Log-correlated Gaussian fields and linear statistics of β-ensembles

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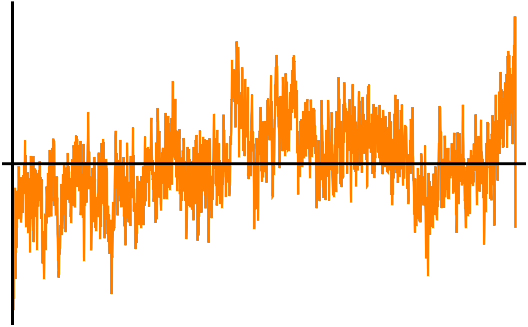
Outline of the talk

- What are log-correlated Gaussian fields?
- Examples of log-correlated fields in RMT.
- Connection between log-correlated fields in RMT and CLTs for linear statistics.
- A sketch of a proof for a CLT for the circular β-ensemble.

What is a log-correlated Gaussian field?

Basically a centered Gaussian process $X(x)$ on a subset of $\mathbb{R}^d$ with a logarithmic singularity in its covariance:

$$\mathbb{E}X(x)X(y) \sim -\log|x-y|, \qquad \text{as} \qquad x \to y.$$



Doesn't make sense as an honest random function.

Examples of log-correlated fields

- Let $A_k \sim N_{\mathbb{C}}(0,1)$ be i.i.d. and

$$X(\theta) = \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \left[A_k e^{ik\theta} + A_k^* e^{-ik\theta}\right].$$

Then formally $\mathbb{E}X(\theta)X(\theta') = -\frac{1}{2}\log|e^{i\theta} - e^{i\theta'}|$.

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- Let $B_k \sim N(0,1)$ be i.i.d. and for $x \in (-1,1)$

$$Y(x) = \sum_{k=1}^{\infty} \sqrt{\frac{1}{k}} B_k T_k(x),$$

  where $T_k(\cos\theta) = \cos k\theta$ (Chebyshev polynomial). Then formally $\mathbb{E}Y(x)Y(y) = -\frac{1}{2}\log(2|x-y|)$.

Examples of log-correlated fields

- Let $C_k \sim N(0,1)$ be i.i.d., $D \subset \mathbb{R}^2$ nice enough, and $\Delta \phi_k = -\lambda_k \phi_k$ on D with zero Dirichlet boundary conditions. Then define the GFF:

$$Z(x) = \sum_{k=1}^{\infty} \frac{C_k}{\sqrt{\lambda_k}} \phi_k(x),$$

Again formally $\mathbb{E} Z(x) Z(y) = G_D(x,y) \sim -\log|x-y|$ as $x \to y$.

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All of these series converge almost surely in suitable spaces of generalized functions (e.g. Sobolev spaces) and one can make precise sense of everything above.

Why care about log-correlated fields?

- Universal objects - show up as asymptotic fluctuations in various models: RMT, growth models, combinatorial models, number theory, lattice models, ...

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- Play a critical role in the mathematics of 2-d quantum gravity (Liouville quantum gravity) - conjectured to be related to a suitable scaling limit of random planar maps.
- Can be used to construct conformally invariant random planar curves (SLE type objects).

Characteristic polynomial of the CUE

Let $U_N \sim CUE(N)$, and

$$X_N(\theta) = \log|\det(I - e^{-i\theta}U_N)|$$
$$= -\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}\left[e^{-ik\theta}\frac{\mathrm{Tr}\,U_N^k}{\sqrt{k}} + e^{ik\theta}\frac{\mathrm{Tr}\,U_N^{-k}}{\sqrt{k}}\right].$$

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Theorem (Diaconis and Shahshahani '94)

For any fixed K, $(TrU_N^k/\sqrt{k})_{k=1}^K \xrightarrow{d} (A_k)_{k=1}^K$, *where* $A_k \sim N_{\mathbb{C}}(0,1)$ *i.i.d..*

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Thus (Hughes, Keating, and O'Connell '01), $X_N \xrightarrow{d} X$ (in a suitable space).

