# The Phase-field Modeling of Hydraulic Fracture

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**Variational Models of Fracture**
**Banff International Research Station**
**for Mathematical Innovation and Discovery**
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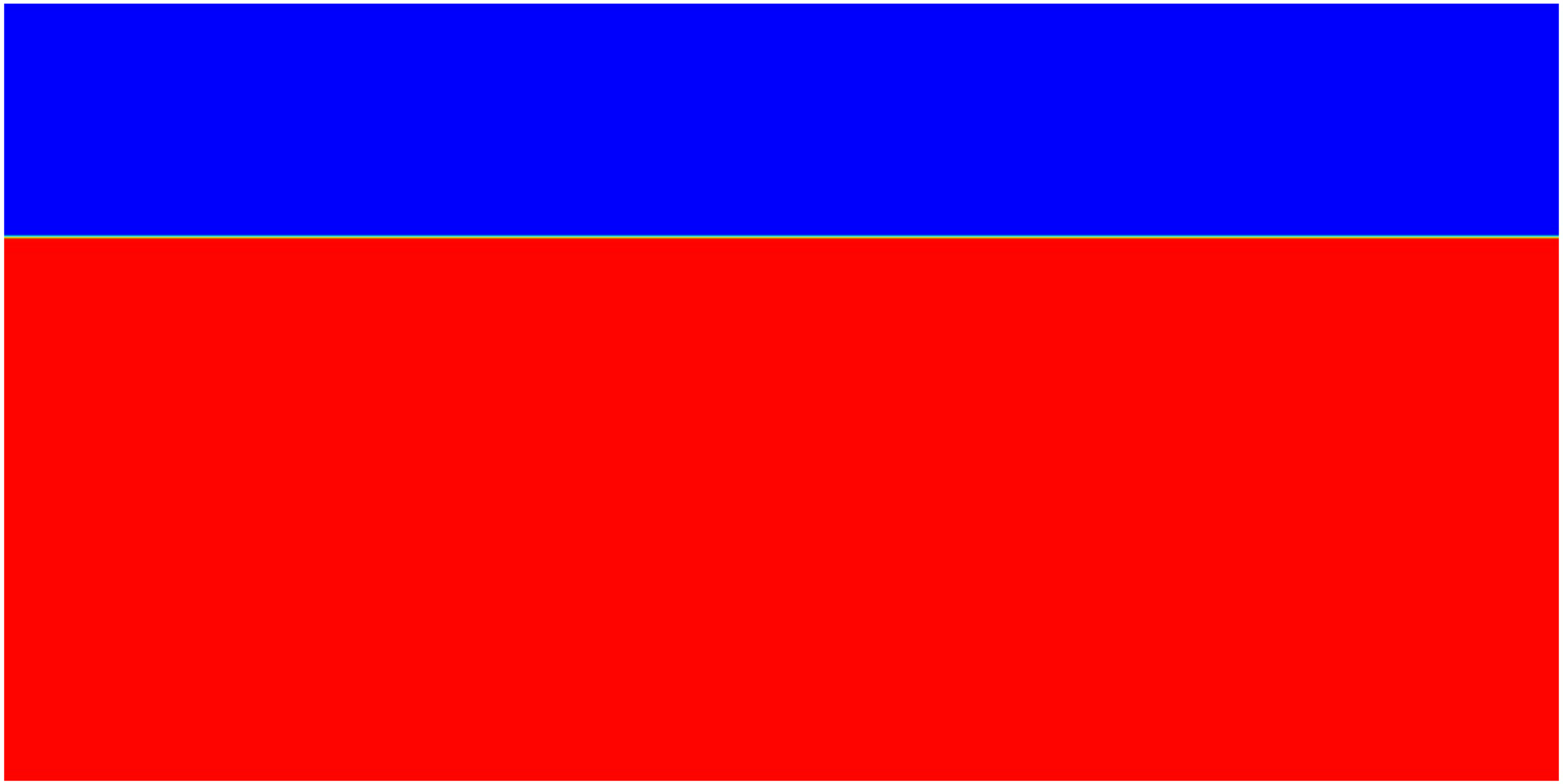
# Boiling



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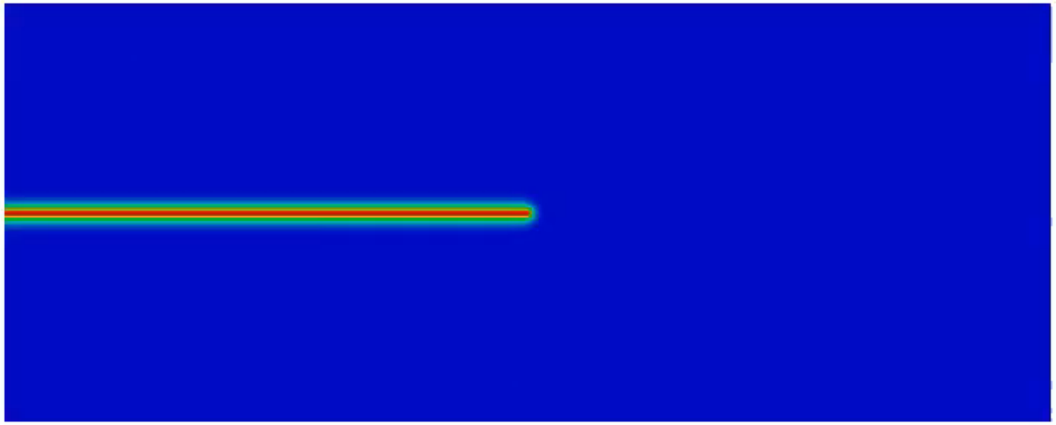
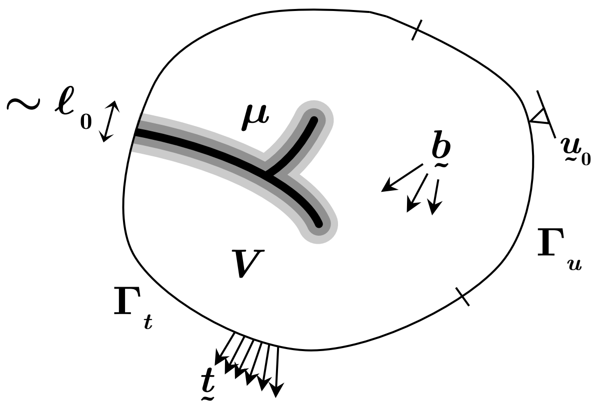


# Micro-force Balance Approach

$\lambda = \xi_i n_i$ - Externally applied surface micro-force

$\gamma$ - Externally applied body micro-force

$\pi$ - Internal micro-force per unit volume




$$\int_S \lambda \, dS + \int_V (\gamma + \pi) \, dV = 0 \quad \rightarrow \quad \xi_{i,i} + \pi + \gamma = 0$$

$$\text{1st Law} \rightarrow \frac{d}{dt}\int_V (\tilde{u} + \tfrac{1}{2}\rho v_i v_i) \, dV = \int_S \lambda \dot{\mu} \, dS + \int_V \gamma \dot{\mu} \, dV + \dots$$

$$\text{2nd Law analysis} \rightarrow \xi_i = \frac{\partial \psi}{\partial \mu_{,i}}, \quad \pi = -\frac{\partial \psi}{\partial \mu} - \beta \dot{\mu} \quad \rightarrow \quad \left( \frac{\partial \psi}{\partial \mu_{,i}} \right)_{,i} - \frac{\partial \psi}{\partial \mu} + \gamma = \beta \dot{\mu}$$

$$\psi(\varepsilon_{ij}, \mu, \mu_{,i}) = f_d(\mu)\psi^+ + \psi^- + \frac{G_c}{4\ell_0}(1-\mu)^2 + G_c \ell_0 \mu_{,i}\mu_{,i}$$

