s(x,t)$



- PDE: $a^s_{tt} = \Delta a^s$ in Ω (wave speed is $c = 1$);
- BC: $a^s + a^i = 0$ on Γ
- TDBIE: a^s can be obtained from surface potential u:

$$\frac{1}{4\pi}\int_\Gamma \frac{u(\mathbf{x}', t-|\mathbf{x}'-\mathbf{x}|)}{|\mathbf{x}'-\mathbf{x}|}\, d\sigma_{\mathbf{x}'} = -a^i(\mathbf{x},t) \quad \mathbf{x}\in\Gamma,\ t>0$$

Approx TDBIE in time & space*

$$\sum_{m=0}^{n} Q^m \underline{U}^{n-m} = \underline{a}^n\,, \quad n = 1 : N_T \quad (^{*}\text{time Galerkin – see later})$$

($\underline{U}^n \in \mathbb{R}^{N_S}$ is spatial approx of u at/near $t^n = n\,h$, Q^m are matrices)

time-stepping scheme: $Q^0\,\underline{U}^n = \underline{a}^n - \sum_{m=1}^{n} Q^m\,\underline{U}^{n-m}$

- Choose space mesh size $\simeq$ time step h, so $N_T \simeq N$, $N_S \simeq N^2$
- Sparsity of matrices Q^m given by (effective) support of time basis functions:
  - Galerkin in time: time BFs are **local**
  - Convolution quadrature (CQ) in time: scheme is very stable, but effective support of BFs is greater and increases with m
  - **Comp complexity** – extra power of $\sqrt{N}$ for both setup & run times for CQ

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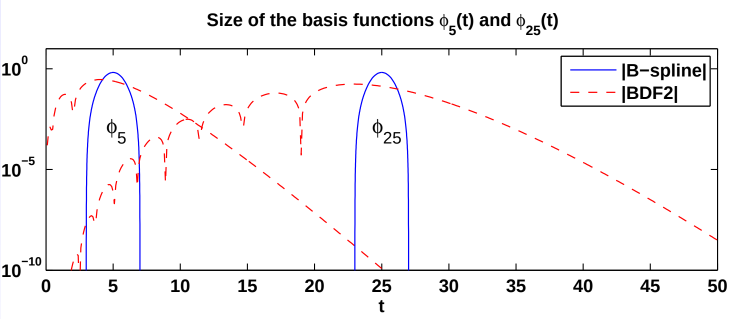
  - **Ideal**: local time BFs with stability of CQ: "convolution spline"

Time basis functions



Basis functions for CQ (BDF2) and cubic splines

- **TDBIE comp complexity** – extra power of $\sqrt{N}$ for both setup & run times for CQ

"Convolution spline" approx in time

- Derive for convolution Volterra integral equation (VIE):

$$\int_0^t K(\tau)\, u(t-\tau)\, d\tau = a(t)$$

- Idea: construct a **backwards-in-time approx** in terms of local basis functions – gives sparse Q^m matrices for TDBIEs

- Approx is:

$$u(t_n - t) \approx \sum_{m=0}^{n} u_{n-m}\, \phi_m(t/h) \quad \textbf{NOT} \quad u(t) \approx \sum_k u_k\, \phi_k(t/h)$$

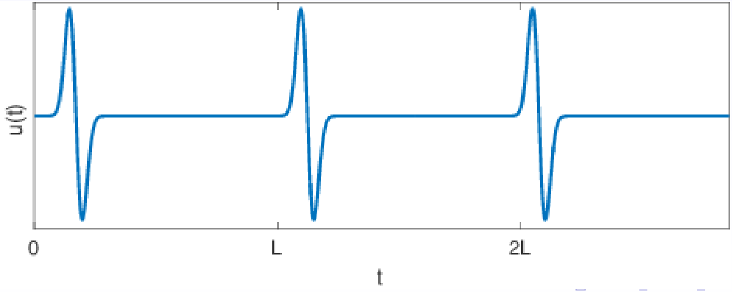
- Basis functions $\phi_m(t)$ are cubic B-splines with parabolic runout conditions at $t = 0$ (so translates for $m \geq 3$)

- New results for VIE: stability and 4th order convergence for kernels K which are piecewise smooth (can be **discontinuous**)