Characteristic polynomial of the GUE

Let $H_N \sim GUE(N)$ (with a suitable normalization). For $x \in (-1,1)$, on the event that $\sigma(H_N) \subset (-1,1)$, one has

$$\begin{aligned} Y_N(x) &= \log|\det(xI - H_N)| \\ &= -\sum_{k=1}^{\infty} \sqrt{\frac{1}{k}}\, T_k(x) \mathrm{Tr}\left[\frac{2}{\sqrt{k}} T_k(H_N)\right]. \end{aligned}$$

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After centering, the random variables $\mathrm{Tr}[\frac{2}{\sqrt{k}} T_k(H_N)]$ *converge jointly in law to i.i.d. standard Gaussians for* $k \leq K$ *fixed.*

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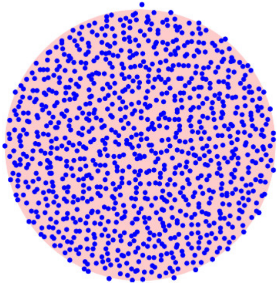
Fyodorov, Khoruzhenko, and Simm '13: $Y_N(x) - \mathbb{E} Y_N(x) \xrightarrow{d} Y(x)$ (in a suitable space).

Characteristic polynomial of the Ginibre ensemble

Let $W_N \sim \mathrm{Ginibre}(N)$ (with a suitable normalization). For $|z| < 1$, on the event that $\sigma(W_N) \subset \mathbb{U} := \{|w| < 1\}$,

$$
\begin{aligned}
Z_N(z) &= \log|\det(zI - W_N)| \\
&= -2\pi \sum_{k=1}^{\infty} \frac{1}{\sqrt{\lambda_k}} \phi_k(z) \mathrm{Tr} \frac{\phi_k(W_N)}{\sqrt{\lambda_k}} - \mathrm{Re} \sum_{k=1}^{\infty} \frac{1}{k} \bar{z}^k \mathrm{Tr} W_N^k.
\end{aligned}
$$



Characteristic polynomial of the Ginibre ensemble

Theorem (Rider and Virág '06)

After centering, $\mathrm{Tr} f(W_N)$ *converges in law to* $N(0, \sigma^2)$,

$$\sigma^2 = \frac{1}{4\pi} \int_{\mathbb{U}} |\nabla f|^2 + \frac{1}{2} \sum_{k \in \mathbb{Z}} |k| |\widehat{f}(k)|^2.$$

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From this, after centering, Z_N converges in law (in a suitable space) to $\widetilde{Z}$ with covariance

$$\mathbb{E} \widetilde{Z}(z) \widetilde{Z}(w) = -\frac{1}{2} \log |z - w|.$$

The eigenvalue counting function for the GUE

For $H_N \sim GUE(N)$ (normalized as before) and

$$\widetilde{X}_N(x) = Im \log \det(xI - H_N) = \pi \sum_{j=1}^{N} \mathbf{1}(x < \lambda_j).$$

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As

$$\int \widetilde{X}_N(x) f'(x) dx = \pi \sum_{j=1}^{N} f(\lambda_j)$$

the CLT implies again that after centering, in the bulk of the spectrum, $\widetilde{X}_N$ converges to a log-correlated field $\widetilde{X}$.

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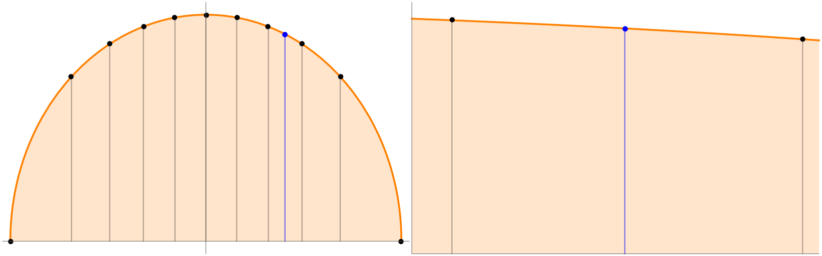
- Then define the "fluctuation field": for $x \in (\gamma_{j-1}, \gamma_j]$,

$$\widetilde{Y}_N(x) = N\sigma(\gamma_j)\left[\lambda_j - N\int_{\gamma_{j-1}}^{\gamma_j} y\sigma(y)dy\right].$$

