Microscopic behavior for β-ensembles: an "energy approach"

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Physical system: N particles $\vec{X}_N = (x_1, \dots, x_N)$ in $\mathbb{R}^d$ $(d = 1, 2)$.
Logarithmic pair interaction $-\log|x - y|$ + confining potential $NV(x)$.

Energy in the state $\vec{X}_N$:

$$\mathcal{H}_N(\vec{X}_N) = \sum_{1 \leq i \neq j \leq N} -\log|x_i - x_j| + \sum_{i=1}^{N} NV(x_i)$$

V "strongly confining" ex. $V(x) \geq (2 + s)\log|x|$ for $|x|$ large.
+ Mild regularity assumptions (see later).

β = "inverse temperature"

Canonical Gibbs measure

$$d\mathbb{P}_{N,\beta}(\vec{X}_N) = \frac{1}{Z_{N,\beta}} \exp\left(-\frac{\beta}{2}\mathcal{H}_N(\vec{X}_N)\right) d\vec{X}_N$$

$Z_{N,\beta}$ = normalization constant = "partition function".

Physical motivation

- $d = 2$: Coulomb systems, fluid mechanics, Ginzburg-Landau.
- $d = 1$: Ground states of some quantum systems.
- Singular *and* long-range interaction.

Random eigenvalues

Hermitian models.

- Gaussian ensembles: $d = 1$, $\beta = 1, 2, 4$, V quadratic.
- β-ensembles: $d = 1$, $\beta > 0$, V quadratic (Dumitriu-Edelman)
- General β-ensemble: V arbitrary.

Non-Hermitian models

- Ginibre ensemble: $d = 2$, $\beta = 2$, V quadratic (Ginibre).
- Random normal matrix model: V arbitrary (Ameur-Hedenmalm-Makarov)

Wigner, Dyson '60 "Statistical Theory of the Energy Levels of Complex Systems"
Boutet de Monvel - Pastur - Shcherbina '95 "On the Statistical Mechanics Approach in the Random Matrix Theory"

Macroscopic scale

Empirical measure

$$\mu_N(\vec{X}_N) := \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i}$$

Re-write the energy as ($\triangle$: diagonal)

$$\mathcal{H}_N(\vec{X}_N) = N^2 \left(\iint_{\triangle^c} -\log|x-y| d\mu_N(x) d\mu_N(y) + \int V(x) d\mu_N(x) \right)$$

Minimizing $\mathcal{H}_N$?

$$I_V(\mu) := \iint_{\mathbb{R}^d \times \mathbb{R}^d} -\log|x-y| d\mu(x) d\mu(y) + \int_{\mathbb{R}^d} V(x) d\mu(x)$$

(weighted logarithmic energy / “free entropy”).

Empirical measure behavior

Classical potential theory

Frostman, Choquet..

I_V is strictly convex on $\mathcal{P}(\mathbb{R}^d)$, unique minimizer μ_{eq} with compact support Σ.

Examples: semi-circle law ($d = 1$), circle law ($d = 2$). (V quadratic).

Theorem: μ_N converges to μ_{eq} almost surely

Theorem (Large deviation principle)

The law of $\{\mu_N\}$ obeys a Large Deviation Principle at speed N^2 with rate function $\frac{\beta}{2}(I_V - I_V(\mu_{\mathrm{eq}}))$.

Ben Arous-Guionnet ('97), Ben Arous-Zeitouni ('98), Hiai-Petz ('98), Chafai-Gozlan-Zitt ('13)

Comments on the macroscopic LDP

- For any test function φ

$$\int \varphi d\mu_N = \int \varphi d\mu + o(1) \text{ with proba } 1 - \exp(-N^2)$$

- The equilibrium measure depends on V, not on β.

