Twisted Calabi-Yau and Artin-Schelter regular properties for locally finite graded algebras

Manuel L. Reyes

BIRS — 9/15/2016

Joint work with Daniel Rogalski

Noncommutative polynomial algebras

What kind of noncommutative graded algebras A deserve to be viewed as "noncommutative polynomials"? ($k =$ an arbitrary field.)

Notes: (1) We allow $\mathrm{GKdim}(A) = \infty$.
(2) Our graded algebras are all $\mathbb{N}$-graded: $A = \bigoplus_{n=0}^{\infty} A_n$.

Noncommutative polynomial algebras

What kind of noncommutative graded algebras A deserve to be viewed as "noncommutative polynomials"? ($k =$ an arbitrary field.)

Notes: (1) We allow $\mathrm{GKdim}(A) = \infty$.
(2) Our graded algebras are all $\mathbb{N}$-graded: $A = \bigoplus_{n=0}^{\infty} A_n$.

Two candidates:

1. Artin-Schelter regular algebras
2. Graded twisted Calabi-Yau algebras

Noncommutative polynomial algebras

What kind of noncommutative graded algebras A deserve to be viewed as "noncommutative polynomials"? ($k =$ an arbitrary field.)

Notes: (1) We allow $\mathrm{GKdim}(A) = \infty$.
(2) Our graded algebras are all $\mathbb{N}$-graded: $A = \bigoplus_{n=0}^{\infty} A_n$.

Two candidates:

1. Artin-Schelter regular algebras
2. Graded twisted Calabi-Yau algebras

Q: How do these compare?

- Same if A is connected: $A_0 = k$.
- **Today's talk:** What happens when A is not connected?

Non-connected algebras: an apology

Why should we care about non-connected algebras?

“Intrinsic” examples: Quivers algebras with relations kQ/I have nontrivial idempotents. (And their associated derived categories can be useful.)

“Extrinsic” examples: Twisted group algebras (or smash products) constructed from A can contain idempotents, even if A does not.

While nontrivial idempotents make these much less “like polynomials,” it’s still useful to understand when they are “homologically nice.”

Preliminaries: the enveloping algebra

The enveloping algebra of A is $A^e = A \otimes A^{\mathrm{op}}$. A left/right A^e-module M is the same as a k-central (A, A)-bimodule:

$$(a \otimes b^{\mathrm{op}}) \cdot m = a \cdot m \cdot b = m \cdot (b \otimes a^{\mathrm{op}})$$

Preliminaries: the enveloping algebra

The enveloping algebra of A is $A^e = A \otimes A^{\mathrm{op}}$. A left/right A^e-module M is the same as a k-central (A, A)-bimodule:

$$(a \otimes b^{\mathrm{op}}) \cdot m = a \cdot m \cdot b = m \cdot (b \otimes a^{\mathrm{op}})$$

Provides a convenient way to discuss homological algebra for bimodules:

- Projective/injective bimodules $\leftrightsquigarrow$ Projective/injective A^e-modules
- Resolutions of (A, A)-bimodules $\leftrightsquigarrow$ resolutions of A^e-modules

Preliminaries: the enveloping algebra

The enveloping algebra of A is $A^e = A \otimes A^{\mathrm{op}}$. A left/right A^e-module M is the same as a k-central (A, A)-bimodule:

$$(a \otimes b^{\mathrm{op}}) \cdot m = a \cdot m \cdot b = m \cdot (b \otimes a^{\mathrm{op}})$$

Provides a convenient way to discuss homological algebra for bimodules:

- Projective/injective bimodules $\leftrightsquigarrow$ Projective/injective A^e-modules
- Resolutions of (A, A)-bimodules $\leftrightsquigarrow$ resolutions of A^e-modules

Def: A is homologically smooth if A has a projective resolution in A^e-Mod of finite length whose terms are finitely generated over A^e. (A is a perfect A^e-module.)

This implies finite global dimension.

