On the quantum grassmannian

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Bridges between Noncommutative Algebra and Algebraic Geometry

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Representation theory / Primitive ideals
of quantum grassmannians

Totally nonnegative (tnn for short)
grassmannians and their tnn cells

Poisson geometry: Symplectic leaves
in Poisson flag varieties

Some quantum algebras

$q \in \mathbb{C}^*$

Quantum affine spaces:

$$O_{q^\bullet}(\mathbb{C}^N) = \mathbb{C}_{q^\bullet}[T_1, \ldots, T_N] = \mathbb{C}\langle T_1, \ldots, T_N\rangle/(T_iT_j = q^{\bullet ij}T_jT_i).$$

Quantum tori:

$$\begin{aligned} O_{q^\bullet}((\mathbb{C}^\times)^N) &= \mathbb{C}_{q^\bullet}[T_1^{\pm 1}, \ldots, T_N^{\pm 1}] \\ &= \mathbb{C}\langle T_1^{\pm 1}, \ldots, T_N^{\pm 1}\rangle/(T_iT_j = q^{\bullet ij}T_jT_i). \end{aligned}$$

Quantum matrices: $O_q(M_N(\mathbb{C}))$ is "designed" to have $O_q(\mathbb{C}^N)$ as a left and right comodule algebra, with $O_q(M_N(\mathbb{C}))$ coacting on $O_q(\mathbb{C}^N)$ in the same way that $O(M_N(\mathbb{C}))$ coacts on $O(\mathbb{C}^N)$.

Quantum 2×2 matrices

The coordinate ring of quantum 2×2 matrices

$$\mathcal{O}_q(\mathcal{M}_2(\mathbb{C})) := \mathbb{C}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is generated by four indeterminates a, b, c, d subject to the following rules:

$$\boxed{\begin{array}{c} ab = qba, \qquad cd = qdc \\ ac = qca, \qquad bd = qdb \\ bc = cb, \qquad ad - da = (q - q^{-1})cb. \end{array}}$$

The **quantum determinant** $ad - qbc$ is a central element

The algebra of $m \times p$ quantum matrices.

$$R = O_q\left(\mathcal{M}_{m,p}(\mathbb{C})\right) := \mathbb{C}\begin{bmatrix} Y_{1,1} & \dots & Y_{1,p} \\ \vdots & & \vdots \\ Y_{m,1} & \dots & Y_{m,p} \end{bmatrix},$$

where each 2×2 sub-matrix is a copy of $O_q\left(\mathcal{M}_2(\mathbb{C})\right)$.

$O_q\left(\mathcal{M}_{m,p}(\mathbb{C})\right)$ is an iterated Ore extension with the indeterminates $Y_{i,\alpha}$ adjoined in the lexicographic order and so is a noetherian integral domain.

In the square case $(m = p = n)$

$$D_q = \sum_{\sigma \in S_n} (-q)^{l(\sigma)} Y_{1,\sigma(1)} \cdots Y_{n,\sigma(n)}$$

is the **quantum determinant**. D_q is a central element.

Quantum minors of $R = \mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C}))$

They are the quantum determinants of square sub-matrices of $\mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C}))$.

More precisely, if $I \subseteq [\![1, m]\!]$ and $\Lambda \subseteq [\![1, p]\!]$ with $\mid I \mid = \mid \Lambda \mid$, the **quantum minor** associated with the rows I and columns Λ is

$$[I \mid \Lambda] := D_q(\mathcal{O}_q(M_{I,\Lambda}(\mathbb{C}))).$$

For example, $[12|23] = Y_{1,2}Y_{2,3} - qY_{1,3}Y_{2,2}$ is the quantum minor of R associated with the rows $1, 2$, and the columns $2, 3$.

The quantum grassmannian $\mathcal{G}_q(k,n)$

The quantum grassmannian $\mathcal{G}_q(k,n)$ is the subalgebra of $\mathcal{O}_q(\mathcal{M}(k,n))$ generated by the maximal $k \times k$ quantum minors

Denote by $[I]$ the quantum minor $[1\dots k|I]$.

Example $\mathcal{G}_q(2,4)$ is generated by the six quantum minors $[12],[13],[14],[23],[24],[34]$.

Most minors $q^{\bullet}$-commute, for example, $[12]\,[34] = q^2\,[34]\,[12]$, however, $[13]\,[24] = [24]\,[13] + (q - q^{-1})\,[14]\,[23]$ and there is a quantum Plücker relation

$$[12]\,[34] - q\,[13]\,[24] + q^2\,[14]\,[23] = 0.$$

Noncommutative dehomogenisation:

$$\mathcal{G}_q(k,n)[[12...k]^{-1}] \simeq \mathcal{O}_q(\mathcal{M}(k,n-k))[Z^{\pm 1};\sigma]$$

The prime spectrum of $\mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C}))$

We now assume that $q \in \mathbb{C}^*$ **is not a root of unity**, and we set $R := \mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C}))$.

• **Goodearl-Letzter** Prime ideals of R are completely prime.

The torus $\mathcal{H} := (\mathbb{C}^*)^{m+p}$ acts by automorphisms on R via :

$$(a_1, \dots, a_m, b_1, \dots, b_p).Y_{i,\alpha} = a_i b_\alpha Y_{i,\alpha}.$$

This action of $\mathcal{H}$ on R induces an action of $\mathcal{H}$ on Spec(R). We denote by $\mathcal{H}$-Spec(R) the set of those prime ideals in R which are $\mathcal{H}$-invariant.

• **Goodearl-Letzter** R has at most 2^{mp} $\mathcal{H}$-primes.

Note that 0 is always an $\mathcal{H}$-prime ideal.

Stratification Theorem (Goodearl-Letzter) :

If $J \in \mathcal{H}$-Spec(R), then we set

$$\text{Spec}_J(R) := \{P \in \text{Spec}(R) \mid \bigcap_{h \in \mathcal{H}} h.P = J\}.$$

1. $\text{Spec}(R) = \bigsqcup_{J \in \mathcal{H}\text{-Spec}(R)} \text{Spec}_J(R)$

2. $\text{Spec}_J(R) \simeq \text{Spec}(\mathbb{C}[Z_1^{\pm 1}, ..., Z_{n(J)}^{\pm 1}])$

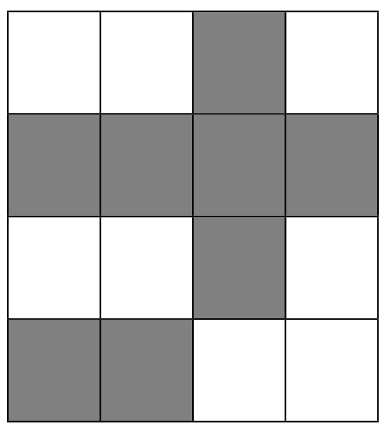
3. The primitive ideals of R are precisely the primes maximal in their $\mathcal{H}$-strata.

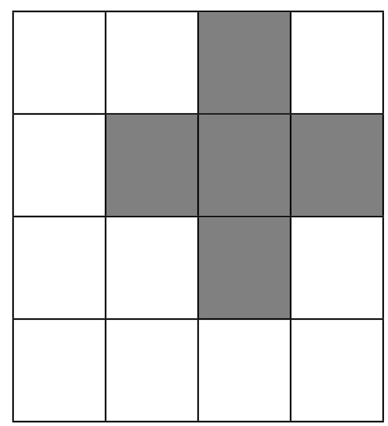
4. The Dixmier-Moeglin Equivalence holds in R.

Cauchon diagrams

A **Cauchon diagram** on an $m \times p$ array is an $m \times p$ array of squares coloured either black or white such that for any square that is coloured black the following holds:
Either each square strictly to its left is coloured black, or each square strictly above is coloured black.

Here are an example and a non-example





Parametrisation of $\mathcal{H}$-Spec$(\mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C})))$

• **Cauchon (2003)** There is a bijection between Cauchon diagrams on an $m \times p$ array and $\mathcal{H}-$ Spec$(\mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C})))$.

If C is a Cauchon diagram, then we denote by J_C the unique $\mathcal{H}$-prime associated to C.

• **L., Yakimov, Casteels** $\mathcal{H}$-primes are generated by quantum minors.

• Dimensions of $\mathcal{H}$-strata and the poset of $\mathcal{H}$-primes are described through another parametrization.

