Connections between collocation (convolution spline) and Galerkin methods for TDBIEs

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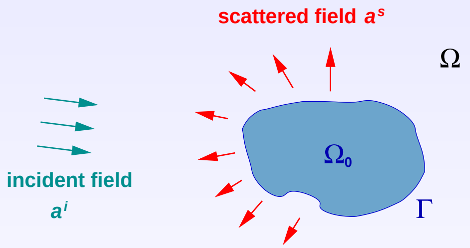
Acoustic scattering – notation

Problem: $\mathbf{a}^{\mathbf{i}}(\mathbf{x},\mathbf{t})$ is incident on Γ for $t > 0$ – find the scattered field $\mathbf{a}^{\mathbf{s}}(\mathbf{x},\mathbf{t})$



- PDE: $\mathbf{a}^{\mathbf{s}}_{\mathbf{tt}} = \mathbf{\Delta a}^{\mathbf{s}}$ in Ω (wave speed is $c = 1$);
- BC: $\mathbf{a}^{\mathbf{s}} + a^i = 0$ on Γ IC: a^i reaches Γ at $t > 0$
- TDBIE: $\mathbf{a}^{\mathbf{s}}$ can be obtained from surface potential u:

$$\frac{1}{4\pi}\int_{\Gamma}\frac{u(\mathbf{x}', t-|\mathbf{x}'-\mathbf{x}|)}{|\mathbf{x}'-\mathbf{x}|}\, d\sigma_{\mathbf{x}'} = -a^i(\mathbf{x}, t) \quad \mathbf{x}\in\Gamma,\ t>0$$

Connections: space–time Galerkin and convolution spline

- Ha Duong: TDBIE variational formulation – stability of full space–time Galerkin approximation
- **But** Galerkin methods are typically **not** in time-marching form – very expensive to implement without modification
- **Strategy:** find a modified variational formulation with the following properties.
  - its exact and (Galerkin) approx solutions are close to those for the unmodified version
  - its Galerkin approx is equivalent to a convolution spline (time-marching) scheme
  - the CS scheme's basis functions are globally smooth enough to make quadrature efficient
- Could then use Ha Duong (Galerkin) analysis for convolution spline

Ha Duong: variational formulation

- TDBIE (single layer potential) for surface potential u:

$$(Su)(\mathbf{x},t) := \frac{1}{4\pi}\int_\Gamma \frac{u(\mathbf{x}',t-|\mathbf{x}'-\mathbf{x}|)}{|\mathbf{x}'-\mathbf{x}|}\,d\sigma_{\mathbf{x}'} = -a^i(\mathbf{x},t) \quad \mathbf{x}\in\Gamma,\ t\in[0,T]$$

- **Equivalent exact variational form:** find $u \in V$, s.t. $\forall q \in V$, $t \in [0,T]$

$$a(u,q;t) := \int_0^t\int_\Gamma q\,\mathbf{S\dot{u}}\,d\sigma_{\mathbf{x}}\,dt = -\int_0^t\int_\Gamma q\,\mathbf{\dot{a}^i}\,d\sigma_{\mathbf{x}}\,d\tau$$

Note: **time differentiated TDBIE** $S\dot{u} = -\dot{a}^i$ **not** $Su = -a^i$

- **Energy** of scattered field a^s is

$$E(u;t) = a(u,u;t) = \frac{1}{2}\int_\Omega \left(|\nabla a^s|^2 + |\dot{a}^s|^2\right)dx$$

Space-time Galerkin approximation

- Approx solution in terms of unknowns U_k^n:

$$u(\mathbf{x},t) \approx u_h(\mathbf{x},t) := \sum_{n=1}^{N_T}\sum_{k=1}^{N_S} U_k^n\,\phi_k(\mathbf{x})\,\boldsymbol{\psi_n(t)} \in V_h$$

- **Approx variational form:** for each $q_h = \phi_j(\mathbf{x})\boldsymbol{\psi_n(t)} \in V_h$

$$a(u_h,q_h;T) = \int_0^{\mathbf{T}}\boldsymbol{\psi_n(t)}\int_\Gamma\int_\Gamma\sum_{k=1}^{N_S}\sum_{m=1}^{N_T}\frac{U_k^m\,\phi_k(\mathbf{x})\,\phi_j(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\;\boldsymbol{\dot\psi_m(t-|x-y|)}\,d\sigma_{\mathbf{x}}\,d\sigma_{\mathbf{y}}\,\mathbf{dt}$$

$$= \sum_{k=1}^{N_S}\sum_{m=1}^{N_T} U_k^m\int_\Gamma\int_\Gamma\frac{\phi_k(\mathbf{x})\phi_j(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\;\boldsymbol{\beta_{m,n}(|x-y|)}\,d\sigma_{\mathbf{x}}d\sigma_{\mathbf{y}} = -\int_0^T\int_\Gamma q_h\dot a^i\,d\sigma_{\mathbf{x}}dt$$

where $\boldsymbol{\beta_{m,n}(r) := \int_0^T \psi_n(t)\,\dot\psi_m(t-r)\,dt = \int_r^T \psi_n(t)\,\dot\psi_m(t-r)\,dt}$