Calabi-Yau and twisted CY algebras

Definition

(i) A is twisted Calabi-Yau of dimension d if it is homologically smooth and there is an invertible (A, A)-bimodule U such that, as A^e-modules,

$$\mathrm{Ext}^i_{A^e}(A, A^e) \cong \begin{cases} 0 & \text{if } i \neq d, \\ U & \text{if } i = d. \end{cases}$$

Calabi-Yau and twisted CY algebras

Definition

(i) A is twisted Calabi-Yau of dimension d if it is homologically smooth and there is an invertible (A, A)-bimodule U such that, as A^e-modules,

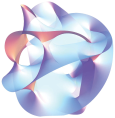
$$\mathrm{Ext}^i_{A^e}(A, A^e) \cong \begin{cases} 0 & \text{if } i \neq d, \\ U & \text{if } i = d. \end{cases}$$

(ii) [Ginzburg] A is Calabi-Yau of dimension d if it twisted CY of dimension d with $U = A$.

The CY condition is "self-duality" of sorts: if $P_\bullet \to A \to 0$ is a projective A^e-resolution, then $\mathrm{Hom}_{A^e}(P_\bullet, A^e)$ is also a resolution of A.

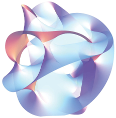
Commutative examples of Calabi-Yau algebras

(1) Calabi-Yau varieties: Coordinate rings of smooth affine Calabi-Yau varieties are CY algebras [Ginzburg]



Commutative examples of Calabi-Yau algebras

(1) Calabi-Yau varieties: Coordinate rings of smooth affine Calabi-Yau varieties are CY algebras [Ginzburg]



We can also consider *graded* Calabi-Yau algebras: take the projective A^e-resolution and Ext isomorphism to be in the graded category.

(2) Graded commutative examples: just direct sums of $k[x_1, \ldots, x_n]$.

We emphasize **(2)**: So graded Calabi-Yau algebras are "noncommutative polynomial rings."

But so are the Artin-Schelter regular algebras. How do these compare?

Artin-Schelter regular algebras

The more standard notion of "noncommutative polynomial algebra."

Def: A connected graded algebra A is Artin-Schelter (AS) regular of dimension d if A has global dimension $d < \infty$ and

$$\mathrm{Ext}^i_A(k, A) \cong \begin{cases} 0, & i \neq d, \\ k(\ell), & i = d \end{cases}$$

in Mod-A, and similarly for $\mathrm{Ext}^i_{A^{\mathrm{op}}}(k, A)$. (We allow $\mathrm{GKdim}(A) = \infty$.)

Many examples already discussed at this conference!

How does this compare with the CY condition?

Artin-Schelter regular algebras

The more standard notion of "noncommutative polynomial algebra."

Def: A connected graded algebra A is Artin-Schelter (AS) regular of dimension d if A has global dimension $d < \infty$ and

$$\mathrm{Ext}^i_A(k, A) \cong \begin{cases} 0, & i \neq d, \\ k(\ell), & i = d \end{cases}$$

in Mod-A, and similarly for $\mathrm{Ext}^i_{A^{\mathrm{op}}}(k, A)$. (We allow $\mathrm{GKdim}(A) = \infty$.)

Many examples already discussed at this conference!

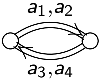
How does this compare with the CY condition?

Theorem [Yekutieli & Zhang], [R., Rogalski, Zhang]: A connected graded algebra is twisted CY-d if and only if it is AS regular of dimension d.

So twisted CY yields the expected "noncommutative polynomial algebras."

Algebras from quivers and potentials

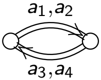
Quiver algebras: Quiver algebras with (twisted) superpotentials tend to give rise to (twisted) CY algebras.

Ex: [Bocklandt] For $Q =$ 

and the superpotential $W = \sum \circlearrowleft (a_1 a_3 a_2 a_4 + a_1 a_4 a_2 a_3)$, the Jacobi algebra $B = \mathbb{C}Q/(\partial_a W)$ is Calabi-Yau of dimension 3.

Algebras from quivers and potentials

Quiver algebras: Quiver algebras with (twisted) superpotentials tend to give rise to (twisted) CY algebras.