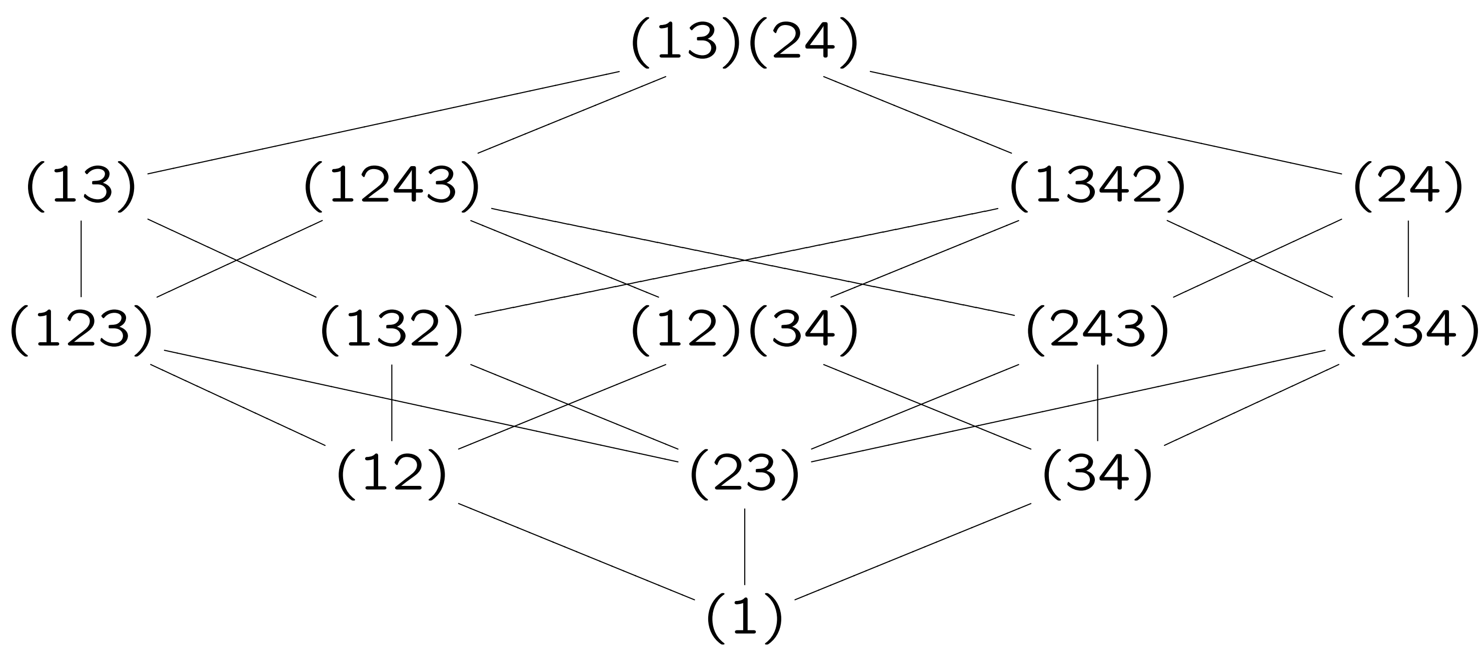
Restricted permutations

$$\mathcal{S} = \{w \in S_{m+p} \mid -p \leq w(i) - i \leq m \text{ for all } i = 1, 2, \ldots, m+p\}.$$

In the 2×2 case, this subposet of the Bruhat poset of S_4 is

$$\mathcal{S} = \{w \in S_4 \mid -2 \leq w(i) - i \leq 2 \text{ for all } i = 1, 2, 3, 4\}.$$

and is shown below.



The poset $\mathcal{H}$-Spec$(\mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C})))$

Set

$$\mathcal{S} := \{\sigma \in S_{m+p} \mid -p \leq \sigma(i) - i \leq m, \ \forall i \in [\![1, m+p]\!]\}$$

and

$$w_0 := \begin{bmatrix} 1 & 2 & \dots & p & p+1 & p+2 & \dots & p+m \\ m+1 & m+2 & \dots & m+p & 1 & 2 & \dots & m \end{bmatrix}.$$

Then

$$\mathcal{S} = \{w \in S_{m+p} \mid w \leq w_0\}$$

and

L. (2007) We have a poset isomorphism

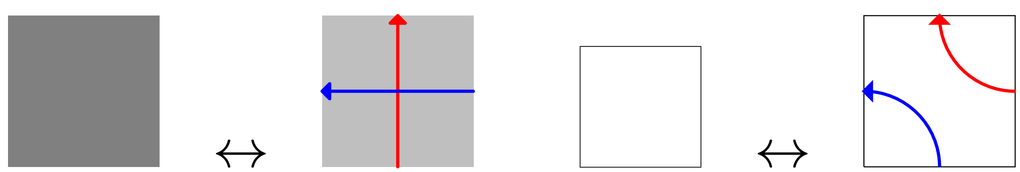
$$\mathcal{H}\text{-Spec}(\mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C}))) \simeq \mathcal{S}.$$

Pipe dreams

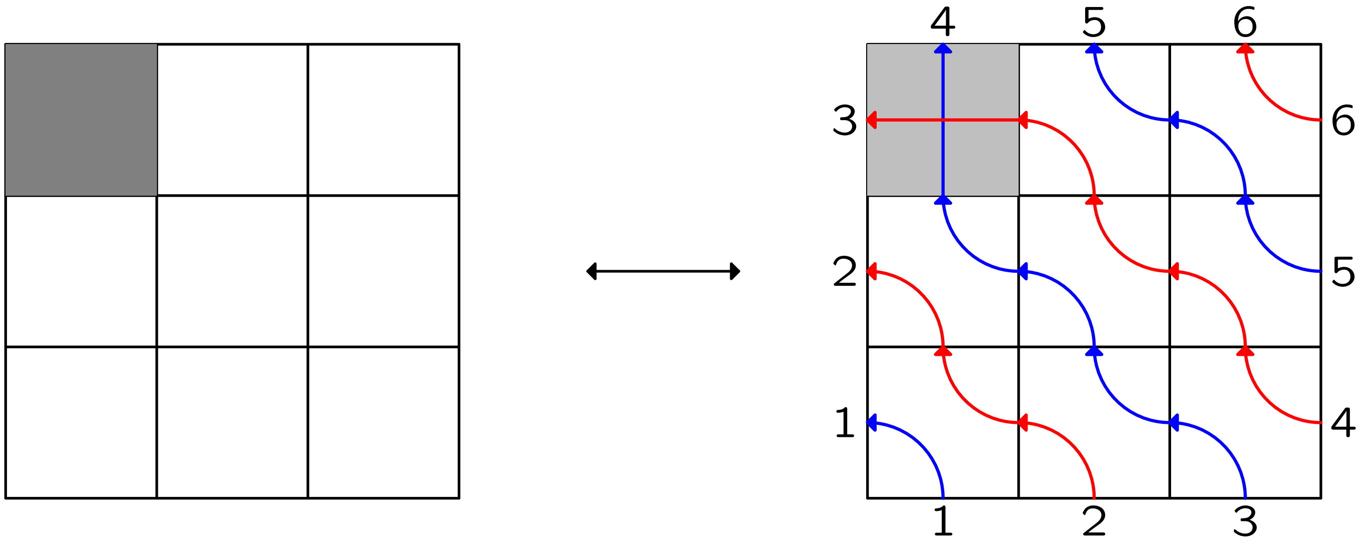
Previous results imply the existence of a bijection between the set of $m \times p$ Cauchon diagrams and the set $\mathcal{S}$ of restricted permutations.

This is no coincidence, and the connection between the two posets can be illuminated by using *Pipe Dreams*.

The procedure to produce a restricted permutation from a Cauchon diagram goes as follows. Given a Cauchon diagram, replace each black box by a cross, and each white box by an elbow joint, that is:



Pipe dreams: an example



So the restricted permutation associated to this Cauchon diagram is (3 4).

Observe that the all black diagram produces the restricted permutation w_0.

- $\dim \mathrm{Spec}_{J_c}(R)$ is equal to the number of odd cycles in ww_0^{-1}.

Dixmier-Moeglin Equivalence

$P \in \text{Spec}A$ is called

- *rational* if $Z(\text{Frac}(A/P))$ algebraic over k;
- *locally closed* if the nonzero prime ideals in A/P have nonzero intersection.