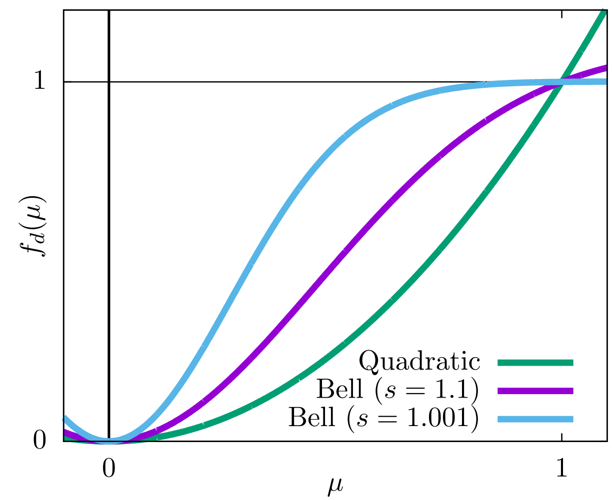
# Degradation Function

**LEFM:**

$$\psi = f_d(\mu)\Omega^+ + \Omega^- + \frac{G_c}{4\ell_0}(1-\mu)^2 + G_c \ell_0 \mu_{,i}\mu_{,i}$$

$$\Omega^+ = \frac{\bar{\lambda}}{2}\langle \varepsilon_{kk} \rangle^2 + \bar{\mu}\left(\langle \varepsilon_1 \rangle^2 + \langle \varepsilon_2 \rangle^2 + \langle \varepsilon_3 \rangle^2\right)$$

$$\Omega^- = -\frac{\bar{\lambda}}{2}\langle -\varepsilon_{kk} \rangle^2 - \bar{\mu}\left(\langle -\varepsilon_1 \rangle^2 + \langle -\varepsilon_2 \rangle^2 + \langle -\varepsilon_3 \rangle^2\right)$$




**Requirements:**

$$f_d(1) = 1, \quad f_d(0) = 0, \quad f_d'(0) = 0, \quad f_d \in C^1$$

**Quadratic Form:**

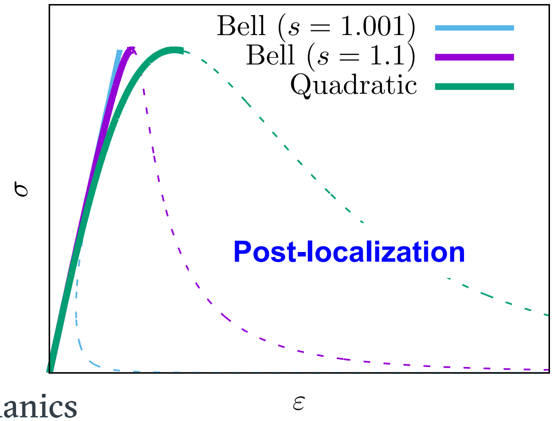
$$f_d(\mu) = \mu^2$$

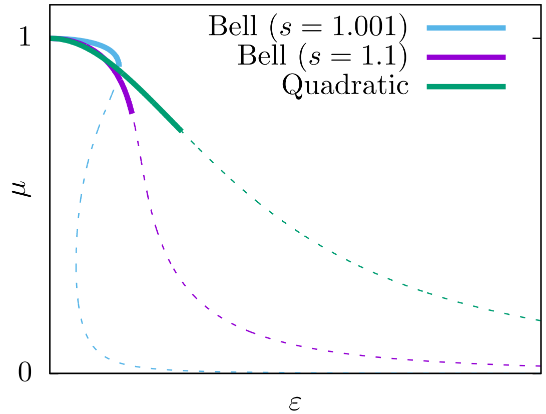
**Modified "Bell Curve" Form:**

$$f_d(\mu) = s\left[1 - \left(\frac{s-1}{s}\right)^{\mu^2}\right]$$

**Homogenous Stress-Strain Response**

**The length scale for each curve has been adjusted to yield the same peak stress**

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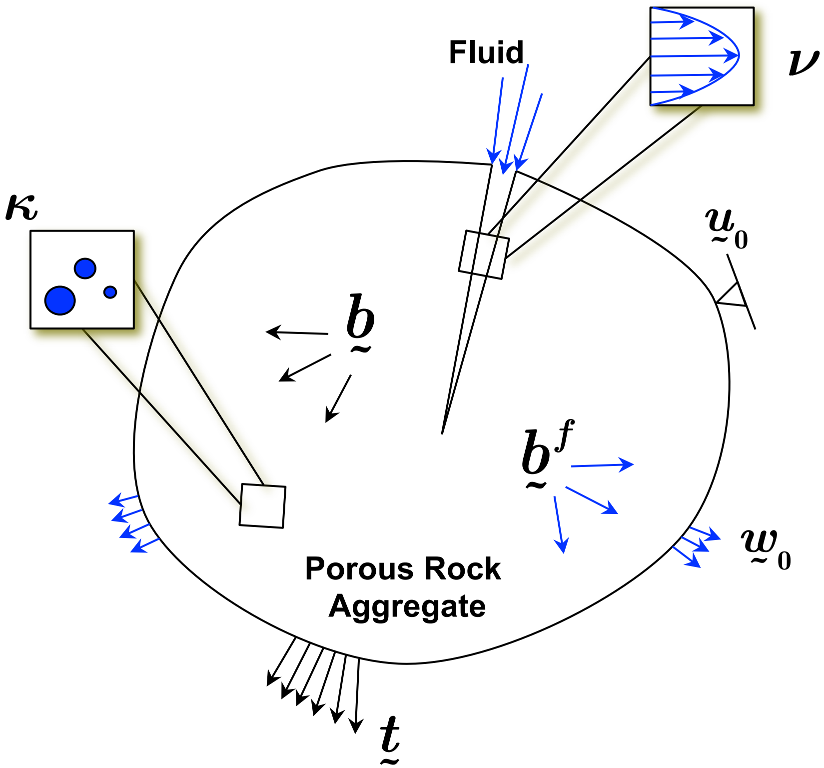
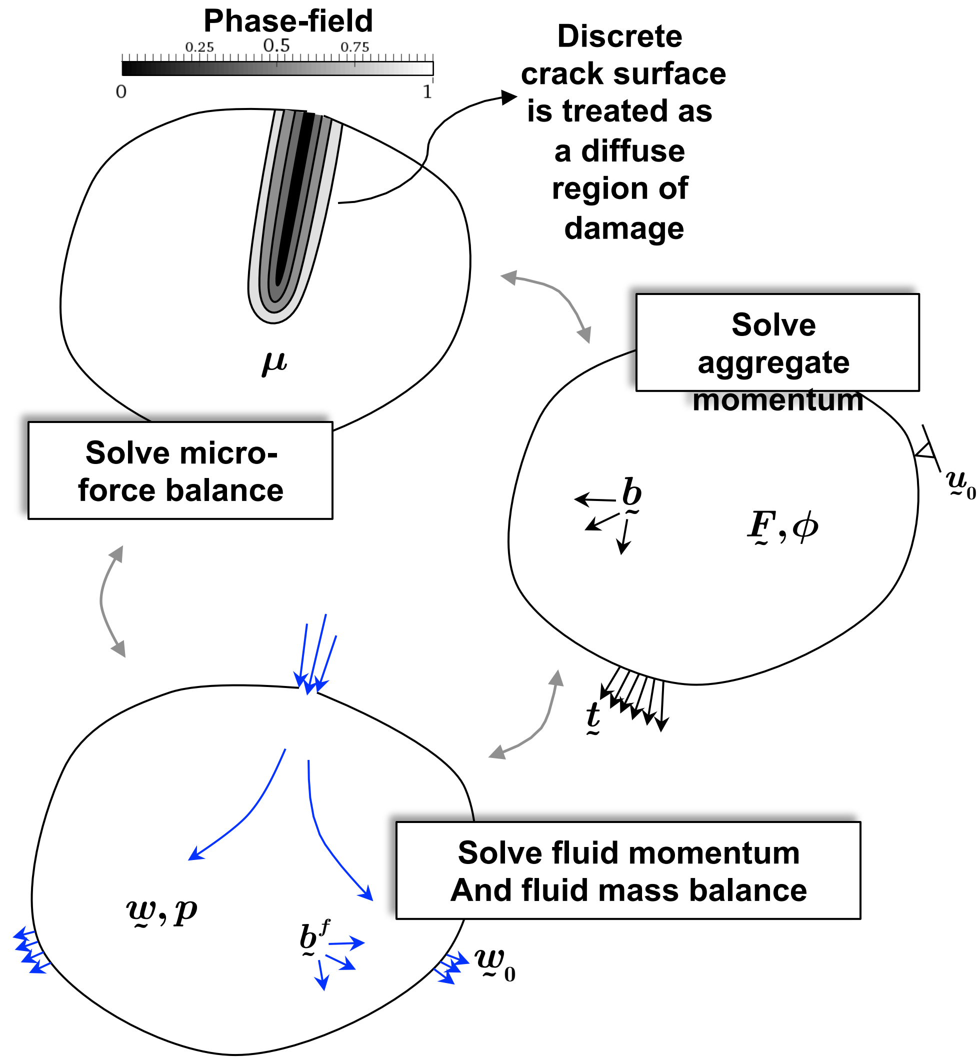
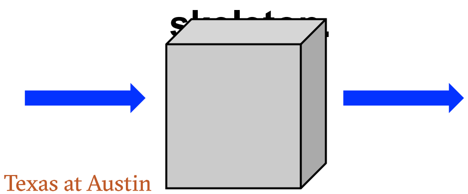
# Phase-field Approach to Hydraulic Fracture