(the time basis functions have support in $[0, T]$)

A nice property of B-splines

- $\beta_{m,n}(r) := \int_0^T \psi_n(t)\,\dot{\psi}_m(t-r)\,dt$

- If $\psi_n(t) = B_\ell(t/h - n)$ then

$$\begin{aligned}
\beta_{m,n}(r) &= \int_0^T B_\ell(t/h - n)\,\dot{B}_\ell(t/h - m - r/h)\,dt \\
&= h\left(B_{2\ell}\left(\frac{r}{h} - \frac{1}{2} + m - n\right) - B_{2\ell}\left(\frac{r}{h} + \frac{1}{2} + m - n\right)\right) \\
&= -h\dot{B}_{2\ell+1}\left(\frac{r}{h} + m - n\right)
\end{aligned}$$

- Note: needs some modification near 0 and T
- Also works for Petrov Galerkin $(B_\ell, B_{\ell'}) \to B_{\ell+\ell'}$

B_ℓ time basis functions for Galerkin

- Approx variational form: for each $q_h = \phi_j(\mathbf{x})\psi_n(t) \in V_h$

$$a(u_h, q_h; T) = -\int_0^T \int_\Gamma q_h \, \dot{a}^i \, d\sigma_{\mathbf{x}} \, dt$$

- Assemble into matrix–vector form for each $n \leq N_T$:

$$\sum_{m=1}^{N_T} \widehat{Q}_{m,n} \, \mathbf{U}^m = \mathbf{a}^n, \quad \widehat{Q}_{m,n} = \int_\Gamma \int_\Gamma \frac{\boldsymbol{\phi}(\mathbf{x})\boldsymbol{\phi}^T(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} \, \boldsymbol{\beta}_{\mathbf{m},\mathbf{n}}(|\mathbf{x}-\mathbf{y}|) \, d\sigma_{\mathbf{x}} d\sigma_{\mathbf{y}}$$

  where $\boldsymbol{\phi}^T = (\phi_1, \ldots, \phi_{N_S})$ and $\widehat{Q}_{m,n} \in \mathbb{R}^{N_S \times N_S}$

- Note: **most $\widehat{\mathbf{Q}}_{\mathbf{m},\mathbf{n}} = \mathbf{Q}^{\mathbf{n}-\mathbf{m}}$** since most $\beta_{m,n} = -h\dot{B}_{2\ell+1}(r/h + m - n)$ – so mainly a convolution sum
- $\beta_{m,n}$ are degree 2ℓ and are globally $C^{2\ell-1}$, so quadrature can be done over space elements only

Galerkin is **not** usually a time-marching scheme. . .

- **Example:** $\psi_m(t/h) = B_1(t/h - m)$ – translates of 1st order B-spline (hat functions)

- Resulting linear system for the $\mathbf{U}^j \in \mathbb{R}^{N_s}$ is: $\mathbf{U}^0 = 0$,

$$Q^\star \mathbf{U^{n+1}} + \sum_{m=0}^{n} Q^m \mathbf{U}^{n-m} = \mathbf{a}^n, \quad n = 1 : N_T - 1 \text{ (modified at } n = N_T)$$

- Use extrapolation: $\mathbf{U}^{n+1} \approx 2\,\mathbf{U}^n - \mathbf{U}^{n-1}$ to get **modified** scheme

Modified (B_1 in time) Galerkin

- Extrapolation is equivalent to modified variational problem:

$$a(u_h, q_h; T) + \underbrace{h^2 \int_0^T \int_\Gamma \int_\Gamma \frac{\dot{q}_h(\mathbf{x}, t) F(|\mathbf{x} - \mathbf{y}|) \dot{u}_h(\mathbf{y}, t)}{|\mathbf{x} - \mathbf{y}|} \, d\sigma_{\mathbf{x}} d\sigma_{\mathbf{y}} dt}_{b(\dot{u}_h, \dot{q}_h; T)} = \text{Galerkin RHS}$$

  where $F(r) = B_2(r/h + 1/2) =$ second order B-spline

- Gives a **time-marching** scheme:

$$\text{i.e.} \quad \left(2Q^\star + Q^0\right) \mathbf{U}^n + \left(Q^1 - Q^\star\right) \mathbf{U}^{n-1} + \sum_{m=2}^{n} Q^m \, \mathbf{U}^{n-m} = \mathbf{a}^n$$