The algebra A is said to satisfy the *Dixmier-Moeglin equivalence* if for every $P \in \text{Spec}A$, we have

$$P\text{primitive} \Leftrightarrow P \text{ locally closed} \Leftrightarrow P \text{ rational.}$$

The H-stratification theorem of Goodearl-Letzter allows to prove that various quantum algebras have the DME.

Strong DME

- $\text{prim.deg}\, P = \inf\{\text{ht}\, Q \mid Q \in \text{Prim}(A/P)\}$.

- $\text{rat.deg}\, P = \text{tr.deg}_k\, Z(\text{Frac}(A/P))$.

- $\text{loc.deg}\, P = \inf\{d \mid \bigcap_{Q \in \text{Spec}_{d+1}(A/P)} Q \neq 0\}$.

($\text{Spec}_{d+1} A/P$ denotes the subspace of $\text{Spec} A/P$ consisting of all those prime ideals of height $d+1$.)

We say that A satisfies the *strong Dixmier-Moeglin equivalence* if every $P \in \text{Spec} A$ satisfies $\text{loc.deg}\, P = \text{prim.deg}\, P = \text{rat.deg}\, P$.

Strong DME: example

The strong Dixmier-Moeglin equivalence is clearly at least as strong as the Dixmier-Moeglin equivalence; in fact, a surprising example shows that it is strictly stronger.

Example: Let us consider the zero ideal of $U(\mathfrak{sl}_2(\mathbb{C}))$: we have $\operatorname{prim.deg}(0) = \operatorname{rat.deg}(0) = 1$ but $\operatorname{loc.deg}(0) = 2$.

Strong DME: general result

• For any $P \in \mathrm{Spec}A$, we have $\mathrm{loc.\,deg}\, P \geq \mathrm{prim.\,deg}\, P$.

• If every prime factor of A has a localisation (by a finitely generated denominator set) which satisfies the SDME, then

$$\mathrm{loc.\,deg}\, P = \mathrm{prim.\,deg}\, P$$

for all $P \in \mathrm{Spec}A$.

Strong DME: quantum matrices

Thm (Bell-L-Nolan): Quantum matrices have the Strong DME

Our strategy is to relate the prime and primitive spectra of the algebras in which we are interested to those of algebras which are more easily understood:

$$\begin{array}{c} \text{Quantum Matrices} \\ \Uparrow \\ \text{Quantum affine spaces} \\ \Uparrow \\ \text{Quantum tori} \\ \Uparrow \\ \text{Commutative affine domains.} \end{array}$$

The quantum grassmannian $\mathcal{G}_q(k, n)$

The quantum grassmannian $\mathcal{G}_q(k, n)$ is the subalgebra of $\mathcal{O}_q(\mathcal{M}(k, n))$ generated by the maximal $k \times k$ quantum minors

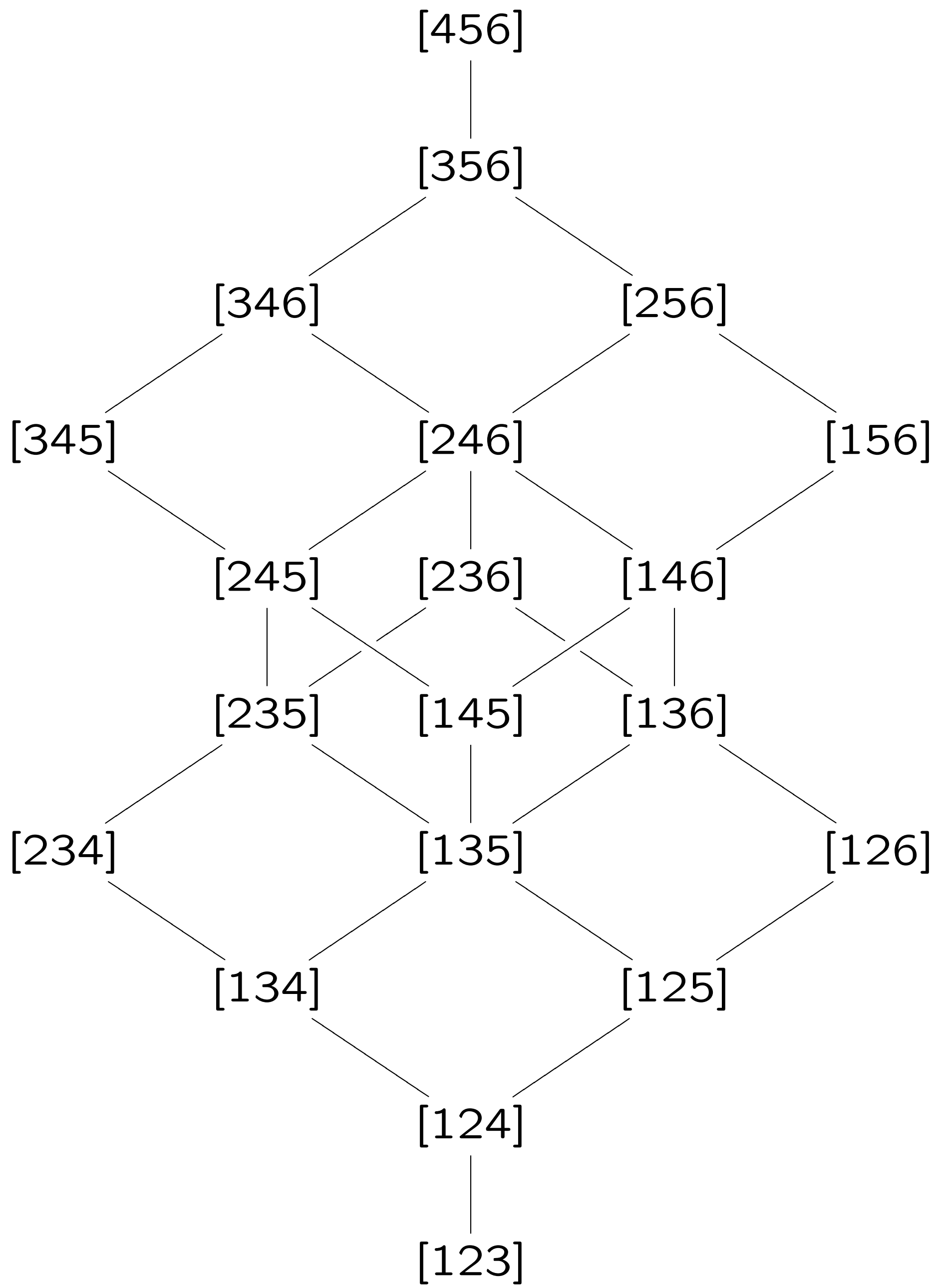
Denote by $[I]$ the quantum minor $[1 \dots k|I]$. There is a torus action of $\mathcal{H} = (\mathbb{C}^*)^n$ given by column multiplication. There are finitely many $\mathcal{H}$-primes.

Example $\mathcal{G}_q(2, 4)$ is generated by the six quantum minors $[12], [13], [14], [23], [24], [34]$.

Most minors $q^\bullet$-commute, for example, $[12]\,[34] = q^2\,[34]\,[12]$, however, $[13]\,[24] = [24]\,[13] + (q - q^{-1})\,[14]\,[23]$ and there is a quantum Plücker relation