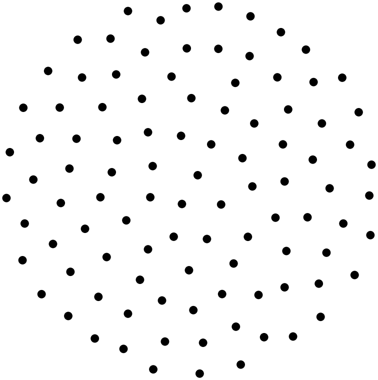

Figure: $\beta = 400$

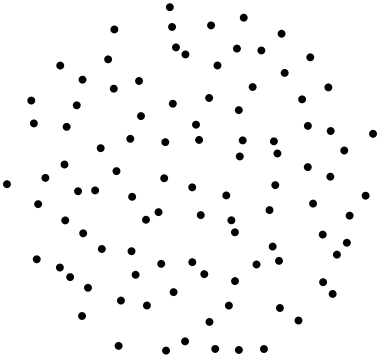

Figure: $\beta = 5$

Microscopic observable I

Non-averaged point process

Let $z \in \mathring{\Sigma}$ be fixed.

$$\mathcal{C}_{N,z} : \vec{X}_N \mapsto \sum_{i=1}^{N} \delta_{N^{1/d}(x_i - z)}.$$

Values in $\mathcal{X}$, the space of point configurations.

Microscopic observable II

Empirical field

Let $\Omega \subset \Sigma$ be fixed.

$$\overline{\mathcal{C}}_{N,\Omega} := \frac{1}{|\Omega|} \int_{\Omega} \delta_{\mathcal{C}_{N,z}} dz$$

Values in $\mathcal{P}(\mathcal{X})$.

- Ω of size independent of N: macroscopic average.
- Ω of size $N^{-\frac{1}{d}+\delta}$ mesoscopic average.

Rate function

Rate function

For $m > 0$, define a free energy functional on $\mathcal{P}(\mathcal{X})$ by

$$\mathcal{F}_{\beta}^{m}(P) := \beta \mathbb{W}_{m}^{\mathrm{elec}}(P) + \mathbf{ent}[P|\Pi^{m}]$$

$\mathbb{W}_{m}^{\mathrm{elec}}(P)$ is an energy functional
$\mathbf{ent}[P|\Pi^{m}]$ is a relative entropy functional.
Π^{m} = Poisson point process.

Good rate function, affine.

Large deviations for the empirical field

Assumptions: Σ is a C^1 compact set, and μ_{eq} has Hölder density.
To simplify, let us assume that $\mu_{\mathrm{eq}}(x) = m$ over Σ.

Theorem

Let Ω = ball of radius ε in $d = 1, 2$.
The law of the empirical field $\overline{\mathcal{C}}_{N,\Omega}$ obeys a LDP at speed $|\Omega|N$ with rate function $\mathcal{F}_\beta^m - \min \mathcal{F}_\beta^m$.
Microscopic behavior after macroscopic average.

Theorem

True for mesoscopic average (i.e. $\varepsilon = N^{-1/2+\delta}$), in dimension 2.

Comments on the microscopic LDP

Corollary (Fluctuation bounds)

$$\int_\Sigma \varphi d\mu_N = \int_\Sigma \varphi d\mu_{\mathrm{eq}} + O\left(\sqrt{\frac{1}{N}}\right) \textit{ with proba } 1 - \exp(-N).$$

+ local laws (cf . Bauerschmidt-Bourgade-Nikula-Yau (d = 2))

- Empirical field concentrates on minimizers of $\mathcal{F}_\beta$ with probability $1 - \exp(-N|\Omega|)$.
- Minimizers of $\mathcal{F}_\beta$ depend on m only through a scaling. The microscopic behavior is thus largely independent of V.
- "Explicit" order N term in $\log Z_{N,\beta}$ (cf. Shcherbina, Borot-Guionnet).

$$\left(1 - \frac{\beta}{2d}\right) \int \mu_{\mathrm{eq}}(x) \log \mu_{\mathrm{eq}}(x) dx + \min \mathcal{F}_\beta^1$$

- Competition between energy and entropy terms.

Non-averaged point processes

Corollary

The Sine_β point processes of Valko-Virag are minimizers of $\mathcal{F}_\beta$ for $\beta > 0$.

(Remark: so is Ginibre for $\beta = 2$)

Theorem (High-temperature)

Minimizers of $\mathcal{F}_\beta$ tend (in entropy sense) to a Poisson point process as $\beta \to 0$.

Allez-Dumaz '14 : convergence of Sine_β to Π^1 (in law) as $\beta \to 0$.

