Commensurated subgroups of finitely generated branch groups

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Permutation Groups, BIRS

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6. Any compact open subgroup of a totally disconnected locally compact (t.d.l.c.) group

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3. *If G is finitely generated, then $G/\!/K$ is compactly generated.*

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Proposition (Le Boudec–W., 16)

Every proper commensurated subgroup of Thompson's group T is finite.

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Preliminaries

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- For $G \leq \mathrm{Aut}(T_\alpha)$ and $s \in T_\alpha$, the **rigid stabilizer** of s is

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- The **n-th level rigid stabilizer** of G is $\mathrm{rist}_G(n) := \langle \mathrm{rist}_G(s) \mid |s| = n \rangle$.

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Commensurated subgroups in branch groups

An infinite group G is **just infinite** if every proper quotient is finite.

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Theorem (W., 15)

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Corollary

Let G be the Grigorchuk group or a Gupta-Sidki group. Every commensurated subgroup of G is either finite or of finite index.

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- Suppose $K \leq G$ is an infinite, infinite index commensurated subgroup.
- Consider the Schlichting completion $G/\!\!/K$. This is a compactly generated t.d.l.c. group that is non-compact and non-discrete.
- Apply results for t.d.l.c. groups to derive a contradiction.

Result 1

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For G a compactly generated t.d.l.c. group, one of the following holds:

1. *G has an infinite discrete quotient.*
2. *G is compact.*
3. *G has a cocompact normal subgroup that admits exactly $0 < n < \infty$ non-discrete topologically simple quotients.*

Result 2

For a closed normal factor K/L of a topological group G, **the centralizer** is

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An equivalence class of non-abelian chief factors under the association relation is called a **chief block**. The set of chief blocks is denoted by $\mathfrak{B}_G$.

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Let G be a Polish group, $\mathfrak{a} \in \mathfrak{B}_G$, *and*

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A question

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Question

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Find finitely generated amenable groups with interesting commensurated subgroups.

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Question

Does the Basilica group have an infinite, infinite index commensurated subgroup with trivial normal core?

Thank you