$$[12]\,[34] - q\,[13]\,[24] + q^2\,[14]\,[23] = 0.$$

Partial order:
$[i_1 < \cdots < i_k] \leq [j_1 < \cdots < j_k]$ whenever $i_s \leq j_s$ for all s.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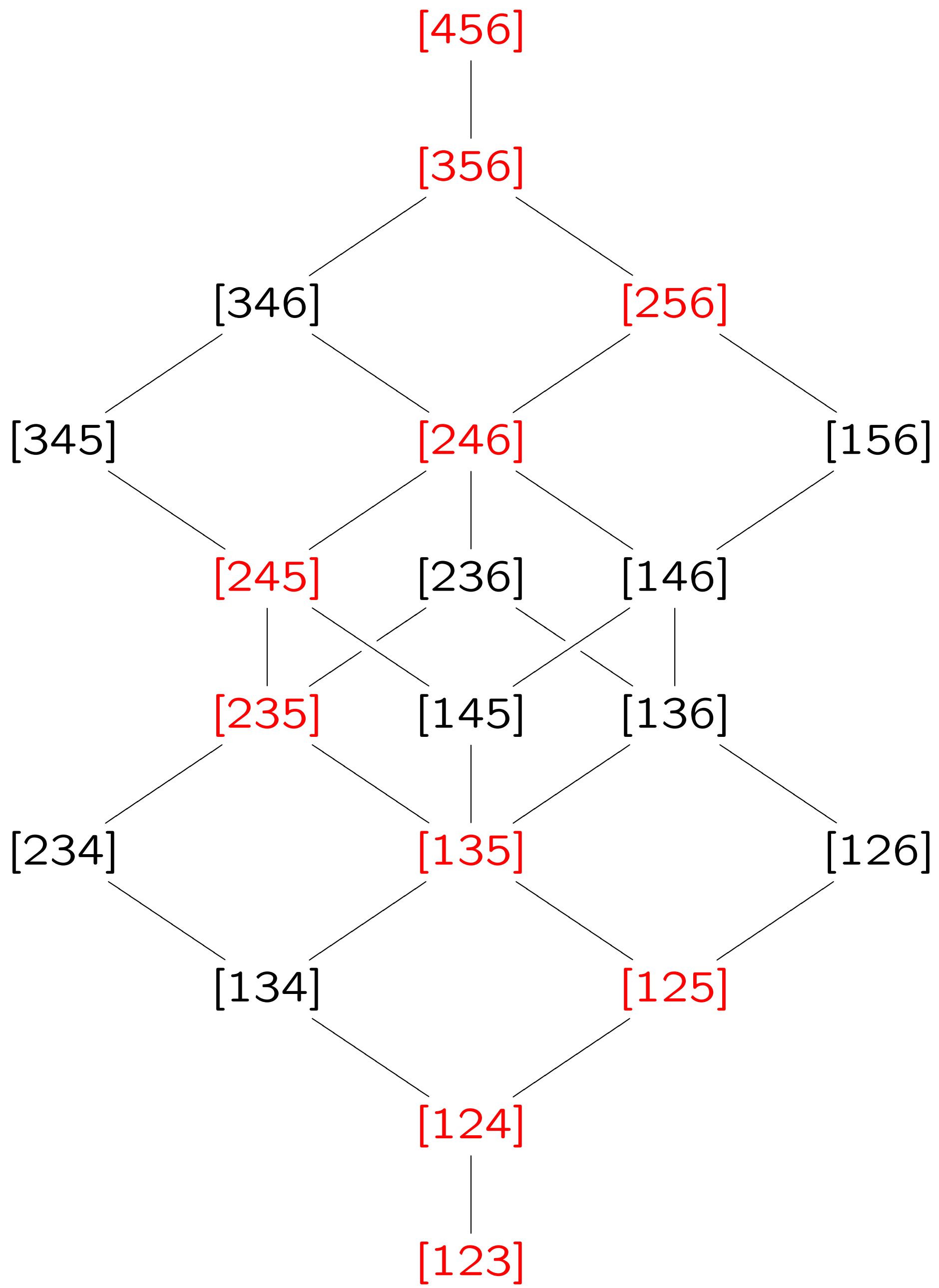
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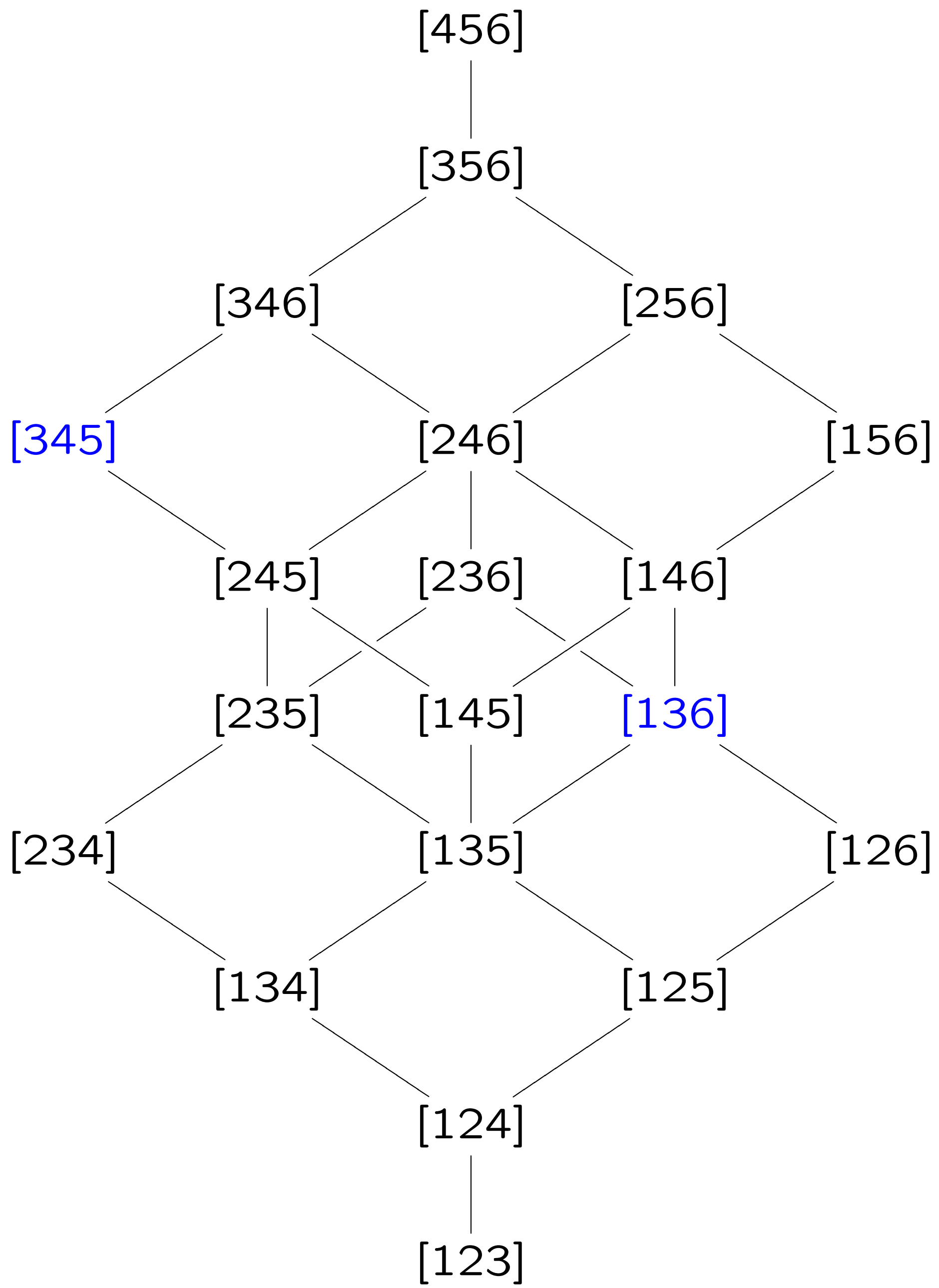