**Fluid mass, momentum, energy, and entropy are convected into and out of deforming control volumes attached to the solid skeleton.**



# Mass Balance

Solid mass balance: $$\frac{D}{Dt}\left[(1-\phi)\rho^s J\right] = 0$$

Fluid mass balance: $$\frac{D}{Dt}\left[\phi\rho^f J + g_m(\mu)\rho^f(1-\phi)\left\langle J-1\right\rangle\right] = -\left(\rho^f \tilde{w}_I\right)_{,I} + \rho^f \tilde{m}$$

$$g_m(\mu) = \begin{cases} 1 & ; \quad \mu < 0 \\ 1-3\mu^2+2\mu^3 & ; \quad 0 \le \mu \le 1 \\ 0 & ; \quad \mu > 1 \end{cases}$$

Effective porosity: $\bar{\phi} = \phi + g_m(\mu)(1-\phi)\left\langle J-1\right\rangle \dfrac{1}{J}$

Fluid mass balance $\Rightarrow$ $$\frac{D}{Dt}\left[\rho^f \bar{\phi} J\right] = -\left(\rho^f \tilde{w}_I\right)_{,I} + \rho^f \tilde{m}$$

$\phi \equiv$ porosity

$\rho^s, \rho^f \equiv$ solid and fluid densities

$\mu \equiv$ phase-field parameter

$J \equiv$ determinant of the deformation gradient $F_{iJ}$

$\tilde{w}_I \equiv$ nominal fluid flux $\left( w_i = \dfrac{\tilde{w}_K F_{iK}}{J} \right)$

$\tilde{m} \equiv$ fluid volume injected per unit reference volume

# Momentum and Micro-force Balances

Effective porosity: $\bar{\phi} = \phi + g_m(\mu)(1-\phi)\langle J-1\rangle \dfrac{1}{J}$

Aggregate momentum: $P_{Ki,K} + \tilde{b}_i = (1-\phi)\rho^s J\dot{v}_i + \bar{\phi}\rho^f J\left(v_i + v_i^{f/s}\right)^{\bullet} + \bar{\phi}\rho^f J\left(v_i + v_i^{f/s}\right)_{,K} F_{Kj}^{-1} v_j^{f/s}$

Fluid momentum: $(\bar{\phi}T_{Ji})_{,J} + \tilde{b}_i^f + \tilde{f}_i^{sf} = \bar{\phi}\rho^f J\left(v_i + v_i^{f/s}\right)^{\bullet} + \bar{\phi}\rho^f J\left(v_i + v_i^{f/s}\right)_{,K} F_{Kj}^{-1} v_j^{f/s}$ (Navier-Stokes)

Solid momentum: $\underbrace{(P_{Ji} - \bar{\phi}T_{Ji})_{,J}}_{\left[(1-\phi)P_{Ji}^s\right]_{,J}} + \tilde{b}_i^s - \tilde{f}_i^{sf} = (1-\phi)J\rho^s\dot{v}_i$ (Newton's 2nd Law for the Solid)

Micro-force balance: $\tilde{\xi}_{I,I} + \tilde{\gamma} + \tilde{\pi} = 0$ (Governs changes and evolution of the phase-field)

$v_i = \dfrac{D}{Dt}\left[u_i\right] \equiv$ velocity of the aggregate

$\dfrac{w_i}{\bar{\phi}} = v_i^{f/s} \equiv$ average fluid velocity relative to the aggregate

$P_{Ji} \equiv$ 1st Piola-Kirchhoff stress for aggregate

$(\tau_{ji} - p\delta_{ij})JF_{Kj}^{-1} = T_{Ki} \equiv$ 1st Piola-Kirchhoff stress for the fluid

# A Biot-type Formulation with Phase-field

$$\left(1-\phi_0\right)J_0\rho_0^s\Omega^s = g_d(\mu)\Omega^+(F_{iJ}) + \Omega^-(F_{iJ}) - \left(1-g_m(\mu)\left\langle J-1\right\rangle^0\right)\left(\alpha(J-1)p + \frac{1}{2M}p^2\right) + \frac{G_c}{4\ell_0}\left[(1-\mu)^2 + 4\ell_0^2\mu_{,I}\mu_{,I}\right]$$

**Constitutive equations developed using the First and Second Laws of Thermodynamics**

$$\phi J = \frac{1-g_m(\mu)\left\langle J-1\right\rangle^0}{1-\left(\frac{1}{J}\right)g_m(\mu)\left\langle J-1\right\rangle}\left(\alpha\left(J-1\right)+\frac{p}{M}\right)$$

$$P_{Ji} = f_d(\mu)\frac{\partial\Omega^+}{\partial F_{iJ}} + \frac{\partial\Omega^-}{\partial F_{iJ}} - \left[\alpha + g_m(\mu)\left\langle J-1\right\rangle^0\left(1-\alpha\right)\right]JF_{Ji}^{-1}p + \bar{\phi}\tilde{\tau}_{Ji}$$

$$\tilde{\pi} = -g_d{}'(\mu)\Omega^+(F_{iJ}) + \frac{G_c}{2\ell_0}\left(1-\mu\right) + g_m{}'(\mu)\left[\left(1-\alpha\right)\left\langle J-1\right\rangle p - \frac{1}{2M}\left\langle J-1\right\rangle^0 p^2\right]$$

$$\tilde{\xi}_I = 2G_c\ell_0\mu_{,I}$$

# Reduced Fluid Momentum (Brinkman Equation)

**Low Reynolds Number Flow (inertial terms are neglected)**

$$(\bar{\phi} T_{Ji})_{,J} + \tilde{b}_i^f + \tilde{f}_i^{sf} = 0$$

**Linearized Compressibility**

$$\psi^f = \frac{1}{c^f}\left(\frac{\rho_0^f}{\rho^f} + \ln(\rho^f)\right)$$

$$p = (\rho^f)^2 \frac{\partial \psi^f}{\partial \rho^f} \quad \Rightarrow \quad \rho^f = \rho_0^f + c^f p$$

**Constitutive relations satisfying the dissipation inequality for the fluid**

$$\tau_{ij} = g_S(\mu)\nu\left[w_{i,j} + w_{j,i} - \tfrac{1}{3} w_{k,k} + \bar{\phi}\left(v_{i,j} + v_{j,i} - \tfrac{1}{3} v_{k,k}\right)\right]$$

$$T_{Ji} = (\tau_{ji} - p\delta_{ij}) J F_{Jj}^{-1}$$

$$g_S(0) = 1, \quad g_S(1) = 0$$
$$g_D(0) = 0, \quad g_D(1) = 1$$

$$\tilde{f}_i^{sf} = p\left(\bar{\phi} J F_{Ji}^{-1}\right)_{,J} - \bar{\phi}_{,J} \tilde{\tau}_{Ji} - g_D(\mu)\bar{\phi} J \frac{\nu}{\kappa} w_i$$

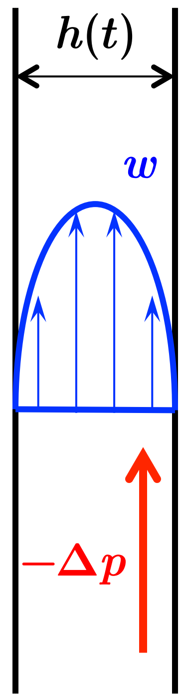
$$\Rightarrow \quad (\tau_{ji} J F_{Jj}^{-1})_{,J} - J F_{Ji}^{-1} p_{,J} - J g_D(\mu) \frac{\nu_{isc}}{\kappa} w_i + \tilde{b}_i^f = 0$$