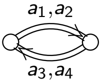
Ex: [Bocklandt] For $Q =$ 

and the superpotential
$W = \sum \circlearrowleft (a_1 a_3 a_2 a_4 + a_1 a_4 a_2 a_3)$, the Jacobi algebra $B = \mathbb{C}Q/(\partial_a W)$ is Calabi-Yau of dimension 3. Relations:

$$\begin{aligned}
\partial_{a_1} W = 0 \quad &\rightsquigarrow \quad a_3 a_2 a_4 = -a_4 a_2 a_3 \\
\partial_{a_2} W = 0 \quad &\rightsquigarrow \quad a_4 a_1 a_3 = -a_2 a_3 a_1 \\
\partial_{a_3} W = 0 \quad &\rightsquigarrow \quad a_2 a_4 a_1 = -a_1 a_4 a_2 \\
\partial_{a_4} W = 0 \quad &\rightsquigarrow \quad a_1 a_3 a_2 = -a_2 a_3 a_1
\end{aligned}$$

Algebras from quivers and potentials

Quiver algebras: Quiver algebras with (twisted) superpotentials tend to give rise to (twisted) CY algebras.

Ex: [Bocklandt] For $Q =$ 

and the superpotential $W = \sum \circlearrowleft (a_1 a_3 a_2 a_4 + a_1 a_4 a_2 a_3)$, the Jacobi algebra $B = \mathbb{C}Q/(\partial_a W)$ is Calabi-Yau of dimension 3. Relations:

$$\begin{aligned} \partial_{a_1} W = 0 \quad &\rightsquigarrow \quad a_3 a_2 a_4 = -a_4 a_2 a_3 \\ \partial_{a_2} W = 0 \quad &\rightsquigarrow \quad a_4 a_1 a_3 = -a_2 a_3 a_1 \\ \partial_{a_3} W = 0 \quad &\rightsquigarrow \quad a_2 a_4 a_1 = -a_1 a_4 a_2 \\ \partial_{a_4} W = 0 \quad &\rightsquigarrow \quad a_1 a_3 a_2 = -a_2 a_3 a_1 \end{aligned}$$

Q: When is a superpotential "nice"? Hard in general, but answered for connected CY-3 algebras by [Mori & Smith], [Mori & Ueyama].

Constructions preserving twisted CY property

Direct and tensor products:

Theorem

Thm: *Let A_1 and A_2 be twisted Calabi-Yau algebras of dimension d_1 and d_2, respectively.*

- *If $d_1 = d_2 = d$, then $A_1 \times A_2$ is twisted CY of dimension d.*
- *$A_1 \otimes A_2$ is twisted CY of dimension $d_1 + d_2$.*

Constructions preserving twisted CY property

Direct and tensor products:

Theorem

Thm: *Let A_1 and A_2 be twisted Calabi-Yau algebras of dimension d_1 and d_2, respectively.*

- *If $d_1 = d_2 = d$, then $A_1 \times A_2$ is twisted CY of dimension d.*
- *$A_1 \otimes A_2$ is twisted CY of dimension $d_1 + d_2$.*

Extension of scalars and Morita equivalence:

Theorem

Thm: *Let A be twisted CY of dimension d.*

- *$A \otimes K$ is twisted CY-d for every field extension K/k.*
- *Every algebra Morita equivalent to A is twisted CY-d.*

An example with U nontrivial

How do we find examples with $U \neq {}^1A^\mu(I)$?

Ex: Set $B = k[x,y] \rtimes \mathbb{Z}_2$: twisted CY-2 with a Nakayama automorphism and $B_0 = ke_1 \oplus ke_2$ (here $\mathrm{char}(k) \neq 2$).

An example with U nontrivial

How do we find examples with $U \neq {}^1A^{\mu}(I)$?

Ex: Set $B = k[x, y] \rtimes \mathbb{Z}_2$: twisted CY-2 with a Nakayama automorphism and $B_0 = ke_1 \oplus ke_2$ (here $\mathrm{char}(k) \neq 2$).

$\mathbb{M}_2(B)$ has $1 = f_1 + f_2 + f_3 + f_4$ for primitive idempotents

$$f_1 = \begin{pmatrix} e_1 & 0 \\ 0 & 0 \end{pmatrix}, f_2 = \begin{pmatrix} e_2 & 0 \\ 0 & 0 \end{pmatrix}, f_3 = \begin{pmatrix} 0 & 0 \\ 0 & e_1 \end{pmatrix}, f_4 = \begin{pmatrix} 0 & 0 \\ 0 & e_2 \end{pmatrix}$$

Set $e = f_1 + f_2 + f_3$ (full idempotent), then $A = e\mathbb{M}_2(B)e$ is Morita equivalent to A and thus is twisted CY-2.