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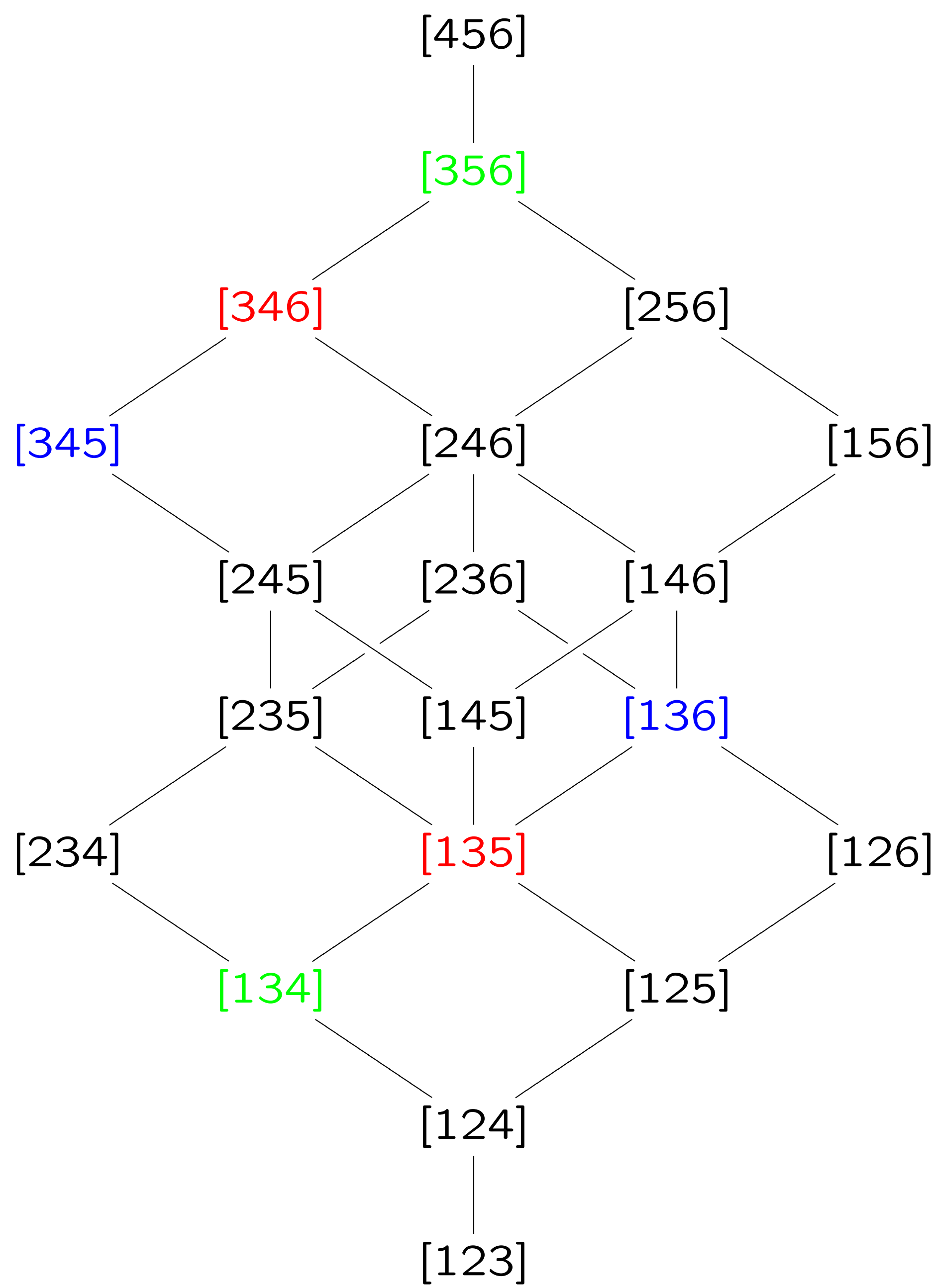
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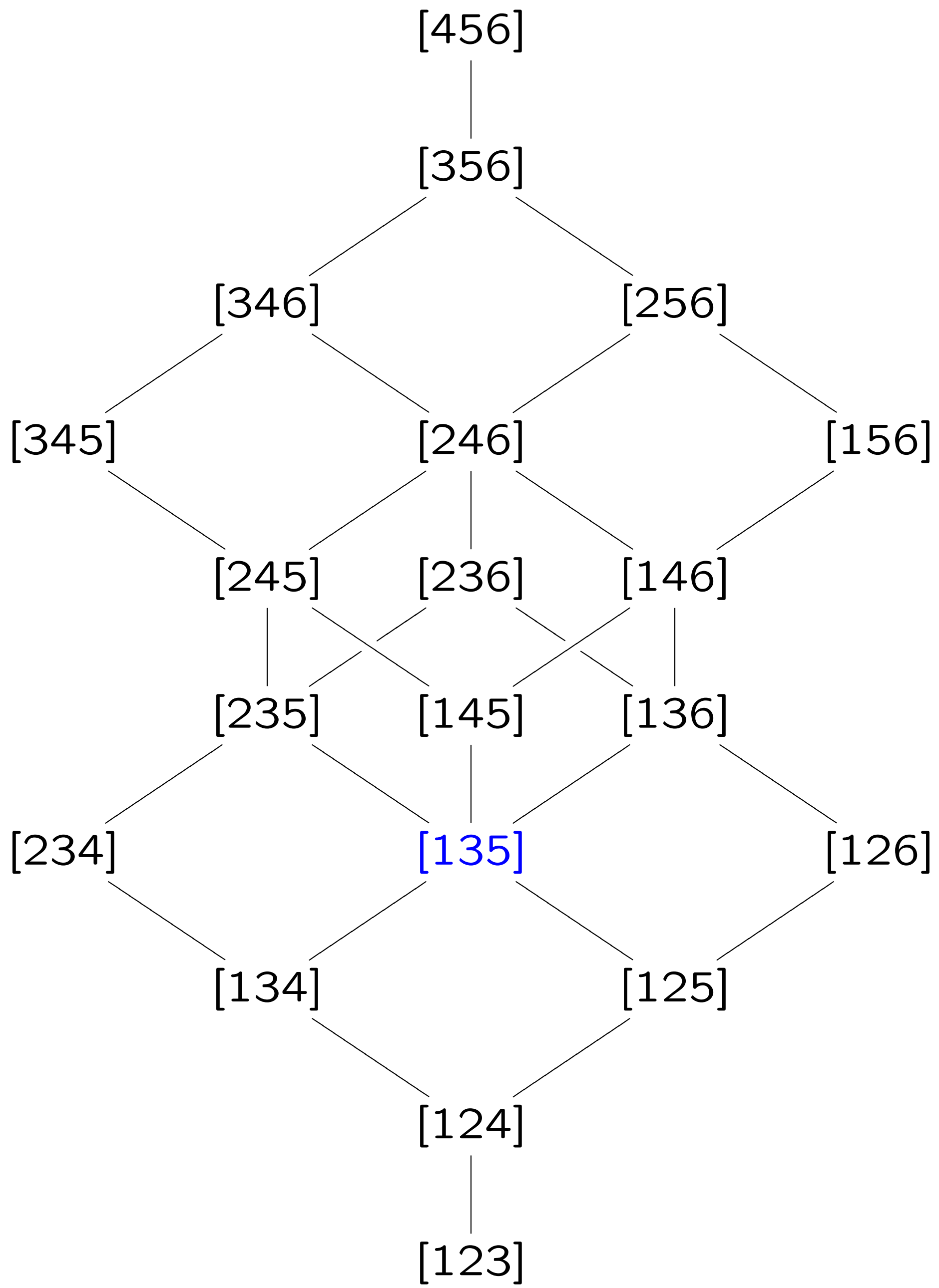
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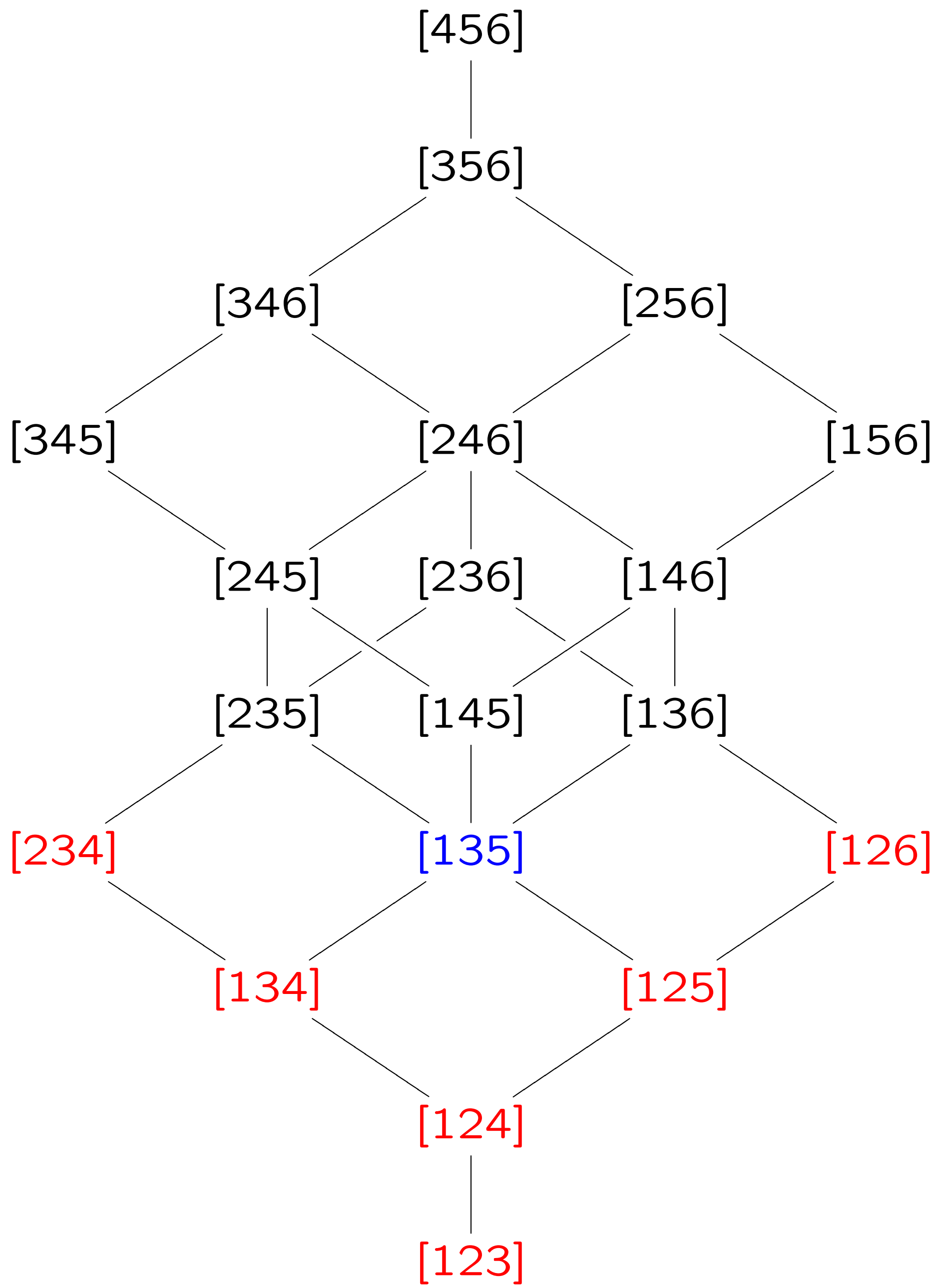
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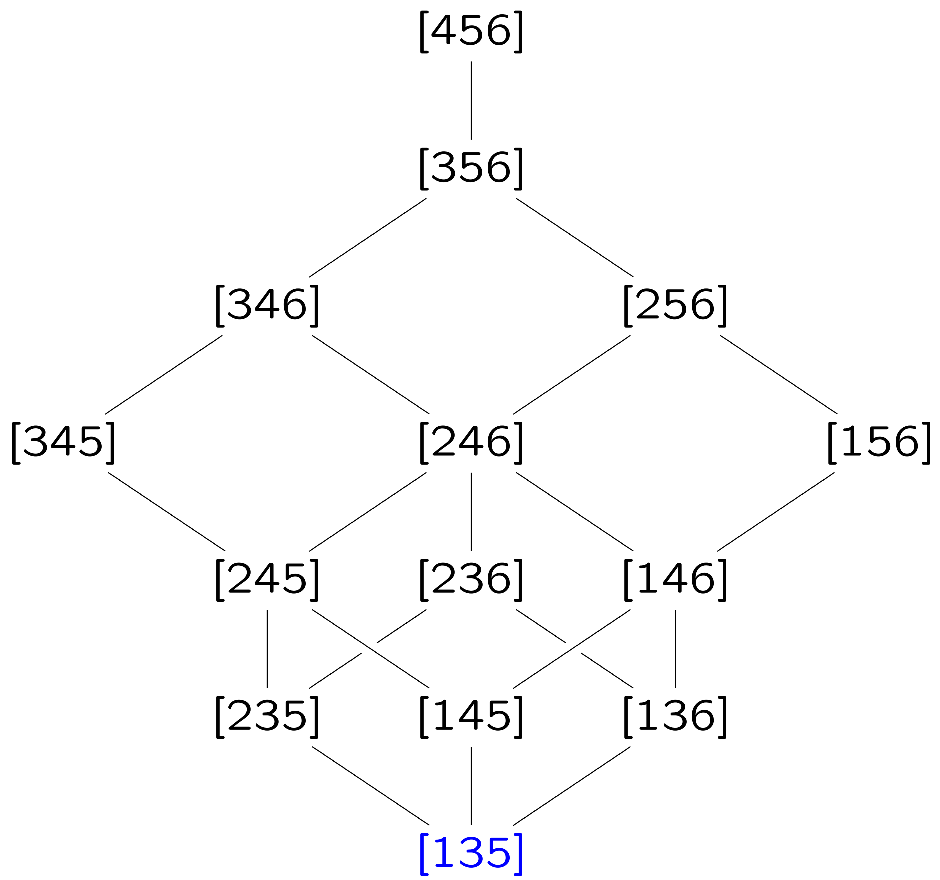
Quantum Schubert variety corresp to [135]



