Construction of Artin-Schelter Regular Algebras —— Homogeneous PBW Deformation

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Joint work with Y. SHEN and G.-S. ZHOU

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- 2-dim'l AS-regular algebras:
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- AS-regular algebras of global dimension 3 were classified by Artin, Schelter, Tate and Van den Bergh.

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- Normal extension of 3-dim'l regular algebras.
- AS-regular algebras with finitely many point modules.
- Ore extension of 3-dim'l regular algebras.
- Quantum 2×2-matrices.
- $q\mathbb{P}^3$ containing a quadric.
- $q\mathbb{P}^3$ related to some Clifford algebras.
- ......

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  The following algebras are AS-regular of dimension 4:

  1. $A(p) := \mathbb{C}\langle x,y\rangle/(xy^2 - p^2y^2x, x^3y + px^2yx + p^2xyx^2 + p^3yx^3)$, where $0 \neq p \in \mathbb{C}$;
  2. $B(p) := \mathbb{C}\langle x,y\rangle/(xy^2 + ip^2y^2x, x^3y + px^2yx + p^2xyx^2 + p^3yx^3)$, where $0 \neq p \in \mathbb{C}$ and $i^2 = -1$;
  3. $C(p) := \mathbb{C}\langle x,y\rangle/(xy^2 + pyxy + p^2y^2x, x^3y + jp^3yx^3)$, where $0 \neq p \in \mathbb{C}$ and $j^2 + j + 1 = 0$;
  4. $D(v,p) := \mathbb{C}\langle x,y\rangle/(xy^2 + vyxy + p^2y^2x, x^3y + (v+p)x^2yx + (pv+p^2)xyx^2 + p^3yx^3)$, where $v, p \in \mathbb{C}$ and $p \neq 0$.

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A complete classification of AS-regular $\mathbb{Z}^2$-graded algebras of all types above are finished in [Zhou-L., 2014].

The 5-dim'l AS-regular algebras of type (4,4,4,5,5)

[Zhou-L., 2014] Let $\mathcal{G} = \{\mathcal{G}(p,j) : p \neq 0,\ j^4 = 1\}$, where $\mathcal{G}(p,j) = k\langle x_1, x_2\rangle/(f_1, f_2, f_3, f_4, f_5)$ is the $\mathbb{Z}^2$-graded algebra with five relations

$$f_1 = x_2x_1^3 + px_1x_2x_1^2 + p^2x_1^2x_2x_1 + p^3x_1^3x_2,$$

$$f_2 = x_2^2x_1^2 + px_2x_1x_2x_1 + p^2x_2x_1^2x_2 + p^2x_1x_2^2x_1 + p^3x_1x_2x_1x_2 + p^4x_1^2x_2^2,$$

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$$f_5 = x_2^2x_1x_2x_1 + px_2x_1x_2^2x_1 + p^2jx_2x_1x_2x_1x_2 + p^3(j-j^2)x_2x_1^2x_2^2 + p^3(j-1)x_1x_2^2x_1x_2 + p^4(j-1)x_1x_2x_1x_2^2 + p^5(-1+j-j^3)x_1^2x_2^3.$$

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Question: Is there a 5-dimensional AS-regular algebra with 2 generators and 4 relations?

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Convention:

- k is a fixed algebraically closed field of characteristic zero.
- all algebras are generated in degree 1.
- r is a positive integer (> 1).

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Admissible order $<$ on $\mathbb{Z}^r$: $\alpha = (a_1, \cdots, a_r), \beta = (b_1, \cdots, b_r)$

$$\alpha < \beta \iff \begin{cases} \|\alpha\| < \|\beta\|, \text{ or} \\ \|\alpha\| = \|\beta\|, \ \exists t, a_i = b_i (i < t), a_t < b_t. \end{cases}$$

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$X = \{x_1, x_2, \cdots, x_n\}$, with a partition $\{X_1, \cdots, X_r\}$ and a $\mathbb{Z}^r$-grading: $\deg x := (\delta_{1i}, \cdots, \delta_{ri})$ for $x \in X_i$.

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$\rightsquigarrow$ $\mathbb{Z}^r$-filtration on $k\langle X\rangle$:

$$F_\alpha(k\langle X\rangle):=\begin{cases}0, & \text{if } \alpha<0;\\ \text{Span}_k\{u\in X^*\mid \deg u\le\alpha\}, & \text{if } \alpha\ge 0.\end{cases}$$

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The associated $\mathbb{Z}$-graded algebra $A^{\mathrm{gr}} = \bigoplus_{i \in \mathbb{Z}} (A^{\mathrm{gr}})_i$, where

$$(A^{\mathrm{gr}})_i = \bigoplus_{\|\alpha\| = i} A_\alpha, \text{ for all } i.$$

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$$p_s = s + \bar{s}, \quad \bar{s} \in k\langle X\rangle,$$

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The new set $P = \{p_s \mid s \in S\}$ products a $\mathbb{Z}$-graded algebra

$$U = k\langle X\rangle/(P).$$

There is a natural $\mathbb{Z}^r$-filtration on U defined by

$$F_\alpha U = \frac{F_\alpha k\langle X\rangle + (P)}{(P)},$$

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Definition

We say that U is a *homogeneous PBW deformation* of A, if φ is an isomorphism.

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2. Double Ore extension $A_P[y_1,y_2;\sigma,\delta,\tau]$ is a homogeneous PBW deformation of trimmed Double Ore extension $A_P[y_1,y_2;\sigma]$.