An example with U nontrivial

How do we find examples with $U \neq {}^1A^\mu(I)$?

Ex: Set $B = k[x, y] \rtimes \mathbb{Z}_2$: twisted CY-2 with a Nakayama automorphism and $B_0 = ke_1 \oplus ke_2$ (here $\mathrm{char}(k) \neq 2$).

$\mathbb{M}_2(B)$ has $1 = f_1 + f_2 + f_3 + f_4$ for primitive idempotents

$$f_1 = \begin{pmatrix} e_1 & 0 \\ 0 & 0 \end{pmatrix}, f_2 = \begin{pmatrix} e_2 & 0 \\ 0 & 0 \end{pmatrix}, f_3 = \begin{pmatrix} 0 & 0 \\ 0 & e_1 \end{pmatrix}, f_4 = \begin{pmatrix} 0 & 0 \\ 0 & e_2 \end{pmatrix}$$

Set $e = f_1 + f_2 + f_3$ (full idempotent), then $A = e\mathbb{M}_2(B)e$ is Morita equivalent to A and thus is twisted CY-2.

Have indecomposable decomposition as projective right modules

$$A_A \cong P \oplus P \oplus Q \qquad \text{but} \qquad U_A \cong P \oplus Q \oplus Q.$$

So U_A not free $\implies$ $U \ncong {}^1A^\mu(\ell)$.

Working with locally finite algebras

We work in the setting of locally finite algebras: $A = \bigoplus A_n$ with all $\dim_k(A_n) < \infty$. So $A_0 =$ (arbitrary!) finite-dimensional algebra

Working with locally finite algebras

We work in the setting of locally finite algebras: $A = \bigoplus A_n$ with all $\dim_k(A_n) < \infty$. So $A_0 =$ (arbitrary!) finite-dimensional algebra

The "good" choice for graded Nakayama's Lemma & minimal graded projective resolutions (after Minamoto & Mori):

Graded Jacobson radical: $J(A) = J(A_0) + A_{\geq 1}$.

We obtain a f.d. semisimple algebra $S = A/J(A) = A_0/J(A_0)$.

First problem: We'd like the f.d. algebra $B := A_0$ to be "well behaved."

Working with locally finite algebras

We work in the setting of locally finite algebras: $A = \bigoplus A_n$ with all $\dim_k(A_n) < \infty$. So $A_0 =$ (arbitrary!) finite-dimensional algebra

The "good" choice for graded Nakayama's Lemma & minimal graded projective resolutions (after Minamoto & Mori):

Graded Jacobson radical: $J(A) = J(A_0) + A_{\geq 1}$.

We obtain a f.d. semisimple algebra $S = A/J(A) = A_0/J(A_0)$.

First problem: We'd like the f.d. algebra $B := A_0$ to be "well behaved."

Recall that twisted CY algebras must be homologically smooth.

Lemma: If A is homologically smooth, then so is A_0.

Graded homologically smooth algebras

Lemma: If A is homologically smooth, then so is $B = A_0$.

How should we think about f.d. homologically smooth algebras?

Fact: If B is a f.d. algebra, then TFAE:

1. B is homologically smooth
2. $\text{pdim}({}_{B^e}B) < \infty$
3. B^e has finite global dimension
4. $B \otimes K$ has finite global dimension for every field extension K/k

Graded homologically smooth algebras

Lemma: If A is homologically smooth, then so is $B = A_0$.

How should we think about f.d. homologically smooth algebras?