Schubert cell: use noncommutative dehomogenisation at [135]

L, Lenagan and Rigal (2008) There is a bijection between $\mathcal{H} - \mathrm{Spec}(\mathcal{G}_q(k,n))$ (ignoring the irrelevant ideal) and Cauchon diagrams on Young diagrams that fit inside a $k \times (n-k)$ array

The theorem is proved by defining quantum algebras with a straightening law, quantum Schubert varieties, quantum Schubert cells, partition subalgebras of quantum matrices and using a non-commutative version of dehomogenisation.

Sketch of proof

Let P be an $\mathcal{H}$-prime (different from the augmentation ideal). Choose a quantum minor γ with $\gamma \notin P$ but $\eta < \gamma \Rightarrow \eta \in P$

Claim: If $\eta \ngeq \gamma$, then $\eta \in P$.

So γ is unique for this P.

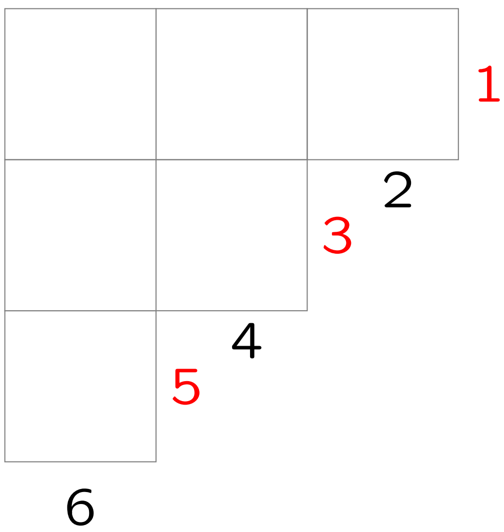
Hence P induces an $\mathcal{H}$-prime of

$$\frac{\mathcal{G}_q(k,n)}{I_\gamma}[\gamma^{-1}]$$

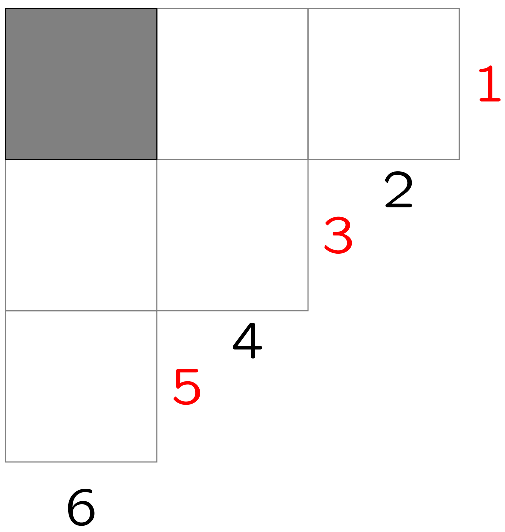
where $I_\gamma := \langle \eta \in \pi | \eta \ngeq \gamma \rangle$