Stokes flow when $\mu=0$ (terms $(\tau_{ji} J F_{Jj}^{-1})_{,J} - J F_{Ji}^{-1} p_{,J}$)

Darcy flow when $\mu=1$ (terms $- J F_{Ji}^{-1} p_{,J} - J g_D(\mu) \frac{\nu_{isc}}{\kappa} w_i$)

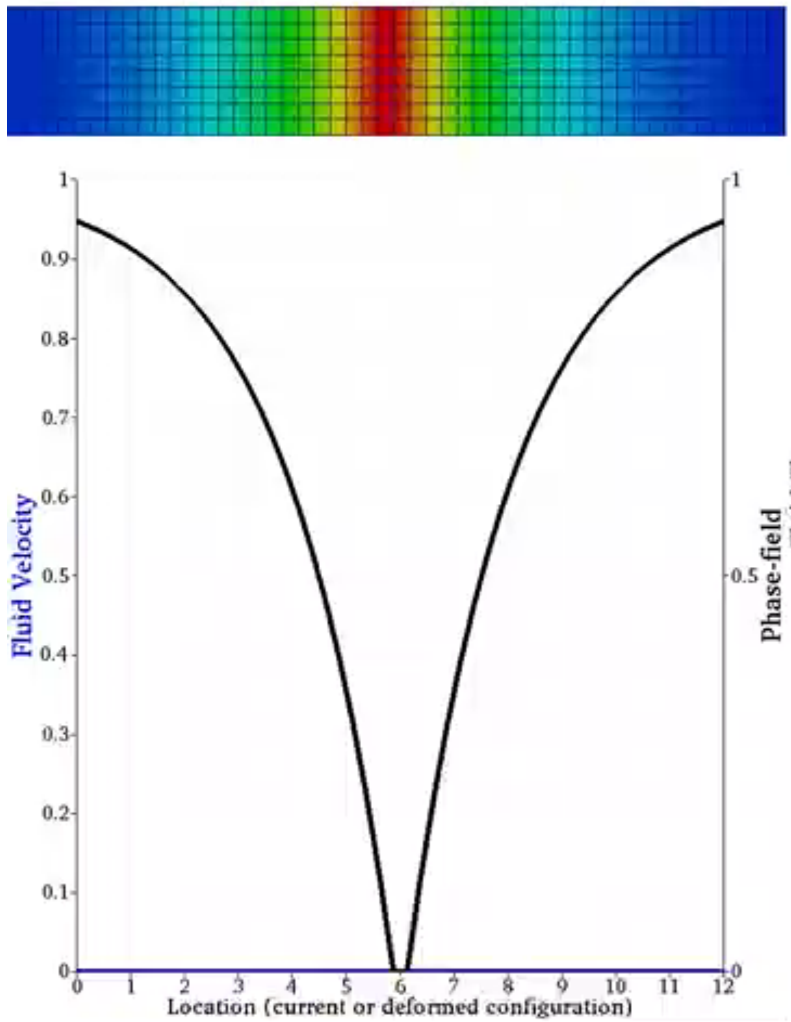
# Numerical Results and Validation

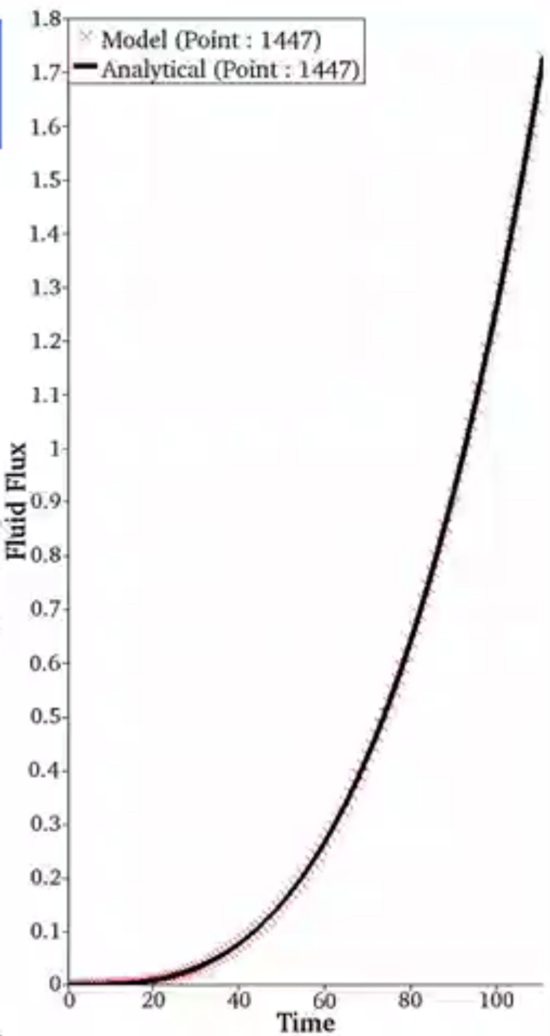
# Flow Through an Opening Channel




$$h(t) = h_0 t$$

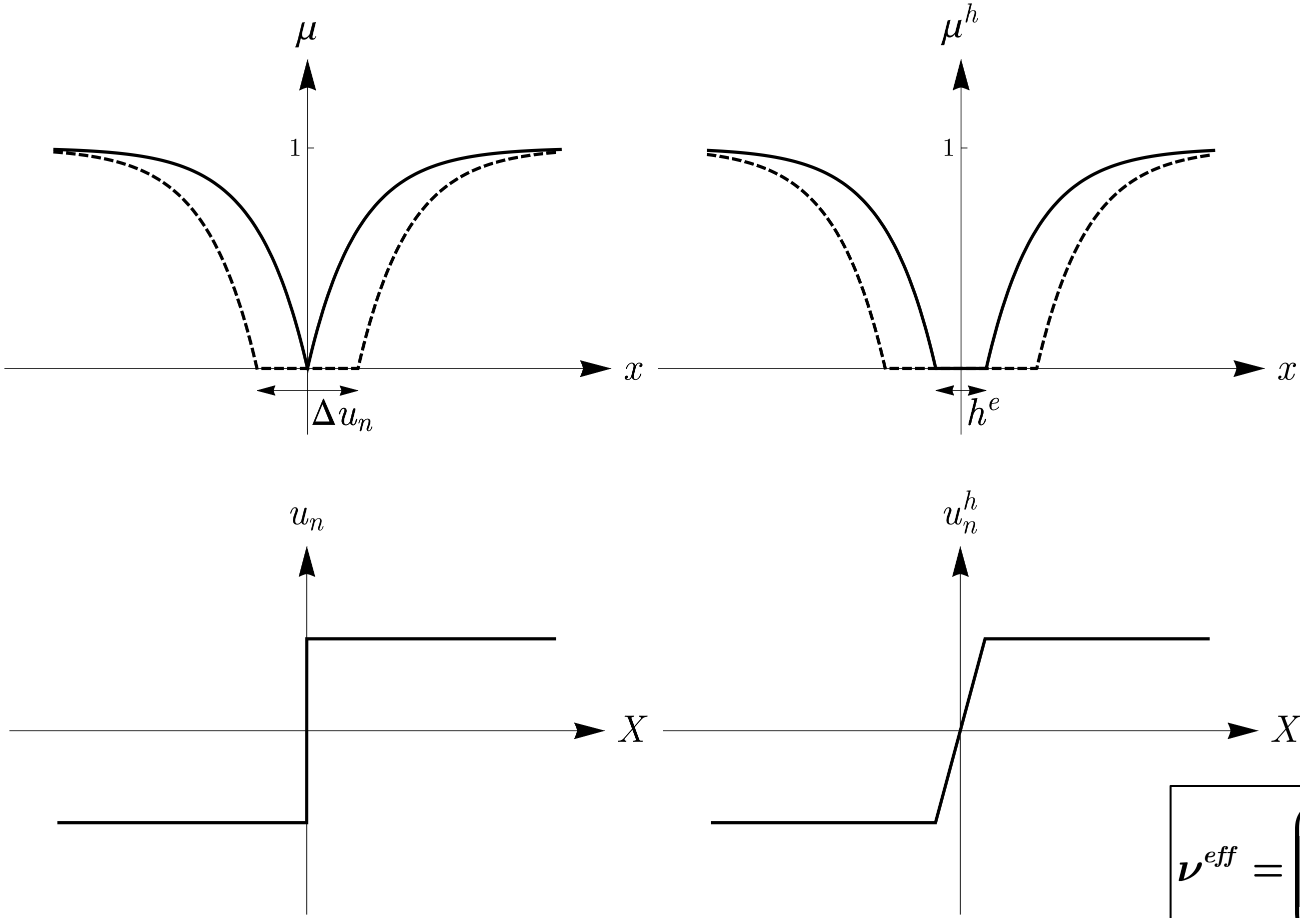
$$q = -\frac{\Delta p \big(h(t)\big)^3}{12\nu}$$