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Denote the *leading homogeneous polynomial* of f as

$$\mathrm{LH}(f) = f_q.$$

Gröbner basis

$k\langle X\rangle$: the $\mathbb{Z}^r$-graded algebra defined above.

For any $f \in k\langle X\rangle$, write $f = f_1 + f_2 + \cdots + f_q$, where$f_i$'s are $\mathbb{Z}^r$-homogeneous polynomials with $\deg f_1 < \cdots < \deg f_q$.

Denote the *leading homogeneous polynomial* of f as

$$\mathrm{LH}(f) = f_q.$$

Theorem (H.-S. Li)

Let $U = k\langle X\rangle/(P)$ and $G^r(U)$ defined as above. Suppose $\mathcal{G}$ is the Gröbner basis of (P). Then

$$G^r(U) \cong k\langle X\rangle/(\mathrm{LH}(\mathcal{G})).$$

- H.-S. Li, *Gröbner bases in ring theory*, Word Scientific Pub. Co. Pte. Ltd., (2012).

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Let $A = k\langle X\rangle/(S)$ be a connected $\mathbb{Z}^r$-graded algebra and $U = k\langle X\rangle/(P)$ be defined as above. Suppose $\mathcal{G}$ is the Gröbner basis of (P). Then the following are equivalent

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- U is a homogeneous PBW deformation of A.
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- $\mathrm{LH}(\mathcal{G})$ is a Gröbner basis of (S).
- $H_{A^{\mathrm{gr}}}(t) = H_U(t)$.

Artin-Schelter regular algebras

Definition (Artin-Schelter)

Let A be a connected graded algebra. We say it is *Artin-Schelter regular* if

1. A has finite global dimension d;
2. A has finite GK-dimension;
3. A is Gorenstein, that is

$$\underline{\mathrm{Ext}}_A^i(k, A) = \begin{cases} 0, & i \neq d, \\ k(\gamma), & i = d. \end{cases}$$

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Proposition

Let A be a connected $\mathbb{Z}^r$-graded algebra. Then A is AS-regular if and only if A^{gr} is AS-regular. Moreover, if A is of Gorenstein parameter γ, then A^{gr} is of Gorenstein parameter $\|\gamma\|$.

Example

The quantum plane $\mathfrak{A}(q) = k\langle x, y\rangle/(xy - qyx)$ $(0 \neq q \in k)$ with $\deg x = (1, 0)$ and $\deg y = (0, 1)$, so $P = \{xy - qyx + py^2\}$.

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Check: P is the Gröbner basis of (P), and $\mathrm{LH}(P) = \{xy - qyx\}$, and so $U(\mathfrak{A}(q)) = k\langle x, y\rangle/(xy - qyx + py^2)$ is a homogeneous PBW-deformation of $\mathfrak{A}(q)$.

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Up to graded isomorphism, there are two cases:

- $q \neq 1$: $U(\mathfrak{A}(q)) \cong A_q$ by the map $x \mapsto x - \frac{p}{1-q}y$, $y \mapsto y$.

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As a consequence, each homogeneous PBW-deformation of $\mathfrak{A}(q)$ is an AS-regular algebra.

Artin-Schelter regular criterion

Theorem

Let $A = k\langle x, y\rangle/(S)$ be a connected $\mathbb{Z}^r$-graded algebra, and U be a homogeneous PBW deformation of A. Then:

- $gl\dim U \leq gl\dim A$;

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- if A is AS-regular, then so is U;
- if A is strongly noetherian, then so is U;
- if A is Auslander regular, then so is U;
- if A is Cohen-Macaulay, then so is U.

The 4-dim'l AS-regular algebra of Jordan type

Take $A = D(-2, -1) = k\langle x, y\rangle/(g_1, g_2)$, the enveloping algebra of positively graded Lie algebra of dimension 4, where

$$\begin{aligned} g_1 &= xy^2 - 2yxy + y^2x, \\ g_2 &= x^3y - 3x^2yx + 3xyx^2 - yx^3. \end{aligned}$$

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A is a $\mathbb{Z}^2$-graded algebra with $\deg x = (1, 0)$ and $\deg y = (0, 1)$. Then the Gröbner basis $\mathcal{G}$ of the ideal (g_1, g_2) is $\{g_1, g_2, g_3\}$, where

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Let $U(A) = k\langle x, y\rangle/(P)$, where P is the reduced form:

$$\left\{ \begin{array}{l} xy^2 - 2yxy + y^2x + ay^3, \\ x^3y - 3x^2yx + 3xyx^2 - yx^3 + b_1xyxy + b_2yx^2y \\ \qquad +b_3yxyx + b_4y^2x^2 + c_1y^2xy + c_2y^3x + dy^4 \end{array} \middle| \begin{array}{l} a, b_i, c_j, d \in k \\ i = 1,2,3,4 \\ j = 1,2 \end{array} \right\}.$$

With the help of Maple, we get

Theorem

The algebra $\mathcal{J} = \mathcal{J}(u, v, w) = k\langle x, y\rangle/(f_1, f_2)$ is an AS-regular algebra of global dimension 4, where

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If k is algebraically closed of characteristic 0, then it is, up to isomorphism, the unique AS-regular algebra of global dimension 4 which is generated by two elements whose Frobenius data is of Jordan type.

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Note: In other cases, the Frobenius data falls within diagonal type.

Thank You!