Simple VIE example

$$\int_0^t K(\tau)\, u(t-\tau)\, d\tau = a(t)\,, \quad \text{where} \quad K(t) = \begin{cases} 1, & t \le L \\ 0, & \text{otherwise} \end{cases}$$

exact solution: $\displaystyle u(t) = \sum_{k=0}^{\lfloor t/L \rfloor} a'(t - k\,L)$



VIE approx: key Gronwall result

Standard Gronwall:

$$x_n \le a + b \sum_{j=0}^{n-1} x_j \quad \Longrightarrow \quad x_n \le a\,(1+b)^n \le a\,e^{bn}$$

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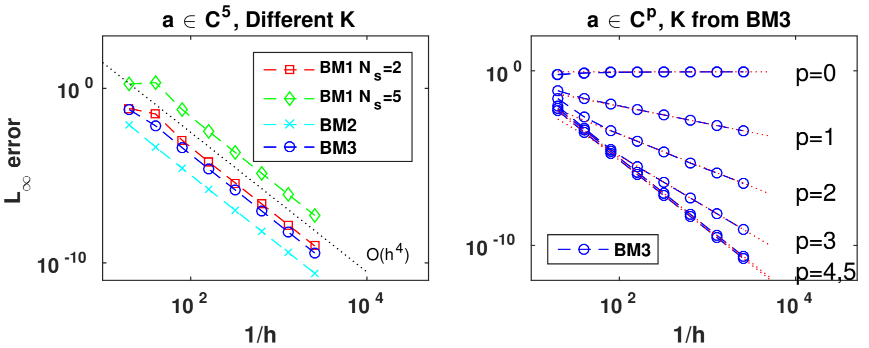
Extension – include contribution from M steps back:

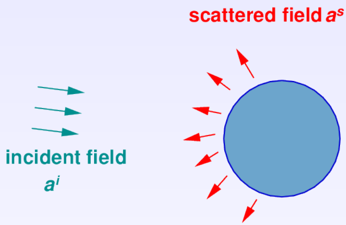
$$x_n \le a + b \sum_{j=0}^{n-1} x_j + \mathbf{c\,x_{n-M}} \quad \Longrightarrow \quad x_n \le a\,(1+b)^n \mathbf{(1+c)^{\lfloor n/M \rfloor}}$$

– Can show that convolution spline scheme is stable and 4th order accurate when kernel K is discontinuous

Numerical results



Scattering from unit sphere $\mathbb{S}$



- Spherical harmonic expansion: $(R = |\boldsymbol{x}|, \ \widehat{\boldsymbol{x}} = \boldsymbol{x}/R)$

$$a^i(\boldsymbol{x}, t) = \sum_{\ell,m} a^i_{\ell,m}(R,t)\, Y^m_\ell(\widehat{\boldsymbol{x}})$$

- Y^m_ℓ are eigenfunctions of the single layer potential operator on $\mathbb{S}$ (e.g. Nédélec) $\implies$ spherical harmonic representation of u and a^s

Potential u on $\mathbb{S}$

- TDBIE for surface potential u:

$$\frac{1}{4\pi}\int_{\mathbb{S}} \frac{u(\boldsymbol{x}', t-|\boldsymbol{x}'-\boldsymbol{x}|)}{|\boldsymbol{x}'-\boldsymbol{x}|}\, d\sigma_{\boldsymbol{x}'} = -a^i(\boldsymbol{x}, t) \quad \boldsymbol{x} \in \mathbb{S},\ t > 0$$

- Spherical harmonic expansion – everything **decouples**:

$$a^i(\boldsymbol{x}, t) = \sum_{\ell,m} a^i_{\ell,m}(R, t)\, Y^m_\ell(\widehat{\boldsymbol{x}}) \quad \Longrightarrow \quad u(\widehat{\boldsymbol{x}}, t) = \sum_{\ell,m} u_{\ell,m}(t)\, Y^m_\ell(\widehat{\boldsymbol{x}})$$

- Separate step-kernel VIE problem for each $u_{\ell,m}$ [Sauter & Veit, 2011]

$$\int_0^t K_\ell(\tau)\, u_{\ell,m}(t-\tau) = -a^i_{\ell,m}(1, t)\,,$$

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$$\int_0^t K_\ell(\tau)\, u_{\ell,m}(t-\tau) = -a^i_{\ell,m}(1,t)\,, \quad K_\ell(t) = \begin{cases} \frac{1}{2}\, P_\ell(1-t^2/2), & t\le 2 \\ 0, & t>2 \end{cases}$$