Theorem (Low-temperature, $d = 1$)

As $\beta \to \infty$, minimizers of $\mathcal{F}_\beta$ converge in law to $P_{\mathbb{Z}}$ (the stationary point process associated to $\mathbb{Z}$).

Killip-Stoicu '09.

Fluctuations (work in progress)

$$\mathrm{fluct}_N = N(\mu_N - \mu_{\mathrm{eq}}) = \sum_{i=1}^{N} \delta_{x_i} - N\mu_{\mathrm{eq}}, \ \mathrm{Fluct}_N[\varphi] := \int_{\mathbb{R}^d} \varphi \, d\mathrm{fluct}_N$$

$d = 2$ Rider-Virag (Ginibre case), Ameur-Hedenmalm-Makarov (Random normal matrix model)
$d = 1$ Johansson, Shcherbina, Borot-Guionnet.

Theorem (CLT, $d = 2$, β arbitrary)

For φ smooth enough (e.g. C^2_c) and V smooth enough (e.g. C^4), $\mathrm{Fluct}_N[\varphi]$ converges to a Gaussian random variable (mean and variance depend on β). The random distribution fluct_N converges to a Gaussian Free Field.
+ Moderate deviations bounds (à la BBNY).
+ Asymptotic independance of the fluctuations if $\int \nabla\varphi_1 \cdot \nabla\varphi_2 = 0$
+ Berry-Esseen?

Energy approach I - Splitting

V, β arbitrary, multi-cut welcome, "elementary" techniques.

First step: "Splitting"

$$\mathcal{H}_N(\vec{X}_N) = N^2 I_V(\mu_{\mathrm{eq}}) - \frac{N \log N}{d} + w_N(\vec{X}_N) + \zeta_N(\vec{X}_N)$$

- $I_V(\mu_{\mathrm{eq}})$: first-order energy.
- ζ_N: confining term
- w_N : interaction energy of the new system

$$w_N(\vec{X}_N) = \iint_{(\mathbb{R}^d \times \mathbb{R}^d)\setminus \triangle} -\log|x-y| (d\nu'_N - d\mu'_{\mathrm{eq}})^{\otimes 2}(x,y)$$

$\nu'_N = \sum_{i=1}^N \delta_{N^{1/d} x_i}$ and $\mu'_{\mathrm{eq}}(N^{1/d} x) = \mu_{\mathrm{eq}}(x)$.

Energy approach II - Electric fields

Second step: Electric fields

$$E^{\mathrm{loc}}(x) := \int -\nabla(\log)(x-y)(d\nu'_N - d\mu'_{\mathrm{eq}})(y)$$

with $d = 2$ or $d = 1+1$ (Cf. Stieltjes transform)

$$-\mathrm{div}\,(E^{\mathrm{loc}}) = 2\pi\left(\nu'_N - d\mu'_{\mathrm{eq}}\right) \text{ (Poisson equation)}$$

$$w_N(\vec{X}_N) \approx \frac{1}{2\pi}\int |E^{\mathrm{loc}}|^2.$$

In fact (Sandier-Serfaty '12, Rougerie-Serfaty '13)

$$w_N(\vec{X}_N) = \frac{1}{2\pi}\lim_{\eta\to 0}\left(|E^{\mathrm{loc}}_\eta|^2 + 2\pi N\log\eta\right).$$

Energy approach III - Is it the right thing?

Third step: controlling the energy

- "Abstract" lower bound $w_N \geq -CN$.
- "By hand" construction $w_N \leq CN$ for a non-tiny volume of configurations.

It implies that

$$\int_{\mathbb{R}^2} |E^{\mathrm{loc}}|^2 = O(N) \text{ with proba } 1 - \exp(-N)$$

Using it!