Fact: If B is a f.d. algebra, then TFAE:

1. B is homologically smooth
2. $\mathrm{pdim}({}_{B^e}B) < \infty$
3. B^e has finite global dimension
4. $B \otimes K$ has finite global dimension for every field extension K/k

Reminiscent of separable algebras S, defined by the equivalent conditions:

1. S is projective as a left S^e-module
2. S^e is semisimple
3. $S \otimes K$ is semisimple for all field extensions K/k
4. $S \cong \prod_{i=1}^{n} \mathbb{M}_{n_i}(D_i)$ with all $Z(D_i)/k$ separable

Graded homolgically smooth algebras

For passage between A-modules and A^e-modules, it's important that A have $S = A/J(A) = A_0/J(A_0)$ separable. One example:

Lemma

If A is locally finite with S separable, then $\mathrm{gl.dim}(A)$ *is equal to*

$$\mathrm{pdim}({}_A S) = \mathrm{pdim}({}_{A^e} A) = \mathrm{pdim}(S_A).$$

Graded homolgically smooth algebras

For passage between A-modules and A^e-modules, it's important that A have $S = A/J(A) = A_0/J(A_0)$ separable. One example:

Lemma

If A is locally finite with S separable, then $\operatorname{gl.dim}(A)$ is equal to

$$\operatorname{pdim}({}_A S) = \operatorname{pdim}({}_{A^e} A) = \operatorname{pdim}(S_A).$$

Fortunately, this holds in our case. (Special thanks to MathOverflow!)

Theorem (Rickard)

If B is a finite-dimensional homologically smooth algebra, then $S = B/J(B)$ is separable.

In particular, A twisted CY $\implies$ S separable.

Dualities and graded socles

It's well known that twisted CY-d algebras satisfy Van den Bergh duality:

$$\operatorname{Ext}^i_{A^e}(A, M) \cong \operatorname{Tor}^{A^e}_{d-i}(A, U \otimes_A M)$$

for left A^e-modules M.

We can use this to deduce further dualities for "1-sided" modules...

Dualities and graded socles

It's well known that twisted CY-d algebras satisfy Van den Bergh duality:

$$\operatorname{Ext}^i_{A^e}(A, M) \cong \operatorname{Tor}^{A^e}_{d-i}(A, U \otimes_A M)$$

for left A^e-modules M.

We can use this to deduce further dualities for "1-sided" modules...

One important consequence allows us to "homologically compute" the socle of a module:

Proposition: For A locally finite twisted CY-d and ${}_AM$ a graded module, there is an isomorphism of graded left S-modules

$$\operatorname{Tor}^A_d(S, M) \cong U^{-1} \otimes_A \operatorname{soc}(M).$$

Twisted CY algebras of dimension 0

Socle formula: $\mathrm{Tor}_d^A(S, M) \cong U^{-1} \otimes_A \mathrm{soc}(M)$

Taking the case $M = A$ yields:

Cor: If A is locally finite twisted CY-d, if $d > 1$ then $\mathrm{soc}(A) = 0$.

Twisted CY algebras of dimension 0

Socle formula: $\mathrm{Tor}_d^A(S, M) \cong U^{-1} \otimes_A \mathrm{soc}(M)$

Taking the case $M = A$ yields:

Cor: If A is locally finite twisted CY-d, if $d > 1$ then $\mathrm{soc}(A) = 0$.

For the $d = 0$ case we have:

Cor: For a (not necessarily graded) algebra A, TFAE:

1. A is (twisted) CY-0
2. A is twisted CY and is a finite-dimensional k-algebra
3. A is a separable k-algebra.

Existing notions of generalized regularity

What should play the role of the AS regular property for non-connected algebras? There is precedent in the work of:

Martinez-Villa: $\operatorname{gl.dim}(A) = d$, the functors $\operatorname{Ext}^d_A(-, A)$ and $\operatorname{Ext}^d_{A^{\mathrm{op}}}(-, A)$ interchange graded simple modules, and other $\operatorname{Ext}^i_A(S, A) = 0 = \operatorname{Ext}^i_{A^{\mathrm{op}}}(T, A)$.

Minamoto & Mori: $\operatorname{gl.dim}(A) = d$ and there is a bimodule isomorphism:

$$\operatorname{Ext}^i_A(A_0, A) \cong \begin{cases} 0, & i \neq d, \\ (A_0^*)^\sigma(\ell), & i = d. \end{cases}$$

Note: The Ext condition can be written as $\operatorname{RHom}_A(A_0, A) \cong (A_0^*)^\sigma(\ell)[d]$.

Generalized regularity properties

For our purposes, we wish to allow a "twist" by a general invertible ${}_AU_A$.