(P_ℓ is Legendre polynomial) **New?**

Scattered field a^s from $\mathbb{S}$

$$a^s(\boldsymbol{x}, t) = \sum_{\ell,m} a^s_{\ell,m}(R,t)\, Y_\ell^m(\widehat{\boldsymbol{x}})$$

- Components ($t \geq 2$):

$$a^s_{\ell,m}(R,t) = \frac{1}{2R} \int_{R-1}^{R+1} P_\ell(1-\tau^2/2)\, u_{\ell,m}(t-\tau)$$

$$\text{where} \quad \frac{1}{2}\int_0^2 P_\ell(1-\tau^2/2)\, u_{\ell,m}(t-\tau) = -a^i_{\ell,m}(1,t)$$

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  - **Problem:** our error bound for approx u involves $\|K_\ell'\| = \ell(\ell+1)\ \ldots$

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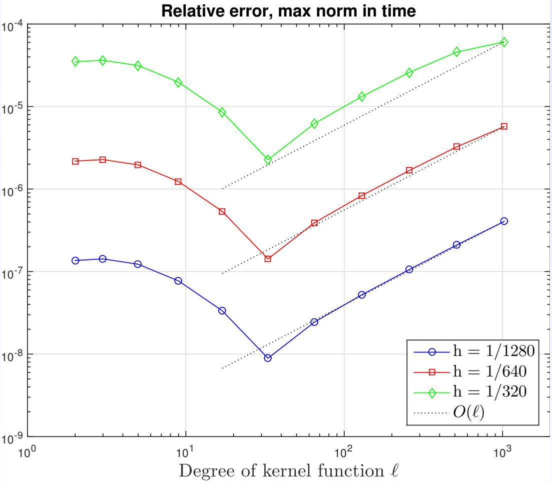
  - **Problem:** our error bound for approx u involves $\|K_\ell'\| = \ell(\ell+1)$ ...
  - **Problem ...** as an exponential!
  - **But:** scheme works much better than this in practice ...

Numerical results for VIE solution $u_{\ell,m}$



Connections: space–time Galerkin and convolution spline

- Ha Duong: TDBIE variational formulation – stability of full space–time Galerkin approximation
- **But** Galerkin methods are typically **not** in time-marching form – very expensive to implement without modification
- **Strategy:** find a modified variational formulation with the following properties.
  - its exact and (Galerkin) approx solutions are close to those for the unmodified version
  - its Galerkin approx is equivalent to a convolution spline (time-marching) scheme
  - the CS scheme's basis functions are globally smooth enough to make quadrature efficient
- Could then use Ha Duong (Galerkin) analysis for convolution spline

Ha Duong: Galerkin variational formulation

- TDBIE (single layer potential) for surface potential u:

$$(Su)(\boldsymbol{x},t) := \frac{1}{4\pi}\int_\Gamma \frac{u(\boldsymbol{x}', t-|\boldsymbol{x}'-\boldsymbol{x}|)}{|\boldsymbol{x}'-\boldsymbol{x}|}\, d\sigma_{\boldsymbol{x}'} = -a^i(\boldsymbol{x},t) \quad \boldsymbol{x}\in\Gamma,\ t\in[0,T]$$

- Approx solution in terms of unknowns U_k^n:

$$u(\boldsymbol{x},t) \approx u_h(\boldsymbol{x},t) := \sum_{n=1}^{N_T}\sum_{k=1}^{N_S} U_k^n\, \psi_k(\boldsymbol{x})\, \phi_n(t/h) \in V_h$$

- Galerkin approx uses **time differentiated TDBIE** $S\dot{u} = -\dot{a}_i$ **not** $Su = -a_i$:

$$a(q_h, u_h; T) := \int_0^T\int_\Gamma q_h\, S\dot{u}_h\, d\sigma_{\boldsymbol{x}}\, dt = -\int_0^T\int_\Gamma q_h\, \dot{a}^i\, d\sigma_{\boldsymbol{x}}\, dt$$

for each $q_h = \psi_j(\boldsymbol{x})\,\phi_m(t/h) \in V_h$

Galerkin is **not** usually a time-marching scheme...