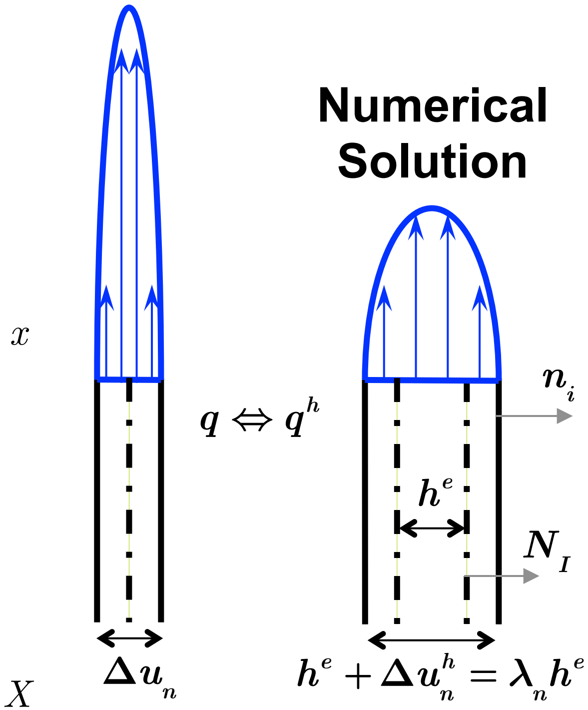


# Viscosity Scaling

**Exact Solution** | **Numerical Solution**



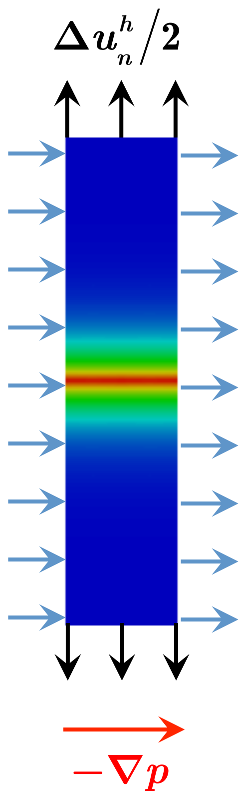



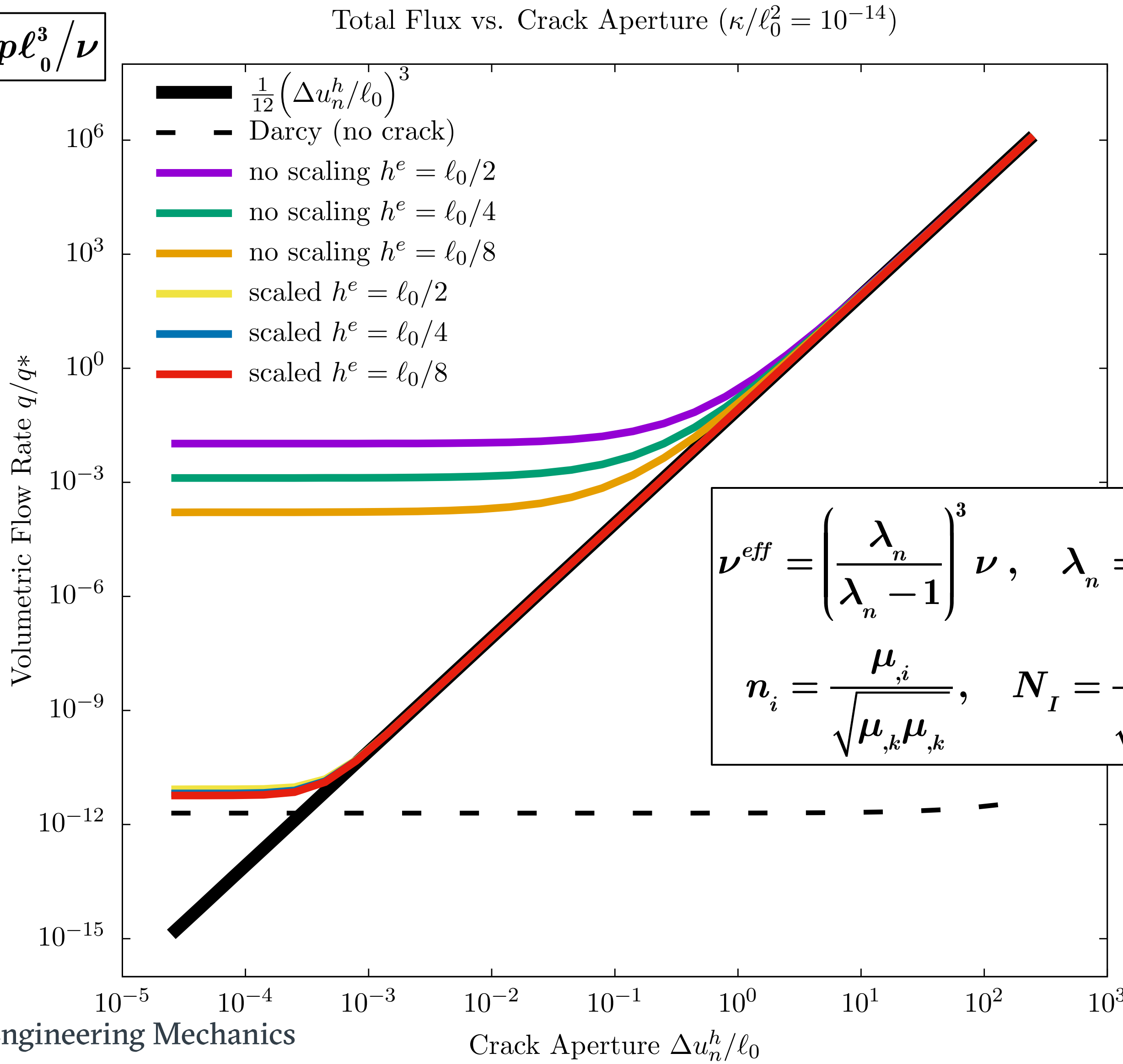

$$\nu^{eff} = \left(\frac{\lambda_n}{\lambda_n - 1}\right)^3 \nu\,, \quad \lambda_n = F_{iJ} n_i N_J$$

$$n_i = \frac{\mu_{,i}}{\sqrt{\mu_{,k}\mu_{,k}}}, \quad N_I = \frac{\mu_{,I}}{\sqrt{\mu_{,K}\mu_{,K}}}$$