The eigenvalue fluctuation field for the GUE



The eigenvalue fluctuation field for the GUE

One can then show that

$$\int \widetilde{Y}_N(x) f'(x) = \sum_{j=1}^{N} [f(\lambda_j) - \mathbb{E} f(\lambda_j)] + error,$$

which implies $Y_N \to c\widetilde{X}$ in a suitable sense (for some constant $c > 0$).

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which implies $Y_N \to c\widetilde{X}$ in a suitable sense (for some constant $c > 0$).
Moral of the story: CLTs equivalent to log-correlated fields describing global fluctuations.

Gaussian multiplicative chaos

Log-correlated fields relevant to Liouville quantum gravity and the construction of SLE type curves through random measures of the form $e^{X(x)-\frac{1}{2}\mathbb{E}X(x)^2}dx$.

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Theorem (W '14, Berestycki, W, Wong '16)

For small enough $\gamma > 0$, $\frac{e^{\gamma X_N(\theta)}}{\mathbb{E}e^{\gamma X_N(\theta)}}d\theta$ (CUE) and $\frac{e^{\gamma Y_N(x)}}{\mathbb{E}e^{\gamma Y_N(x)}}dx$ (GUE) converge to chaos measures.

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Proof through RHP estimates (due to Deift, Its, and Kraosvsky; and others).

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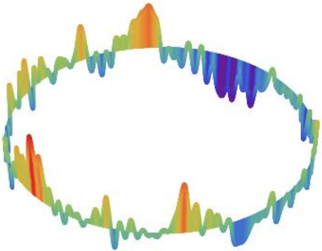

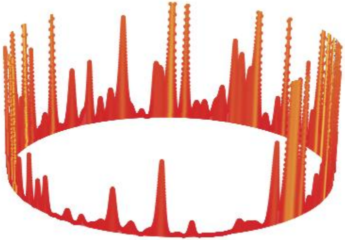

Central limit theorems

- CLTs proven in great generality and with different methods for one-cut regular β-ensembles - even in the discrete case (Diaconis and Shahshahani; Johansson; Pastur; Shcherbina; Rider and Virág; Ameur, Hedenmalm, and Makarov; Borot and Guionnet; Dumitriu and Paquette; Döbler and Stolz; Forrester and Witte; Jiang and Matsumoto; Borodin, Gorin, and Guionnet,...).

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- Log-correlated objects appearing generically in β-ensembles (though 2d for $\beta \neq 2$ open?).
- Chaos measures and behavior of the maximum of the fields universal?
- I can't prove it, but the following came out of an attempt.

A Gaussian approximation result

- Assume that $W \sim N(0, \sigma^2)$ and W' is a "small perturbation" of this preserving the law: $W' \sim N(0, \sigma^2)$ and $\mathbb{E}(WW') = \sigma^2(1 - \epsilon)$.

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- Then one has

$$\mathbb{E}(W' - W|W) = -\epsilon W \tag{1}$$

$$\mathbb{E}[(W' - W)^2|W] = 2\epsilon\sigma^2 + \mathcal{O}(\epsilon^2) \tag{2}$$

$$\mathbb{E}|W' - W|^3 = \mathcal{O}(\epsilon^{3/2}) \tag{3}$$

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- Perhaps if this nearly holds, W is nearly Gaussian?