$H^{\text{loc}} := \int -\log|x-y|(d\nu'_N - d\mu'_{\text{eq}})(y)$ (electric potential)

Fluctuations

$$\text{Fluct}_N[\varphi] = \int_{\mathbb{R}^2} \varphi \Delta H^{\text{loc}} = \int_{\mathbb{R}^2} \nabla\varphi \cdot E^{\text{loc}} \leq \|\nabla\varphi\|_{L^2}\sqrt{N}$$

Discrepancy

$$D_R := \int_{B(0,R)} 1(d\nu'_N - d\mu'_{\text{eq}}) = \int_{B(0,R)} \Delta H^{\text{loc}} = \int_{\partial B(0,R)} E^{\text{loc}} \cdot \vec{n}.$$

Mean value theorem + a priori bounds $\Longrightarrow$ control on the discrepancy.

$$\min\left(\frac{D_R^3}{R^d}, D_R^2\right) \leq \int_{B(0,2R)} |E^{\text{loc}}|^2$$

Infinite-volume objects : Energy I

Energy of a field

$d = 2$

$$\mathcal{W}(E) := \limsup_{R \to \infty} \frac{1}{|C_R|} \int_{C_R} |E|^2$$

$d = 1$ (dimension extension)

$$\mathcal{W}(E) := \limsup_{R \to \infty} \frac{1}{R} \int_{[-R/2, R/2] \times \mathbb{R}} |E|^2$$

Energy of a point configuration

$$\mathbb{W}(\mathcal{C}) = \inf \mathcal{W}(E),$$

among "compatible" E satisfying the associated Poisson equation

$$\mathrm{div}\, E = 2\pi \left(\mathcal{C} - \text{background}\right)$$

Infinite-volume objects: Energy II

"Electric" energy of a random point process P

$$\mathbb{W}^{\mathrm{elec}}(P) := \mathbf{E}_P[\mathbb{W}(\mathcal{C})]$$

Using it ? Discrepancy estimates:

$$\mathbf{E}_P[D_R^2] \leq C(C + \mathbb{W}^{\mathrm{elec}}(P))R^d$$

$+$ Markov's $\longrightarrow P(D_R \approx R^d) \leq \frac{1}{R^d}$.
Versus exponential tails for Sine_β (Holcomb-Valko), predictions of physicists (Jancovici-Lebowitz-Manificat)...

Infinite-volume objects: Energy III

A more explicit formulation? Inspired by Borodin-Serfaty.
For stationary random point processes P, define

$$\mathbb{W}^{\mathrm{int}}(P) := \liminf_{R\to\infty} \frac{1}{R^d} \int_{[-R,R]^d} - \log|v|(\rho_{2,P}(v) - 1) \prod_{i=1}^{d} (R - |v_i|) dv,$$

$$\mathcal{D}^{\log}(P) := C \limsup_{R\to\infty} \left(\frac{1}{R^d} \iint_{C_R^2} (\rho_{2,P}(x,y) - 1) dxdy + 1 \right) \log R,$$

Theorem

1. *($d = 1$) $\mathbb{W}^{\mathrm{elec}}$ is the l.s.c. regularization of $\mathbb{W}^{\mathrm{int}} + \mathcal{D}^{\log}$*
2. *($d = 2$) $\mathbb{W}^{\mathrm{elec}} \leq \mathbb{W}^{\mathrm{int}} + \mathcal{D}^{\log}$.*

Infinite-volume objects: Entropy I

P stationary random point process, we define

$$\mathbf{ent}[P|\Pi] = \lim_{R\to\infty} \frac{1}{R^d} \mathrm{Ent}[P_R|\Pi_R]$$

$P_R, \Pi_R =$ restrictions to $[-R/2, R/2]^d$.
"Specific relative entropy". $\mathbf{ent}[\cdot|\Pi]$ is lower semi-continuous, non-negative, and has its only zero at Π. It is **affine**.
Computable in some cases: renewal processes in $1d$, periodic processes...

Infinite-volume objects: Entropy II

Occurs in "Sanov-like" large deviation principle for empirical fields **without interaction**.

$$\lim_{\varepsilon \to 0} \lim_{R \to \infty} \frac{1}{|C_R|} \log \Pi_{C_R} \left(\text{Empirical field } \in B(P, \varepsilon) \right) = -\mathbf{ent}[P|\Pi]$$

Föllmer, Föllmer-Orey, Georgii-Zessin

Analogous to Sanov's theorem for empirical measure of i.i.d samples.