- Using B_3-basis functions also applied in convolution spline form to time differentiated TDBIE $\quad S\dot{u} = -\dot{a}^i$ gives **same** matrices and slightly altered RHS

Compare **exact** solutions of variational formulations

- **Original:** find $u \in V$ such that for all $q \in V$, $t \in [0, T]$

$$a(u, q; t) = -\int_0^t \int_\Gamma q(\mathbf{x}, \tau)\, a^i(\mathbf{x}, \tau)\, d\sigma_{\mathbf{x}}\, d\tau$$

- **Modified:** find $v \in V$ such that for all $q \in V$, $t \in [0, T]$

$$a(v, q; t) + h^2\, b(\dot{v}, \dot{q}; t) = -\int_0^t \int_\Gamma q(\mathbf{x}, \tau)\, a^i(\mathbf{x}, \tau)\, d\sigma_{\mathbf{x}}\, d\tau$$

- So $\quad a(v - u, q; t) + h^2\, b(\dot{v} - \dot{u}, \dot{q}; t) = -h^2 b(\dot{u}, \dot{q}; t)$

- Set $q = v - u$ to get an **energy-like** expression:

$$a(v - u, v - u; t) + h^2\, b(\dot{v} - \dot{u}, \dot{v} - \dot{u}; t) = -h^2\, b(\dot{u}, \dot{v} - \dot{u}; t)$$

What's known about energy $E(u,t) = a(u,u;t)$?

- $E(u,t) \geq 0$
- Ha Duong's results concern its **time integral**:

$$\alpha \|u\|^2_{\mathcal{H}^{-1/2}} \leq \int_0^T a(u,u;t)dt = \int_0^T E(u;t)dt \leq \beta \|u\|_{\mathcal{H}^{-1/2}} \|\dot{a}^i\|_{\mathcal{H}^{1/2}}$$

$$\Rightarrow \quad \|u\|_{\mathcal{H}^{-1/2}} \leq \frac{\beta}{\alpha} \|\dot{a}^i\|_{\mathcal{H}^{1/2}}$$

- What is the space $\mathcal{H}^{-1/2}$? Ha Duong uses
  - $(H_{00}^{1/2,1/2})'$ (from Lions & Magenes) in earlier work, including PhD thesis
  - $\mathcal{H}^{-1/2} = H^{-1/2}(0,T;L^2(\Gamma)) \cap L^2(0,T;H^{-1/2}(\Gamma))$ in 2003 survey article

- Note: rearranging gives:

$$\int_0^T a(u,u;t)\,dt = \int_0^T \int_0^t \int_\Gamma u\,(S\dot{u})\,d\sigma_{\mathbf{x}}\,d\tau\,dt = \int_0^T \mathbf{(T-t)} \int_\Gamma u\,(S\dot{u})\,d\sigma_{\mathbf{x}}\,dt$$

Stability in modified variational problem – exact

- **Exact:** find $v \in V$ s.t.

$$a(v, q; t) + h^2 b(\dot{v}, \dot{q}; t) = -\int_0^t \int_\Gamma q \dot{a}^i \, d\sigma_{\mathbf{x}} \, d\tau$$

for all $q \in V$ and all $t \in [0, T]$

- **Exact:** Ha Duong coercivity and upper bound give stability

$$\alpha \|v\|^2_{\mathcal{H}^{-1/2}} \leq \int_0^T \Big(a(v, v; t) + h^2 \underbrace{b(\dot{v}, \dot{v}; t)}_{\geq 0}\Big) dt$$

$$= -\int_0^T \int_0^t \int_\Gamma v \dot{a}^i \, d\sigma_{\mathbf{x}} \, d\tau \, dt \leq \beta \|v\|_{\mathcal{H}^{-1/2}} \|\dot{a}^i\|_{\mathcal{H}^{1/2}}$$

$$\Rightarrow \quad \|v\|_{\mathcal{H}^{-1/2}} \leq \frac{\beta}{\alpha} \|\dot{a}^i\|_{\mathcal{H}^{1/2}}$$

Stability in modified variational problem - Galerkin approx

- **Approx:** find $v_h \in V_h$ s.t. for all $q_h \in V_h$

$$a(v_h, q_h; T) + h^2 b(\dot{v}_h, \dot{q}_h; T) = -\int_0^T \int_\Gamma q_h \dot{a}^i \, d\sigma_{\mathbf{x}} d\tau$$