Definitions: Let A be a locally finite graded algebra of (graded) global dimension $d < \infty$.

(a) A is MV-regular if there is a bijection π from the iso-classes of graded simple left modules to the graded simple right modules with $\mathrm{RHom}_A(M, A) \cong \pi(M)[d]$ for all graded simple ${}_AM$.

(b) A is MM-regular if $\mathrm{RHom}_A(A_0, A) \cong A_0^* \otimes_A U[d]$ as A^e-complexes for some invertible U.

(c) A is J-regular if $\mathrm{RHom}_A(S, A) \cong S \otimes_A U[d]$ as A^e-complexes for some invertible U, where $S = A/J(A)$.

Equivalence of twisted CY and AS regular properties

These properties are exactly what we need to characterize twisted CY algebras:

Theorem

Let A be a locally finite graded algebra, and set $S = A/J(A)$. Then TFAE:

1. *A is twisted Calabi-Yau of dimension d.*
2. *A is MV-regular of dimension d and S is separable.*
3. *A is MM-regular of dimension d and S is separable.*
4. *A is J-regular of dimension d and S is separable.*

So the twisted CY property (involving *bimodules*) can be verified using one of these AS regular properties (involving *one-sided modules*).

AS regular properties for quivers with relations

Q: Suppose $A = kQ/I$ is a graded quotient for a connected quiver Q. How do these regularity properties translate?

Here $S = A_0 = ke_1 \oplus \cdots \oplus ke_n$, with non-iso. simple modules $S_i = ke_i$.

Lemma: For such A, every invertible ${}_A U_A$ is "boring": $U = {}^1 A^{\mu}(\ell)$.

Such μ permutes the vertices $\{1, \ldots, n\}$; call this permutation μ also.

AS regular properties for quivers with relations

Q: Suppose $A = kQ/I$ is a graded quotient for a connected quiver Q. How do these regularity properties translate?

Here $S = A_0 = ke_1 \oplus \cdots \oplus ke_n$, with non-iso. simple modules $S_i = ke_i$.

Lemma: For such A, every invertible ${}_A U_A$ is "boring": $U = {}^1 A^\mu(\ell)$.

Such μ permutes the vertices $\{1, \dots, n\}$; call this permutation μ also.

The regularity conditions (and equivalently, the twisted CY condition) amount to:

$$\mathrm{Ext}^i_A(S_j, A) \cong \begin{cases} S_{\mu(j)}(\ell), & i = d, \\ 0, & i \neq d, \end{cases}$$

for all the graded simple S_j, and similar condition as simple right modules.

The noetherian property for low-dimensional algebras

If A is twisted CY-d of finite GK dimension, then is A noetherian?

We can answer yes when $d \leq 2$.

The noetherian property for low-dimensional algebras

If A is twisted CY-d of finite GK dimension, then is A noetherian?

We can answer yes when $d \leq 2$.

How to show an algebra is noetherian? The following is essentially part of "Cohen-type" arguments to establish that a ring is (left) noetherian.

Lemma: A graded algebra A is left noetherian if (and only if) every graded left noetherian A-module is finitely presented.

The noetherian property for low-dimensional algebras

If A is twisted CY-d of finite GK dimension, then is A noetherian?

We can answer yes when $d \leq 2$.

How to show an algebra is noetherian? The following is essentially part of "Cohen-type" arguments to establish that a ring is (left) noetherian.

Lemma: A graded algebra A is left noetherian if (and only if) every graded left noetherian A-module is finitely presented.

Proof: Suppose A weren't left noetherian. "Zornify" to obtain left ideal I maximal w.r.t. not being finitely generated.

The noetherian property for low-dimensional algebras

If A is twisted CY-d of finite GK dimension, then is A noetherian?

We can answer yes when $d \leq 2$.

How to show an algebra is noetherian? The following is essentially part of "Cohen-type" arguments to establish that a ring is (left) noetherian.

Lemma: A graded algebra A is left noetherian if (and only if) every graded left noetherian A-module is finitely presented.

Proof: Suppose A weren't left noetherian. "Zornify" to obtain left ideal I maximal w.r.t. not being finitely generated. Then A/I is not finitely presented (Schanuel's Lemma).