# Viscosity Scaling

$$q^* = -\nabla p \ell_0^3 / \nu$$

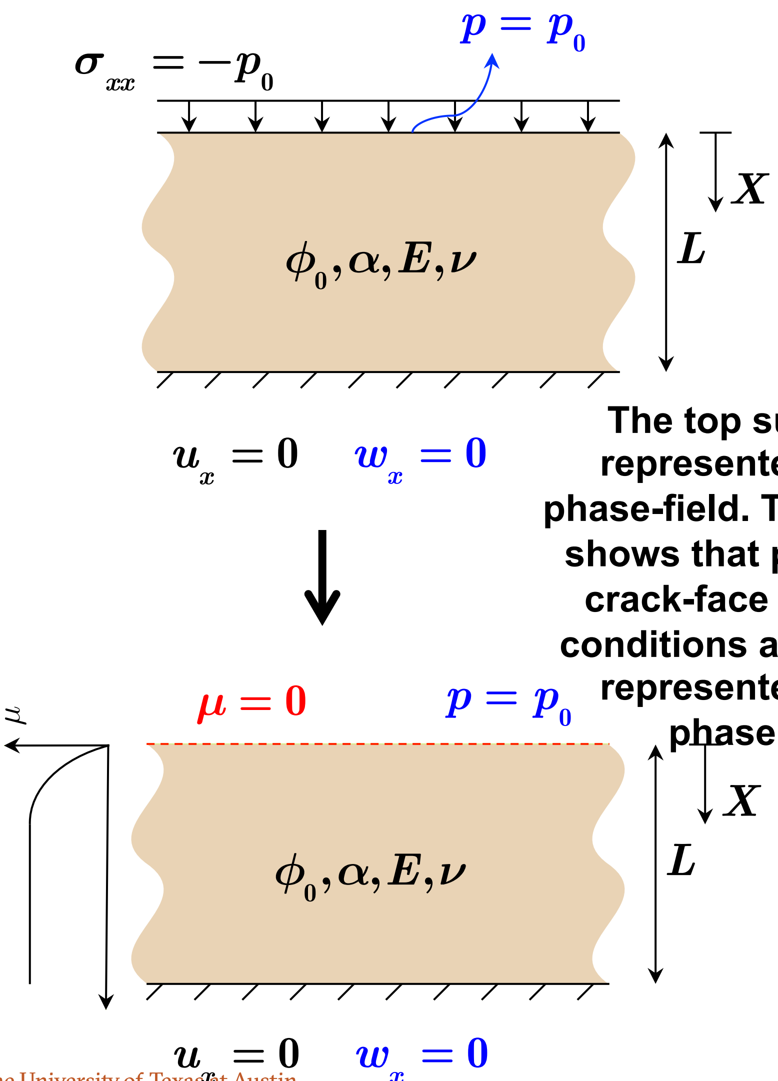


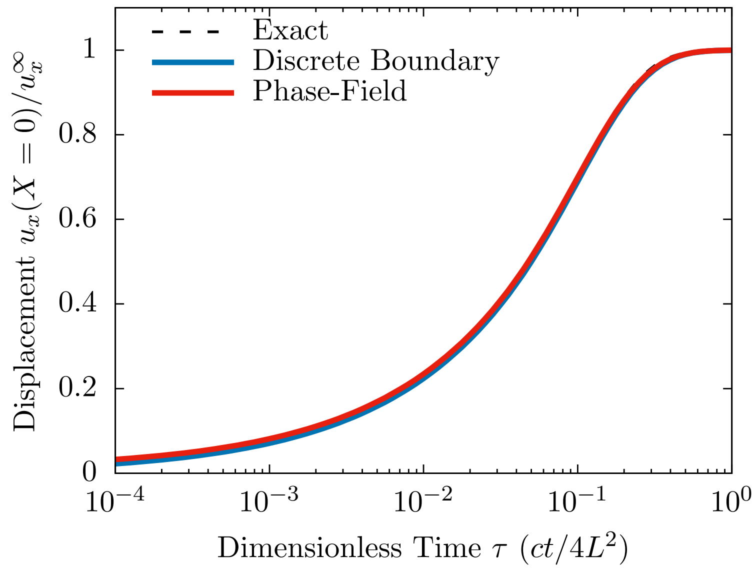

$$\nu^{eff} = \left(\frac{\lambda_n}{\lambda_n - 1}\right)^3 \nu\,, \quad \lambda_n = F_{iJ} n_i N_J$$

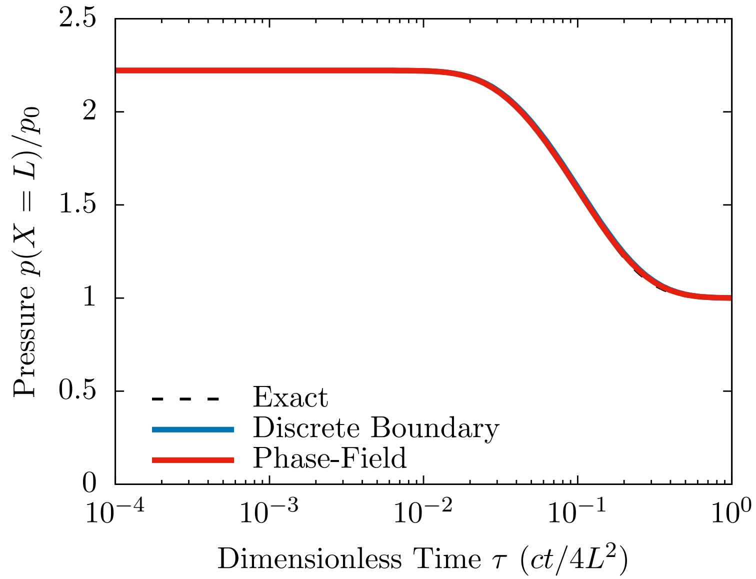
$$n_i = \frac{\mu_{,i}}{\sqrt{\mu_{,k}\mu_{,k}}}, \quad N_I = \frac{\mu_{,I}}{\sqrt{\mu_{,K}\mu_{,K}}}$$

# Consolidation with Fluid Loading

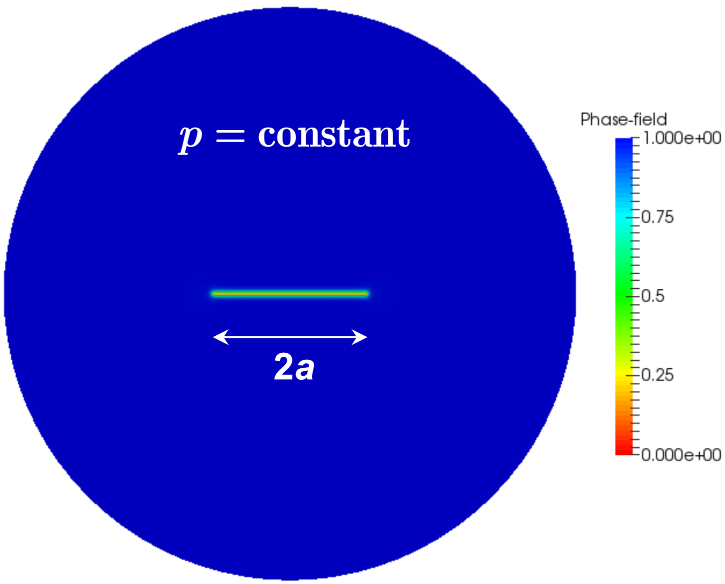



**The top surface is represented by the phase-field. This solution shows that poroelastic crack-face boundary conditions are properly represented by the phase-field.**