- It is when ϕ_m are **piecewise constants** in time, but not in general
- **Example:** $\phi_m(t/h) = B_1(t/h - m)$ – translates of 1st order B-spline (hat functions)
- Resulting linear system for the $\underline{U}^j \in \mathbb{R}^{N_S}$ is: $\underline{U}^0 = 0$,

$$Q^\star \, \underline{\boldsymbol{U}}^{\boldsymbol{n+1}} + \sum_{m=0}^{n} Q^m \, \underline{U}^{n-m} = \underline{a}^n \,, \quad n = 1 : N_T - 1 \text{ (modified at } n = N_T)$$

Picture for $N_T = 4$ ($*$ is a non-zero block $N_S \times N_S$ matrix):

$$\begin{pmatrix} * & * & 0 & 0 \\ * & * & * & 0 \\ * & * & * & * \\ * & * & * & * \end{pmatrix} \begin{pmatrix} \underline{U}^1 \\ \underline{U}^2 \\ \underline{U}^3 \\ \underline{U}^4 \end{pmatrix} = \begin{pmatrix} \underline{a}^1 \\ \underline{a}^2 \\ \underline{a}^3 \\ \underline{a}^4 \end{pmatrix}$$

. . . it can be **modified** to give a time-marching scheme

- Can **either** apply Galerkin (B_1 in time) to modified variational problem

$$a(q_h, u_h; T) + h^2 \int_0^T \int_\Gamma \int_\Gamma \frac{\dot{q}_h(\boldsymbol{x}, t) F(|\boldsymbol{x} - \boldsymbol{y}|) \dot{u}_h(\boldsymbol{y}, t)}{|\boldsymbol{x} - \boldsymbol{y}|} \, d\sigma_{\boldsymbol{x}} \, d\sigma_{\boldsymbol{y}} \, dt = \text{Galerkin RHS}$$

  where $F(r) = B_2(r/h + 1/2) =$ second order B-spline

- **Or** modify Galerkin scheme using extrapolation $\underline{U}^{n+1} \approx 2\,\underline{U}^n - \underline{U}^{n-1}$
- Both approaches give the same **time-marching** scheme:

$$-Q^\star(\cancel{\underline{U}^{n+1}} - 2\,\underline{U}^n + \underline{U}^{n-1}) + Q^\star \cancel{\underline{U}^{n+1}} + \sum_{m=0}^{n} Q^m \, \underline{U}^{n-m} = \underline{a}^n$$

$$\text{i.e.} \quad \left(2Q^\star + Q^0\right) \underline{U}^n + \left(Q^1 - Q^\star\right) \underline{U}^{n-1} + \sum_{m=2}^{n} Q^m \, \underline{U}^{n-m} = \underline{a}^n$$

Connections: B_1-Galerkin and convolution spline

- Modified Galerkin scheme based on B_1 (linears) in time:

$$\left(2Q^\star + Q^0\right)\underline{U}^n + \left(Q^1 - Q^\star\right)\underline{U}^{n-1} + \sum_{m=2}^{n} Q^m\,\underline{U}^{n-m} = \underline{a}^n$$

- Convolution spline scheme with B_3-basis functions also applied to time differentiated TDBIE $\quad S\dot{u} = -\dot{a}^i$ is

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  – **same matrices** and $\underline{a}_C^n \approx \underline{a}^n$

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- **Shown:** direct connection with convolution spline scheme with B_2-basis functions applied to $Su = -a^i$, via

$$\dot{B}_3(t) = B_2(t+1/2) - B_2(t-1/2)$$

Connections: Galerkin and convolution spline

Galerkin time basis (app to $S\dot{u} = -\dot{a}^i$)	Convolution spline (app to $Su = -a^i$)	Conv spline stable for VIE?
$\boldsymbol{B_1(t)}$	$\boldsymbol{B_2(t)}$	Yes
?	$\boldsymbol{B_3(t)}$	Yes
$B_2(t)$	$B_4(t)$	Yes
$B_3(t)$	$B_6(t)$	No

- Ha Duong: standard (unmodified) Galerkin scheme should be stable
- **Shown:** Modified B_1-Galerkin scheme is equivalent to B_2-convolution splines
- Shown (probably!): exact solutions of standard and modified B_1 variational formulation differ by $\mathcal{O}(h^2)$. Approximate solutions are harder...
- B_4-convolution spline looks similar to B_2-Galerkin, but details are messy
- B_3-convolution spline? Fractional spline Galerkin? Petrov-Galerkin method??
- Petrov-Galerkin approach: [Elwin van 't Wout et al, 2016]

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- **Scattering from Γ:** strategy is to exploit connection between convolution spline and (modified) Galerkin to use Ha Duong stability analysis