- **Approx:** stability would follow from coercivity (OK) and upper bound (???)

$$\alpha \|v_h\|^2_{\mathcal{H}^{-1/2}} \leq \int_0^T a(v_h, v_h; t) + h^2 \underbrace{b(\dot{v}_h, \dot{v}_h; t)}_{\geq 0} \; dt \leq \boldsymbol{\beta \|v_h\|_{\mathcal{H}^{-1/2}} \|\dot{a}^i\|_{\mathcal{H}^{1/2}} ???}$$

- Upper bound appears to need

$$a(v_h, v_h; \mathbf{t}) + h^2 b(\dot{v}_h, \dot{v}_h; \mathbf{t}) = -\int_0^{\mathbf{t}} \int_\Gamma v_h \dot{a}^i \, d\sigma_{\mathbf{x}} d\tau \quad \forall \mathbf{t} \in [0, T]$$

to work in simple way – **but this is not true in general, only for** t=T.

Bound difference between exact solns of original and modified variational problems

- Set $\mathbf{q} = \mathbf{v} - \mathbf{u}$, the difference in solutions. We have

$$\underbrace{a(q, q; t) + h^2 b(\dot{q}, \dot{q}; t)}_{\text{LHS}} = \underbrace{-h^2 b(\dot{u}, \dot{q}; t)}_{\text{RHS}}$$

- Can show that $b(\dot{q}, \dot{q}; t) \geq 0$, so using Ha Duong coercivity

$$\begin{aligned} \alpha \|q\|^2_{\mathcal{H}^{-1/2}} &\leq \int_0^T \text{LHS}\, dt = \int_0^T \text{RHS}\ dt \\ &\leq h^2 C_1 \|q\|_{\mathcal{H}^{-1/2}} \left(\|\partial_t u\|_{\mathcal{H}^{-1/2}} + T \|\partial_t^2 u\|_{\mathcal{H}^{-1/2}} \right) \\ &\leq h^2 C_2 \|q\|_{\mathcal{H}^{-1/2}} \left(\|\partial_t^2 a^i\|_{\mathcal{H}^{1/2}} + T \|\partial_t^3 a^i\|_{\mathcal{H}^{1/2}} \right) \end{aligned}$$

provided a^i is well-enough behaved.

- Finally

$$\|v - u\|_{\mathcal{H}^{-1/2}} \leq h^2 C_2 \left(\|\partial_t^2 a^i\|_{\mathcal{H}^{1/2}} + T \|\partial_t^3 a^i\|_{\mathcal{H}^{1/2}} \right)$$

Difference between approximate solutions

- Would like to bound the difference between the Galerkin approximate solutions of the two variational problems.
- Galerkin approx: find $u_h, v_h \in V_h$ such that, for all $q_h \in V_h$

$$\underbrace{a(u_h, q_h; T)}_{\text{original}} = -\int_0^T \int_\Gamma q_h \dot{a}^i \, d\sigma_{\mathbf{x}} \, dt = \underbrace{a(v_h, q_h; T) + h^2 b(v_h, q_h; T)}_{\text{modified}}$$

- Set $\mathbf{q_h = v_h - u_h}$ and subtract original from modified:

$$a(q_h, q_h; T) + h^2 b(\dot{q}_h, \dot{q}_h; T) = -h^2 b(\dot{u}_h, \dot{q}_h; T)$$

- Coercivity same as for the exact solutions:

$$\alpha \|q_h\|^2_{\mathcal{H}^{-1/2}} \leq \int_0^T \left(a(q_h, q_h; t) + h^2 b(\dot{q}_h, \dot{q}_h; t) \right) dt$$

but the upper bound is not clear, since we do not know a way to bound $\|\partial_t u_h\|_{\mathcal{H}^{-1/2}}$ and $\|\partial_t^2 u_h\|_{\mathcal{H}^{-1/2}}$ in terms of derivatives of a^i.

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  - equivalent to B_2 "convolution spline" method with explicit time marching
  - modified variational problem inherits coercivity property
  - Galerkin approx of modified variational problem stable ???
  - solutions of original and modified variational problems differ by $\mathcal{O}(h^2)$
  - Galerkin approx of original and modified variational problems differ by ???

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  – simple formula for core time calculation in Galerkin approx
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- **Outlook:** dig deeper in Ha Duong's orginal work to understand and fix gaps above (or ask the audience)
  – tidy up B_2 spline Galerkin and explicit time marching B_4 "convolution spline" counterpart