The noetherian property for low-dimensional algebras

If A is twisted CY-d of finite GK dimension, then is A noetherian?

We can answer yes when $d \leq 2$.

How to show an algebra is noetherian? The following is essentially part of "Cohen-type" arguments to establish that a ring is (left) noetherian.

Lemma: A graded algebra A is left noetherian if (and only if) every graded left noetherian A-module is finitely presented.

Proof: Suppose A weren't left noetherian. "Zornify" to obtain left ideal I maximal w.r.t. not being finitely generated. Then A/I is not finitely presented (Schanuel's Lemma). But every $J \supsetneq I$ finitely generated implies A/I is noetherian, a contradiction. □

The noetherian property

Original Goal: Show A left noetherian.

New Goal: Show noetherian A-modules are finitely presented.

The noetherian property

Original Goal: Show A left noetherian.

New Goal: Show noetherian A-modules are finitely presented.

But how? The "socle formula" $\mathrm{Tor}_d^A(S, M) \cong U^{-1} \otimes \mathrm{soc}(M)$ is handier than it might seem...

Fact: If M has minimal resolution $\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$, then each P_i is f.g. (or zero) if and only if $\mathrm{Tor}_i^A(S, M)$ is f.d. (or zero).

The noetherian property

Original Goal: Show A left noetherian.

New Goal: Show noetherian A-modules are finitely presented.

But how? The "socle formula" $\mathrm{Tor}_d^A(S, M) \cong U^{-1} \otimes \mathrm{soc}(M)$ is handier than it might seem...

Fact: If M has minimal resolution $\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$, then each P_i is f.g. (or zero) if and only if $\mathrm{Tor}_i^A(S, M)$ is f.d. (or zero).

Cor: If A is twisted CY-d and ${}_A M$ is graded noetherian, then the term P_d in the resolution above is finitely generated.

Proof: M is noetherian $\Rightarrow \mathrm{soc}(M)$ is f.d. $\Rightarrow \mathrm{Tor}_d^A(S, M)$ is f.d. □

Twisted CY-1 algebras

This is already enough for algebras of dimension 1:

Theorem: If A is locally finite twisted Calabi-Yau of dimension 1, then A is noetherian.

Twisted CY-1 algebras

This is already enough for algebras of dimension 1:

Theorem: If A is locally finite twisted Calabi-Yau of dimension 1, then A is noetherian.

Proof: If M is noetherian with minimal resolution $0 \to P_1 \to P_0 \to M$, we have P_1 f.g. by the "socle argument." So M is finitely presented. □

Twisted CY-1 algebras

This is already enough for algebras of dimension 1:

Theorem: If A is locally finite twisted Calabi-Yau of dimension 1, then A is noetherian.

Proof: If M is noetherian with minimal resolution $0 \to P_1 \to P_0 \to M$, we have P_1 f.g. by the "socle argument." So M is finitely presented. □

By the way, what do twisted CY-1 algebras look like?

Theorem: A locally finite graded algebra A is twisted Calabi-Yau of dimension 1 if and only if $A \cong T_S(V)$ is a tensor algebra, where S is a separable algebra and V is an invertible positively graded S^e-module.

But note that the noetherian result is proved *without* the structure theorem!

The noetherian argument in dimension 2

We don't expect *all* twisted CY-2 algebras to be noetherian:

Zhang: studied non-noetherian AS regular algebras A of dimension 2. He found that such A is noetherian $\iff$ $\mathrm{GKdim}(A) < \infty$

The noetherian argument in dimension 2

We don't expect *all* twisted CY-2 algebras to be noetherian:

Zhang: studied non-noetherian AS regular algebras A of dimension 2. He found that such A is noetherian $\iff$ $\mathrm{GKdim}(A) < \infty$

We found that a similar result holds for twisted CY algebras:

Theorem

Let A be a locally finite twisted Calabi-Yau algebra of dimension 2. Then A is noetherian if and only if A has finite GK dimension.

Again, this is proved *without* first classifying the iso-types of A.

The noetherian argument in dimension 2

Theorem

Let A be a locally finite twisted Calabi-Yau algebra of dimension 2. Then A is noetherian if and only if A has finite GK dimension.