"Process-level/type III LDP"

Scheme of the proof I : Setting of a LDP

$d = 2$, V quadratic. $\mu_{\mathrm{eq}} = \frac{1}{\pi} dx$ on unit disk $D(0,1)$.
Empirical field $\bar{\mathcal{C}}_N$ averaged on $D(0,1)$.
Let $P \in \mathcal{P}(\mathcal{X})$.

$$\mathbb{P}_{N,\beta}(\bar{\mathcal{C}}_N \in B(P,\varepsilon)) = \frac{1}{Z_{N,\beta}} \int_{\bar{\mathcal{C}}_N \in B(P,\varepsilon)} \exp(-\beta \mathcal{H}_N(\vec{X}_N)) d\vec{X}_N$$

$$\approx \frac{1}{K_{N,\beta}} \int_{\bar{\mathcal{C}}_N \in B(P,\varepsilon)} \exp(-\beta w_N(\vec{X}_N)) \mathbf{1}_{D(0,1)^N}(\vec{X}_N) d\vec{X}_N$$

We used the splitting $\mathcal{H}_N(\vec{X}_N) = N^2 I(\mu_{\mathrm{eq}}) - \frac{N \log N}{2} + w_N(\vec{X}_N) + \zeta_N(\vec{X}_N)$

Scheme of the proof II : Ideal case

If $\bar{\mathcal{C}}_N \approx P \Longrightarrow w_N(\vec{X}_N) \approx N\mathbb{W}^{\mathrm{elec}}(P)$

$$\frac{1}{K_{N,\beta}} \int_{\bar{\mathcal{C}}_N \in B(P,\varepsilon)} \exp(-\beta w_N(\vec{X}_N)) \mathbf{1}_{D(0,1)^N}(\vec{X}_N) d\vec{X}_N$$
$$\approx \frac{1}{K_{N,\beta}} \exp(-\beta N \mathbb{W}^{\mathrm{elec}}(P)) \int_{\bar{\mathcal{C}}_N \in B(P,\varepsilon)} \mathbf{1}_{D(0,1)^N}(\vec{X}_N) d\vec{X}_N.$$

+ plug in the LDP for empirical fields without interaction.

Scheme of the proof - III : Tools

- Lower bound on the energy

$$\bar{\mathcal{C}}_N \approx P \Longrightarrow w_N(\vec{X}_N) \geq N\mathbb{W}^{\mathrm{elec}}(P)$$

"Two-scale Γ-convergence approach" (Sandier-Serfaty)
Elementary abstract functional analysis.

- LDP for empirical fields without interaction
- Upper bound constructions?

Scheme of the proof - IV : construction

Want a volume $\exp\left(-N\mathbf{ent}[P|\Pi^1]\right)$ of configurations $\vec{X}_N$ s.t.

1. Empirical field $\bar{\mathcal{C}}_N \approx P$
2. Energy upper bound $w_N(\vec{X}_N) \leq N\mathbb{W}^{\mathrm{elec}}(P) + o(N)$

What is the strategy?

1. Generating microstates: Lower bound of Sanov-like result yields a volume $\exp\left(-N\mathbf{ent}[P|\Pi^1]\right)$ of microstates $\{\vec{X}_N\}$ s.t. $\bar{\mathcal{C}}_N \approx P$.
2. Screening
3. Regularization

Other settings

- Different pair interaction $g(x-y) = \frac{1}{|x-y|^s}$ (Riesz gases)
- Two-component plasma (L.-Serfaty-Zeitouni)

Could be applied to:

- Laguerre, Jacobi, Circular Unitary ensemble?
- Zeroes of random polynomials?

Open problems

- Edge case?
- Low-temperature behavior for $d \geq 2$? Crystallization conjecture.
- Limiting point processes for $d = 2$, $\beta \neq 2$ ("Ginibre-β")?
- Uniqueness of minimizers for $\mathcal{F}_\beta$ vs. phase transition?
- Description of minimizers (DLR theory)? Rigidity of minimizers?
- Phase diagram? Liquid/solid transition at finite β for two-dimensional β-ensemble? (Brush-Sahlin-Teller '66, Hansen-Pollock '73, Caillol-Levesque-Weis-Hansen '82 . . .).

Thank you for your attention!