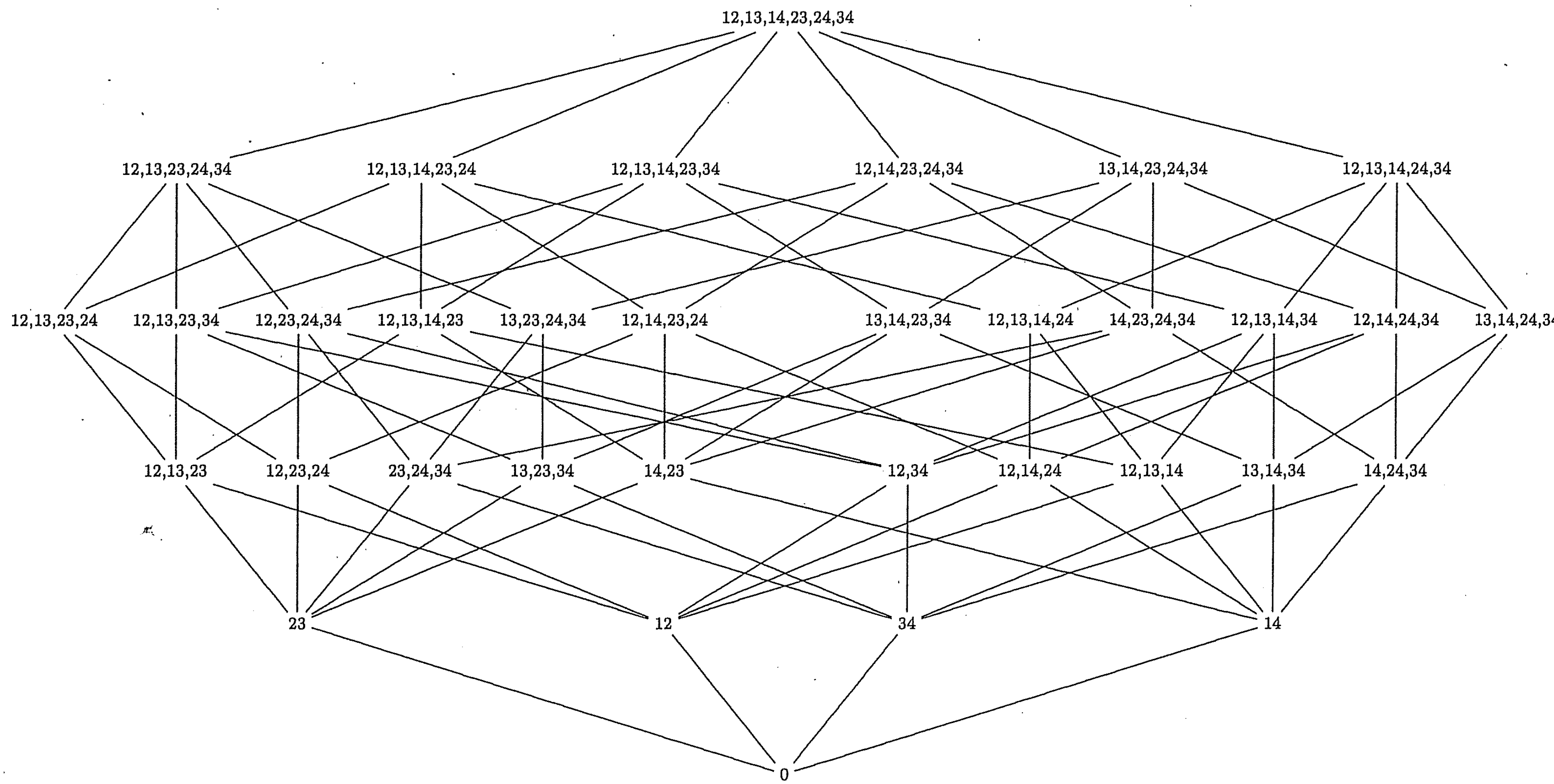
This algebra is isomorphic via noncommutative dehomogenisation to $A_\lambda[Z^{\pm 1}; \sigma_\gamma]$, where λ is a partition that fits into the partition $(n-k)^k$ and A_λ is the subalgebra of $O_q(M_{k,n-k})$ generated by those Y_{ij} with $j \leq \lambda_i$.

Schubert cell for [135]



$\mathcal{H}$**-prime in Schubert cell** [135]





$\mathcal{H} - \mathrm{Spec}(\mathcal{O}_q(G_{2,4}(\Bbbk)))$

The quantum grassmannian $\mathcal{G}_q(k,n)$: open questions

Conjecture (2007): $\mathcal{H}$-primes are generated by quantum Plücker coordinates.

Questions:

1. Can we specify the quantum Plücker coordinates in a given H-prime?
2. Can we describe the poset of $\mathcal{H}$-primes in $\mathcal{G}_q(k,n)$? (Yakimov's conjecture)
3. Does $\mathcal{G}_q(k,n)$ have the SDME?

The totally nonnegative world

Totally nonnegative (tnn for short) matrices

A matrix A in $\mathcal{M}_{m,p}$ is **totally nonnegative** if each of its minors is nonnegative.

A point P in the grassmannian $\mathcal{G}_{kn}(\mathbb{R})$ is **totally nonnegative** if its Plücker coordinates can be represented by the $k \times k$ minors of a $k \times n$ matrix A such that each of these $k \times k$ minors are nonnegative.

Cells are specified by stating precisely which minors are zero. If Z is a subset of minors then S_Z° is the cell where minors in Z are zero (and those not in Z are nonzero, so positive).

minors of an $n \times n$ matrix $= \sum_{k=1}^{n} \binom{n}{k}^2 = \binom{2n}{n} - 1 \approx \frac{4^n}{\sqrt{\pi n}}$

Example In $\mathcal{M}_2^{\mathsf{tnn}}$ the cell $S^{\circ}_{\{[2|2]\}}$ is empty.

For, suppose that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is tnn and $d = 0$.

Then $a, b, c \geq 0$ and also $ad - bc \geq 0$.

Thus, $-bc \geq 0$ and hence $bc = 0$ so that $b = 0$ or $c = 0$.

It is often easier to work in the totally non-negative grassmannian.

Example The 2×2 minors of

$$\begin{pmatrix} 0 & 1 & a & b \\ -1 & 0 & c & d \end{pmatrix}$$

are

$$[12] = 1, [13] = a, [14] = b, [23] = c, [24] = d, [34] = ad - bc,$$

so this matrix represents a point in the nonnegative 2×4 real grassmannian if and only if the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is totally nonnegative.

Postnikov has obtained a bijection between the non-empty tnn cells in $\mathcal{G}_{kn}^{\mathrm{tnn}}$ and various combinatorial objects such as Le-diagrams, decorated permutations, network diagrams, etc.

A Young diagram with entries either 0 or 1 is said to be a **Le-diagram** if it satisfies the following rule: if there is a 0 in a given square then either each square to the left is also filled with 0 or each square above is also filled with 0.

An example of a Le-diagram on the Young diagram representing the partition $\lambda = (5, 4, 3, 3, 1)$ is

1	1	0	1	0
0	0	0	1	
1	1	1		
0	0	1		
1				

A non-example is

1	1	0	1	0
0	0	1	0	
1	1	1		
0	0	1		
1				

• **Postnikov (arXiv:math/0609764)** There is a bijection between Le-diagrams (=Cauchon diagrams) on partitions that fit into a $k \times (n-k)$ array and non-empty cells S_Z° in $\mathcal{G}_{kn}^{\mathrm{tnn}}$.

(A Young diagram representing a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ fits inside a $k \times (n-k)$ array provided that $(n-k) \geq \lambda_1 \geq \lambda_2 \geq \lambda_k \geq 0$.)

For 2×2 **matrices**, this says that there is a bijection between Le-diagrams on 2×2 arrays and non-empty cells in $\mathcal{M}_2^{\mathrm{tnn}}$.

The next page is from Lauren Williams' thesis and shows the non-empty cells in the totally non-negative 2×4 real grassmannian.