Idea of Proof: Suppose M is a graded noetherian A-module.

Projective resolution: $0 \to P_2 \to P_1 \to P_0 \to M \to 0$.

Clearly P_0 is f.g.

The noetherian argument in dimension 2

Theorem

Let A be a locally finite twisted Calabi-Yau algebra of dimension 2. Then A is noetherian if and only if A has finite GK dimension.

Idea of Proof: Suppose M is a graded noetherian A-module.

Projective resolution: $0 \to P_2 \to P_1 \to P_0 \to M \to 0$.

Clearly P_0 is f.g. "Socle argument" $\implies$ P_2 is also f.g.

The noetherian argument in dimension 2

Theorem

Let A be a locally finite twisted Calabi-Yau algebra of dimension 2. Then A is noetherian if and only if A has finite GK dimension.

Idea of Proof: Suppose M is a graded noetherian A-module.

Projective resolution: $0 \to P_2 \to P_1 \to P_0 \to M \to 0$.

Clearly P_0 is f.g. "Socle argument" $\implies$ P_2 is also f.g.

Exactness above with P_2, P_0 f.g. and $\mathrm{GKdim}(A) < \infty$ implies $\mathrm{GKdim}(P_1) < \infty$.

Deduce that P_1 is finitely generated. □

The noetherian argument in dimension 2

Theorem

Let A be a locally finite twisted Calabi-Yau algebra of dimension 2. Then A is noetherian if and only if A has finite GK dimension.

Idea of Proof: Suppose M is a graded noetherian A-module.

Projective resolution: $0 \to P_2 \to P_1 \to P_0 \to M \to 0$.

Clearly P_0 is f.g. "Socle argument" $\implies$ P_2 is also f.g.

Exactness above with P_2, P_0 f.g. and $\mathrm{GKdim}(A) < \infty$ implies $\mathrm{GKdim}(P_1) < \infty$.

Deduce that P_1 is finitely generated. □

Open Q: If A above is not graded, must A still be noetherian?

Presentation of twisted CY-2 algebras

So what do twisted CY-2 algebras actually look like?

Zhang: AS regular algebras of dimension 2 are "free algebras in $n \geq 2$ indeterminates modulo twisted potentials."

If A is a graded quotient of a quiver algebra kQ, obtain a similar description as follows.

Presentation of twisted CY-2 algebras

So what do twisted CY-2 algebras actually look like?

Zhang: AS regular algebras of dimension 2 are "free algebras in $n \geq 2$ indeterminates modulo twisted potentials."

If A is a graded quotient of a quiver algebra kQ, obtain a similar description as follows.

Denote:

- Vertex space: $(kQ)_0 = ke_1 \oplus \cdots \oplus ke_n$
- Arrow space: $V = (kQ)_1$; space of arrows $j \to i$ is $e_i V e_j$

Required data:

- Permutation μ of $\{1, \ldots, n\}$
- Linear automorphism τ of V such that $\tau(e_j V e_i) = e_{\mu(i)} V e_j$

Presentation of twisted CY-2 algebras

Required data:

- Permutation μ of $\{1, \ldots, n\}$
- Linear automorphism τ of V such that $\tau(e_j V e_i) = e_{\mu(i)} V e_j$

Def: $A(Q, \tau) = kQ/(\sum_a \tau(a)a)$. (Algebra with "mesh relations.")

This construction generalizes preprojective algebras (where $\mu = \text{id}$)

Presentation of twisted CY-2 algebras

Required data:

- Permutation μ of $\{1, \ldots, n\}$
- Linear automorphism τ of V such that $\tau(e_j V e_i) = e_{\mu(i)} V e_j$

Def: $A(Q, \tau) = kQ/(\sum_a \tau(a)a)$. (Algebra with "mesh relations.")

This construction generalizes preprojective algebras (where $\mu = \mathrm{id}$)

Theorem: Every twisted CY-2 algebra that is a graded quotient of kQ is of the form $A = A(Q, \tau)$, such that the incidence matrix M of Q has spectral radius $\rho(M) \geq 2$. $(\mathrm{GKdim}(A) < \infty \iff \rho(M) = 2.)$

(The converse should hold, too.)

Thank you!