A Gaussian approximation result

Theorem (Meckes '09;Döbler and Stolz '11)

Assume: $(W, W_t) \in \mathbb{C}^{2d}$ *exchangeable,* $Z \sim N_{\mathbb{C}}(0, I_{d\times d})$, $\exists$ *deterministic* $\Lambda \in \mathbb{C}^{d\times d}$ *and* $\Sigma \in \mathbb{C}^{d\times d}$ *positive definite, and random* $R \in \mathbb{C}^d$ *and* $S, T \in \mathbb{C}^{d\times d}$*:*

$$\lim_{t\to 0} \frac{1}{t}\mathbb{E}(W_t - W|W) = -\Lambda W + R \tag{1}$$

$$\lim_{t\to 0} \frac{1}{t}\mathbb{E}((W_t - W)(W_t - W)^*|W) = 2\Lambda\Sigma + S \tag{2}$$

$$\lim_{t\to 0} \frac{1}{t}\mathbb{E}((W_t - W)(W_t - W)^T|W) = T \tag{2'}$$

$$\lim_{t\to 0} \frac{1}{t}\mathbb{E}|W_t - W|^3 = 0. \tag{3}$$

$\Rightarrow d(W, \sqrt{\Sigma}Z) \lesssim ||\Lambda^{-1}||_{op}\left[\mathbb{E}||R||_2 + ||\Sigma^{-1/2}||_{op}\mathbb{E}(||S||_{HS} + ||T||_{HS})\right].$

A CLT for the CβE

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Theorem (W '15)

Let $(e^{ix_j})_{j=1}^N \sim C\beta E(N)$ $(\beta > 0)$, $T_K = \left(\sum_{j=1}^N e^{ikx_j}\right)_{k=1}^K$, *and* $G_K = \left(\sqrt{\frac{2}{\beta}j}Z_j\right)_{j=1}^K$ $(Z_j \sim N_{\mathbb{C}}(0,1)$ *i.i.d.*$)$. *Then*

$$d(T_K, G_K) = \mathcal{O}\left(\frac{K^{7/2}}{N}\right).$$

Remarks

- Gives a (far from optimal?) rate of convergence for the CLT.
- K can increase with N!
- Implies CLTs for smooth functions through Fourier expanding.
- For the proof, only need to estimate (mixed) moments up to order 4.
- Not strong enough to estimate maximum of X_N: would need to have $K \sim N$ for this.

Sketch of proof

- Take $W = T_K$, and $W_t = \left(\sum_{j=1}^{N} e^{ikx_j(t)}\right)_{k=1}^{K}$, where

$$dx_j(t) = \frac{\beta}{2}\sum_{i\neq j}\cot\frac{x_j(t)-x_i(t)}{2}dt + \sqrt{2}dB_j(t)$$

(circular DBM) started from (x_j).

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  (circular DBM) started from (x_j).
- (W, W_t) exchangeable as CβE is reversible for the dynamics.
- Limits of the conditional expectations can be expressed in terms of the generator of cDBM

$$L_\beta = \frac{\beta}{2} \sum_j \sum_{i \neq j} \cot \frac{x_j - x_i}{2} \partial_{x_j} + \sum_{j=1}^{N} \partial_{x_j}^2$$

  acting on power sums $p_k(x) = \sum_{j=1}^{N} e^{ikx_j}$ (and their products).

Sketch of proof

$$e.g. \qquad \lim_{t\to 0} \frac{1}{t}\mathbb{E}(W_t - W|W) = (L_\beta p_k(x))_{k=1}^K .$$

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Doing the calculations:

$$\Lambda_{k,l} = \delta_{k,l} N k \frac{\beta}{2} \tag{1}$$

$$\Sigma_{k,l} = \frac{2}{\beta} k \delta_{k,l} \tag{2}$$

$$R_k = -k^2 \left[\frac{\beta}{2} - 1\right] p_k(x) - k\frac{\beta}{2}\sum_{l=1}^{k-1} p_l(x) p_{k-l}(x) \tag{3}$$

$$S_{k,l} = (1-\delta_{k,l}) 2kl p_{k-l}(x) \tag{4}$$

$$T_{k,l} = -2kl p_{k+l}(x). \tag{5}$$

Sketch of proof

- One concludes by bounding relevant moments.
- Best ones I know of (for the CβE): Jiang and Matsumoto '11.
- Weaker ones would suffice if you're happy with weaker K.
- Approach should work for other models too (works at least for the Gaussian β-ensemble).