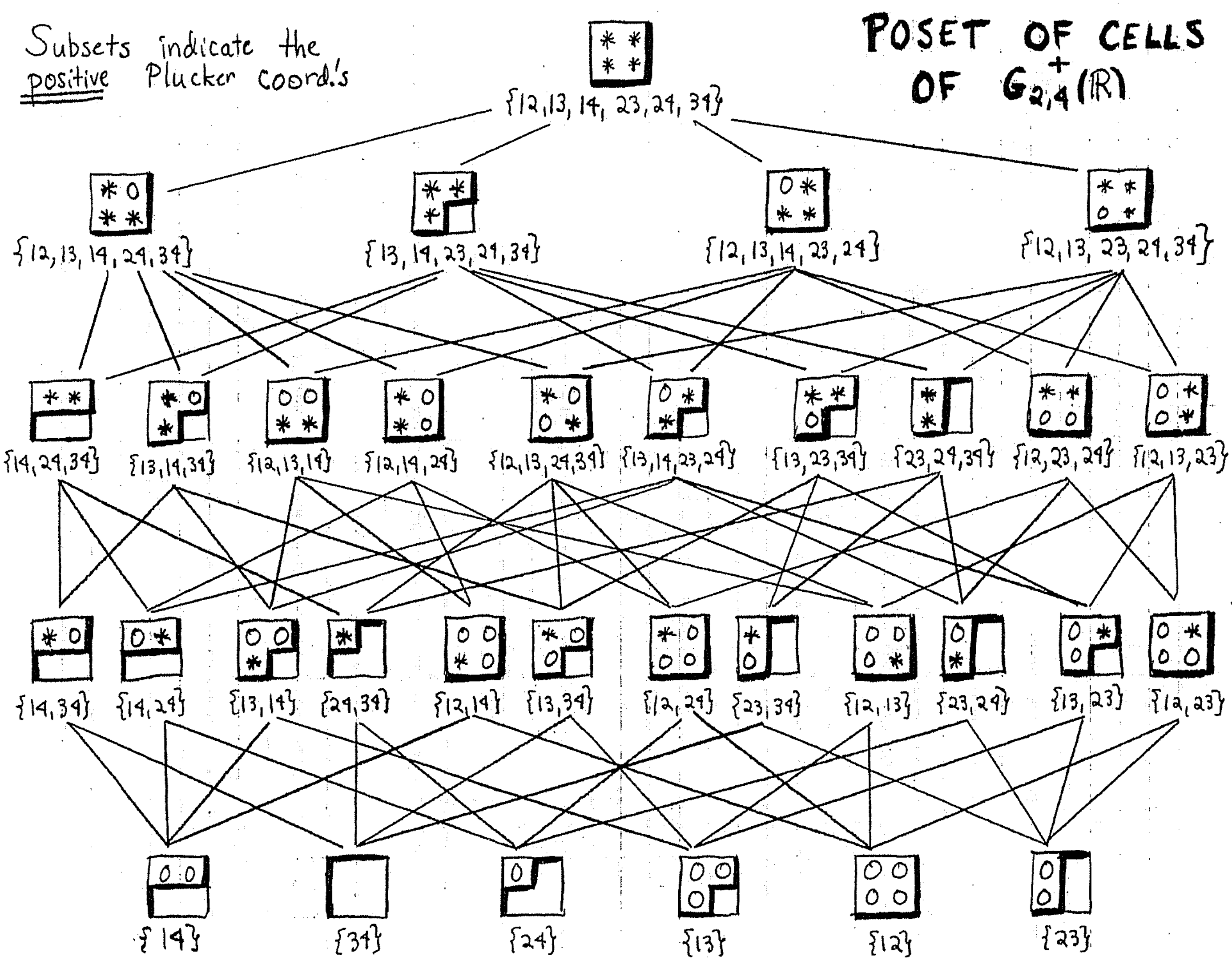


Figure A-1: L-diagrams

TNN versus Quantum

Goodearl-L.-Lenagan (2011) Let $\mathcal{F}$ be a family of minors in the coordinate ring of $\mathcal{M}_{m,p}(\mathbb{C})$, and let $\mathcal{F}_q$ be the corresponding family of quantum minors in $\mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C}))$. Then the following are equivalent:

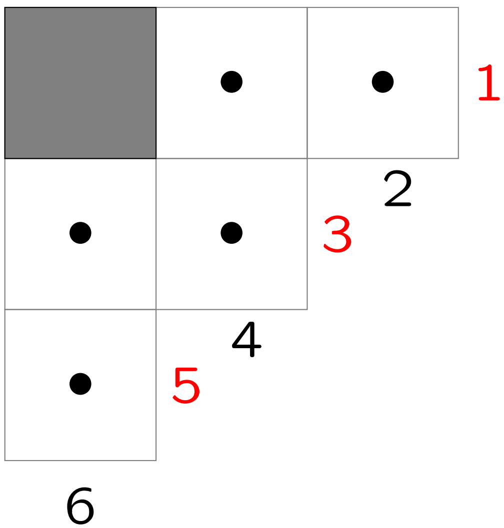
1. The totally nonnegative cell associated to $\mathcal{F}$ is non-empty.

2. $\mathcal{F}_q$ is the set of all quantum minors that belong to torus-invariant prime in $\mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C}))$.

Positroid varieties

Knutson-Lam-Speyer Let $\mathcal{F}$ be a family of Plücker coordinates that defines a nonempty cell in the tnn grassmannian. Then $\langle F \rangle$ is a prime ideal.

There are several descriptions of the above family F using combinatorial tools such as grassmann necklaces, planar networks, etc.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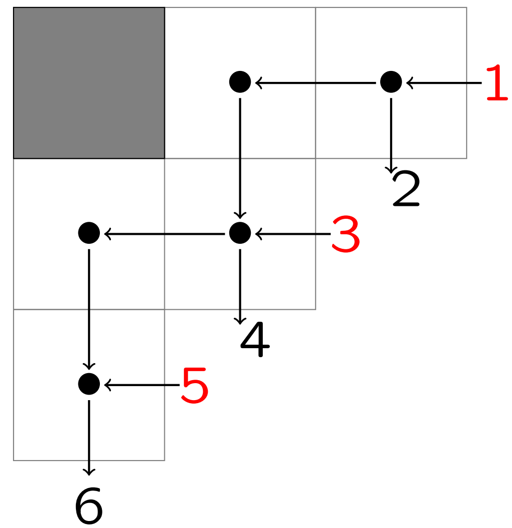
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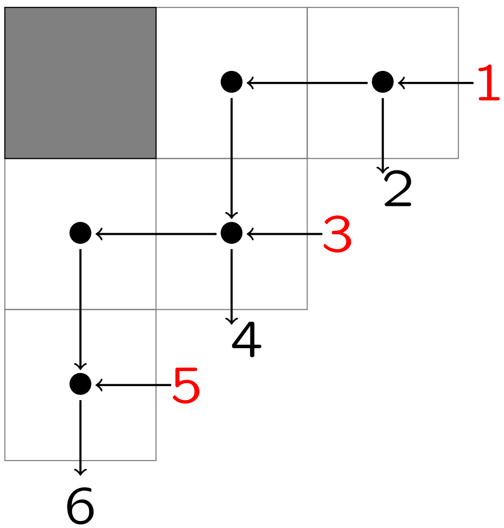


There is a vertex disjoint set of paths from $\{1, 3\}$ to $\{2, 4\}$ so [245] is positive on this corresponding cell.

There is no vertex disjoint set of paths from $\{1, 3\}$ to $\{4, 6\}$ so [456] is zero on this cell.

Quantum Plücker coordinates in a given $\mathcal{H}$-prime

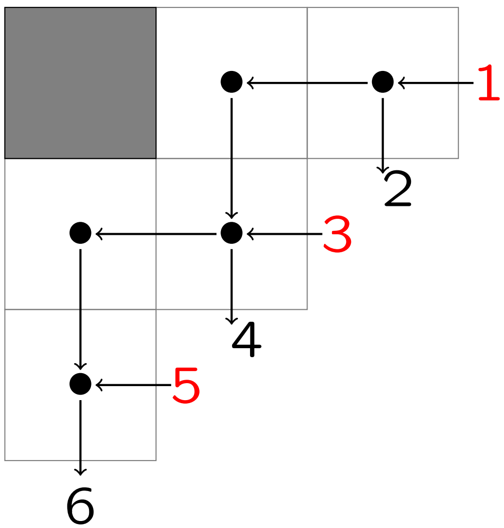
L-Lenagan-Nolan Let $\mathcal{F}$ be a family of Plücker coordinates and $\mathcal{F}_q$ the corresponding family of quantum Plücker coordinates. TFAE

• The totally nonnegative cell associated to $\mathcal{F}$ in $\mathcal{G}_{kn}^{\text{tnn}}$ is non-empty.

• $\mathcal{F}_q$ is the set of all quantum minors that belong to torus-invariant prime in $\mathcal{G}_q(k,n)$.

Strategy: Let C be a Cauchon diagram of shape λ. Then we prove that $[I] \in J_C$ iff there are no vertex disjoint set of paths from $\lambda \setminus I$ to $I \setminus \lambda$ in the Postnikov graph.

Consequence: We have an explicit description of these families thanks to work of Oh.



There is a vertex disjoint set of paths from $\{1, 3\}$ to $\{2, 4\}$ so [245] is not in the prime.

There is no vertex disjoint set of paths from $\{1, 3\}$ to $\{4, 6\}$ so [456] is in the prime.