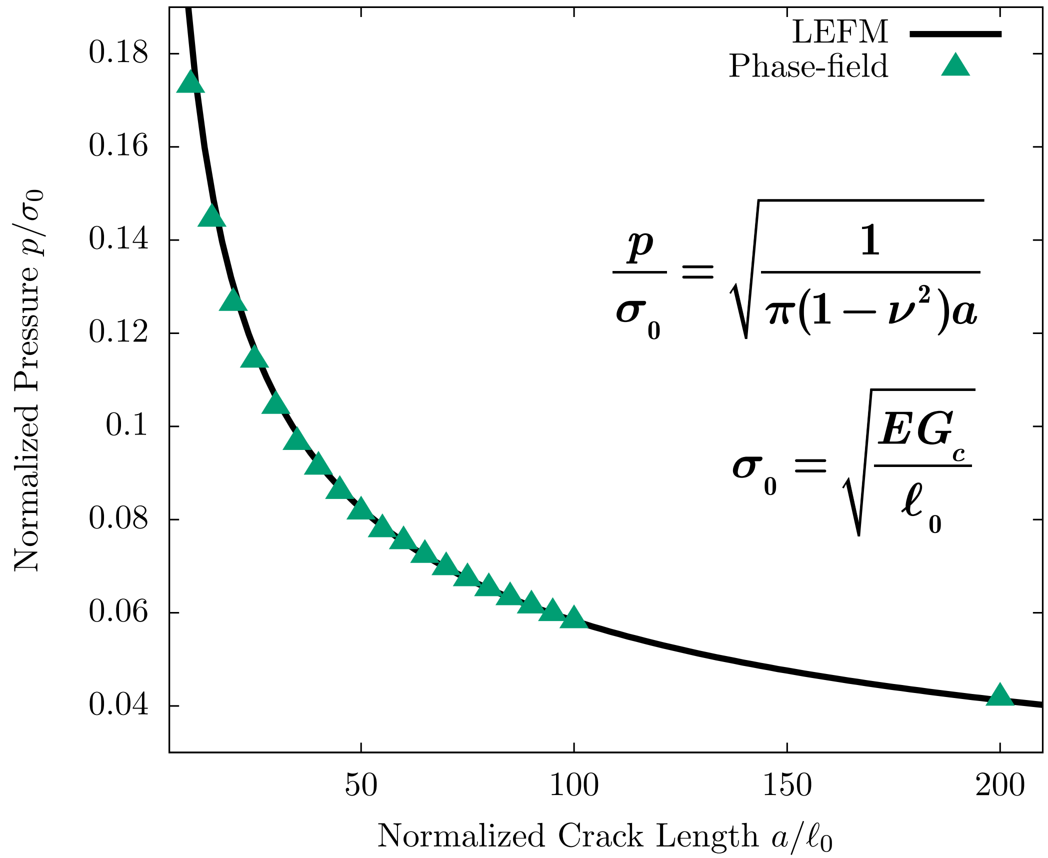


# Pressurized Center Crack




**DtN Mapping on boundary elements**



# Plane Strain Crack

**Two Regimes of Propagation:**

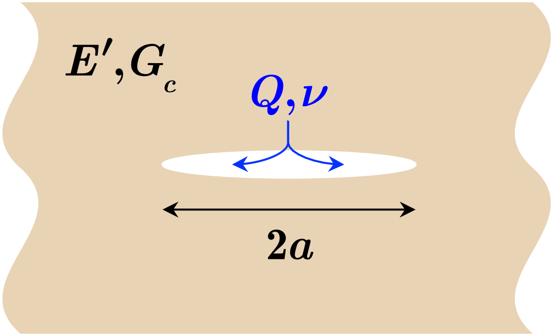
$\mathcal{K} \to 0 \Rightarrow$ **Viscosity Dominated**

$\mathcal{K} \to \infty \Rightarrow$ **Toughness Dominated**

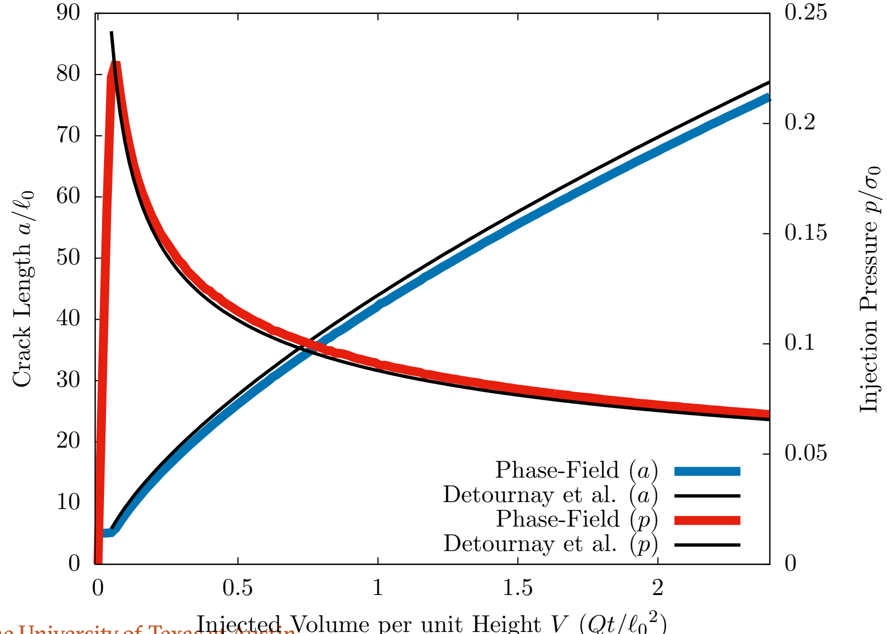
**Dimensionless Toughness (Detournay):**

$$\mathcal{K} = \frac{\left(4\sqrt{\frac{2}{\pi}}K_{Ic}\right)}{\left[\left(E'\right)^3\left(12\nu\right)Q\right]^{1/4}} \quad , \quad K_{Ic} = \sqrt{G_c E'}$$

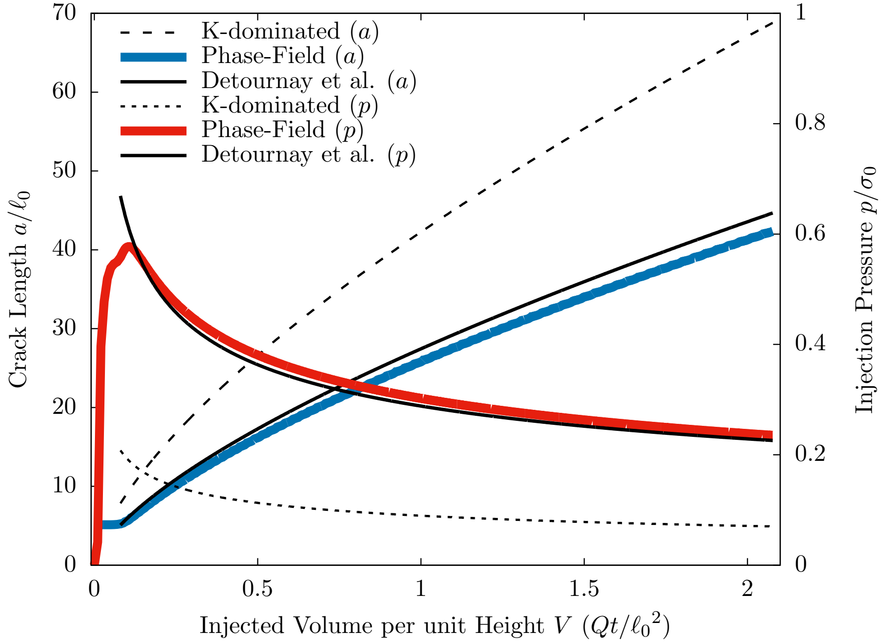
**KGD crack geometry**



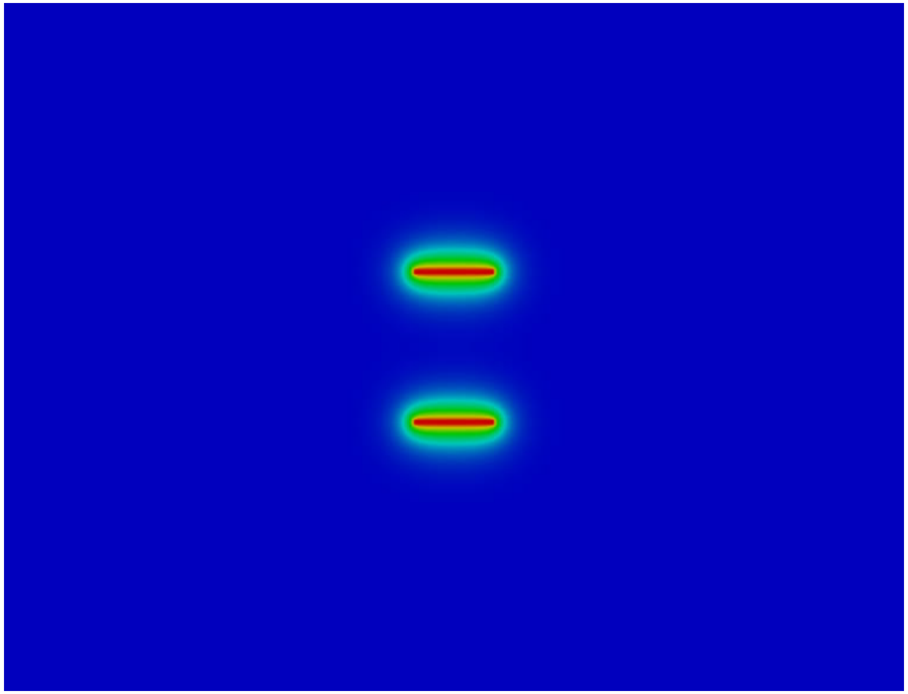

TOUGHNESS DOMINATED REGIME

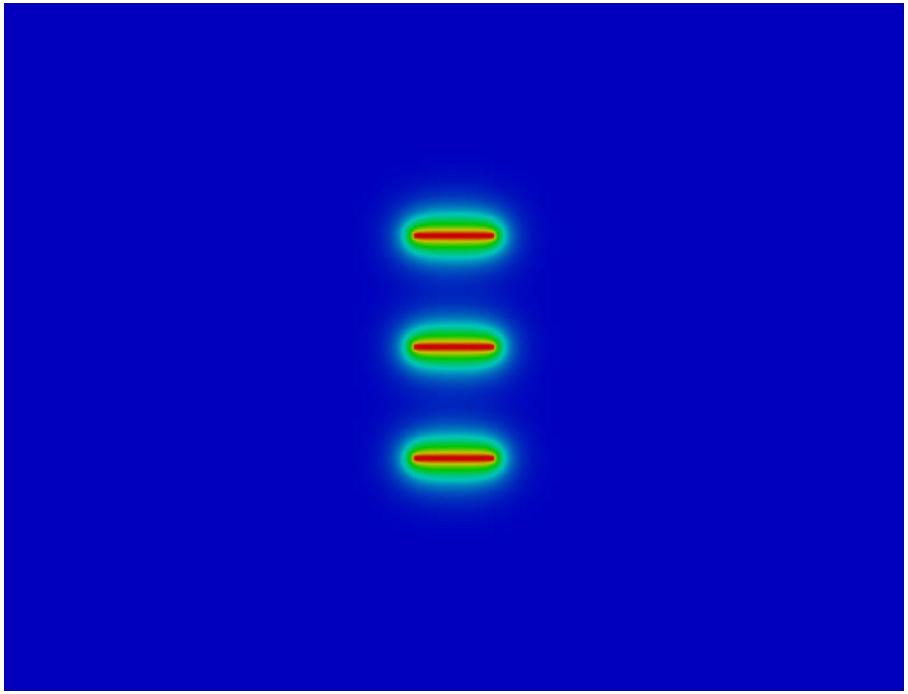



VISCOSITY DOMINATED REGIME



# Parallel Cracks



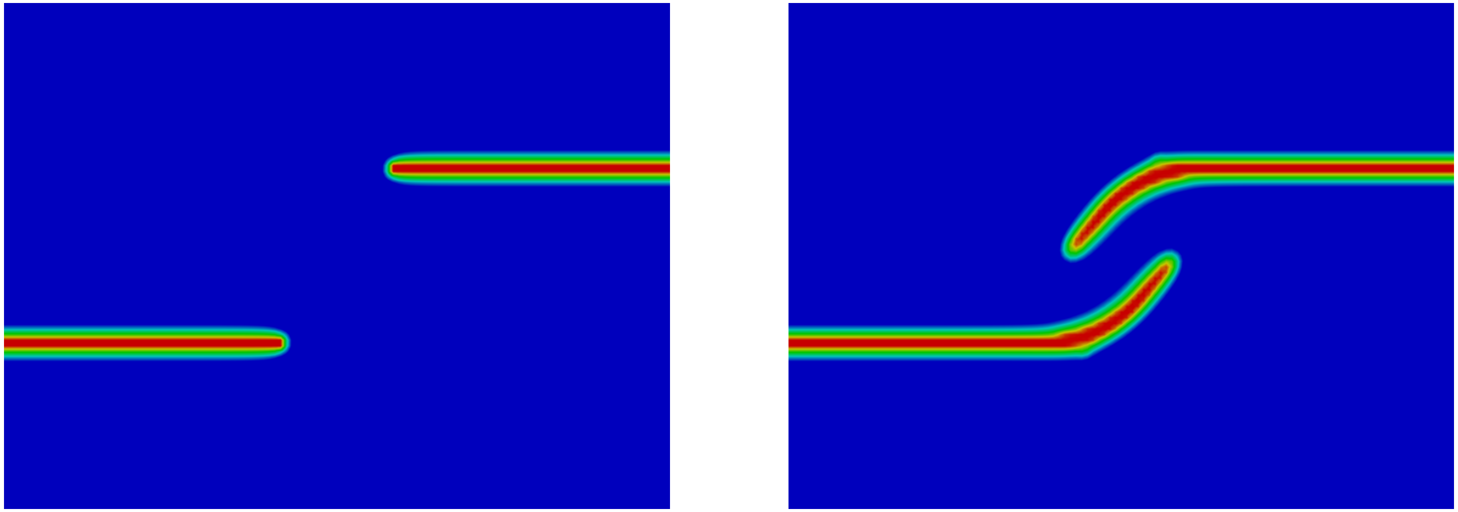


# Merging Cracks

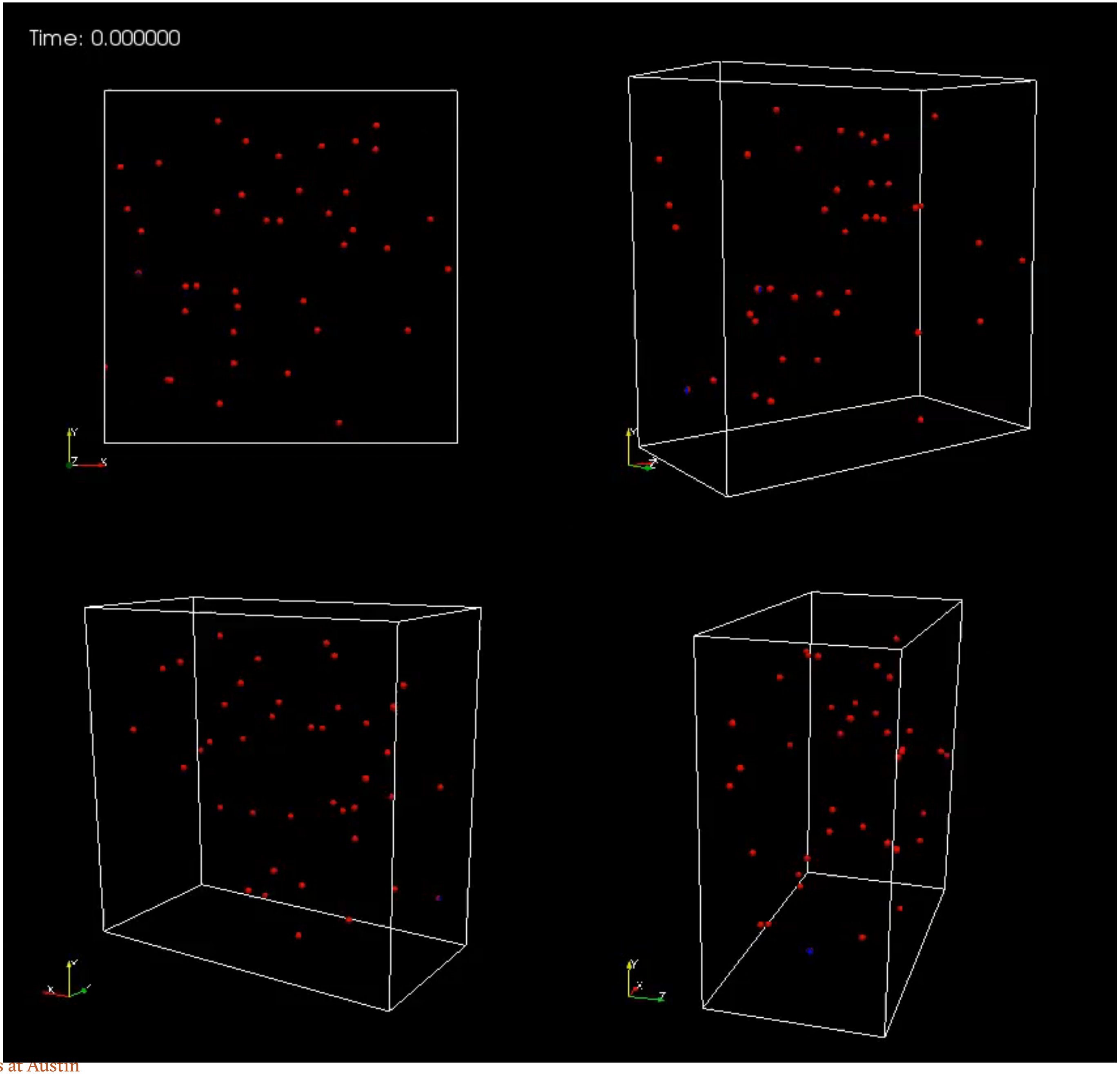
# Summary

- Cracks are represented by a diffuse phase-field and have been shown to be capable of representing interesting geometrical evolution.
- Thermodynamically consistent models have been developed using a balance law approach to model crack propagation fluid saturated poroelastic media
- The approach has been shown to compare favorably to several simple benchmark solutions.

# Thank You for Your Attention



- **Support from Statoil on this project is gratefully acknowledged.**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# 3D Boiling