Unidirectional gradient flow and its application to a crack propagation model

Masato Kimura

Kanazawa University, Japan

Variational Models of Fracture
Banff International Research Station
May 8-13 2016

Collaborators:
G. Akagi (Kobe Univ.), T. Takaishi (Hiroshima Kokusai Gakuin U.)

Irreversible diffusion system and crack propagation model

- Irreversible diffusion equation (Unidirectional evolution)

$$u_t = \big(\Delta u + f(x,t)\big)_+ \quad x \in \Omega \subset \mathbb{R}^n,\ \ t > 0$$

  - Irreversibility $u_t \geq 0 \quad (a)_+ := \max(a,0))$
  - Gradient flow structure $\dfrac{d}{dt}E(u(\cdot,t)) = -\displaystyle\int_\Omega |u_t|^2\,dx \leq 0$

$$E(u) := \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega fu\,dx$$

    $(\text{if } u|_{\partial\Omega} = 0,\ \ f = f(x))$

- A crack propagation model [Takaishi-Kimura 2009]
  - A phase field variable (damage variable) $z(x,t) \in [0,1]$ for crack position: $z \approx 0$: no crack, $z \approx 1$: crack
  - Derived as a gradient flow of [elastic energy + surface energy].
  - Non-repairability of crack is expressed as $z_t = (\Delta z + g(z, |\nabla u|))_+$.

Contents

1. A phase field mdel for crack propagation
   (joint work with Takeshi Takaishi, [Takaishi-Kimura 2009])
   - Derivation of the model and gradient flow structure (mode III crack model)
   - Numerical examples
   - A numerical example for 3D elasticity crack propagation model

2. Mathematical analysis of irreversible diffusion equation
   (joint work with Goro Akagi, preprint in arXiv)
   - Known results
   - Main results (unique existence of a global solution, comparison principle, asymptotic behavior)
   - Implicit time discretization and construction of a strong solution
   - Improvement of regularity estimate for a variational inequality (obstacle problem)
   - Stefan problem

Crack propagation model

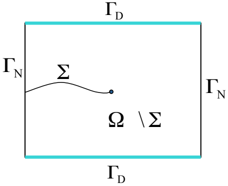
mode III crack propagation model [Takaishi-Kimura 2009]

$\Omega : \mathbb{R}^2$: bdd domain $\partial\Omega = \Gamma = \Gamma_D \cup \Gamma_N$: smooth
$u(x,t) \in \mathbb{R}$: antiplane displacement, $z(x,t) \in [0,1]$: damage variable
$\gamma(x) > 0$: fracture toughness $g(x,t)$: $\alpha,\ \epsilon > 0$,

$$\begin{cases} \mathrm{div}\left((1-z)^2 \nabla u\right) = 0 & (x \in \Omega,\ t > 0) \\ \alpha z_t = \left(\epsilon\, \mathrm{div}\left(\gamma(x)\nabla z\right) - \dfrac{\gamma(x)}{\epsilon} z + |\nabla u|^2 (1-z)\right)_+ & (x \in \Omega,\ t > 0) \\ u = g(x,t) & (x \in \Gamma_D,\ t > 0) \\ \dfrac{\partial u}{\partial n} = 0 & (x \in \Gamma_N,\ t > 0) \\ \dfrac{\partial z}{\partial n} = 0 & (x \in \Gamma,\ t > 0) \\ z(x,0) = z_0(x) \in [0,1] & (x \in \Omega) \end{cases}$$

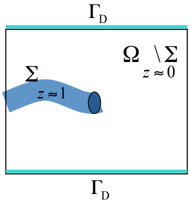
Elasticity eq.(anti-plane displ.) in a cracked domain

$$-\mu \Delta u = f \quad \text{in } \Omega \setminus \Sigma$$
$$u = g \quad \text{on } \Gamma_D$$
$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_N \ \& \ \Sigma_{\pm}$$



Approximation by z

$$-\mu \, \mathrm{div}((1-z)^2 \nabla u) = f \quad \text{in } \Omega$$
$$u = g \quad \text{on } \Gamma_D$$
$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_N$$



Irreversibility (Non-repairability) and gradient flow

- Non-repairability of crack is expressed by $z_t = (\cdots)_+$.
- Ambrosio-Tortorelli approximation of Griffith-Francfort-Marigo energy:

$$\mathcal{E}(z) := \min_{u|_{\Gamma_D}=g} \left(\frac{1}{2}\int_\Omega (1-z)^2 |\nabla u|^2 \, dx \right) + \frac{1}{2}\int_\Omega \gamma(x) \left(\epsilon |\nabla z|^2 + \frac{1}{\epsilon} z^2 \right) dx$$

elastic energy　　　　regularized surface energy

- Gradient flow structure (if $g_t = 0$)

$$\frac{d}{dt}\mathcal{E}(z(\cdot,t)) = -\int_\Omega \left\{ \epsilon \, \mathrm{div}\,(\gamma(x)\nabla z) - \frac{\gamma(x)}{\epsilon} z + |\nabla u|^2 (1-z) \right\} z_t \, dx$$

$$= -\alpha \int_\Omega |z_t|^2 \, dx \le 0$$

Numerical examples

Method and parameters

- Numerical method
  - Implicit scheme
  - ALBERTA : Adaptive mesh FEM
- Parameters

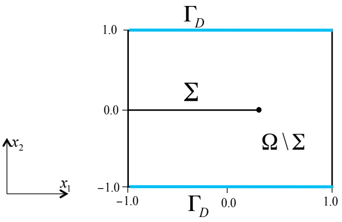
$\varepsilon = 10^{-3}$

$\alpha = 10^{-3}$

$\gamma = \gamma_0 = 0.5, \mu = 1$

$f(x,t) = 0, g(x,t) = 10tx_2$

$0 \leq t \leq 3$



A straight crack

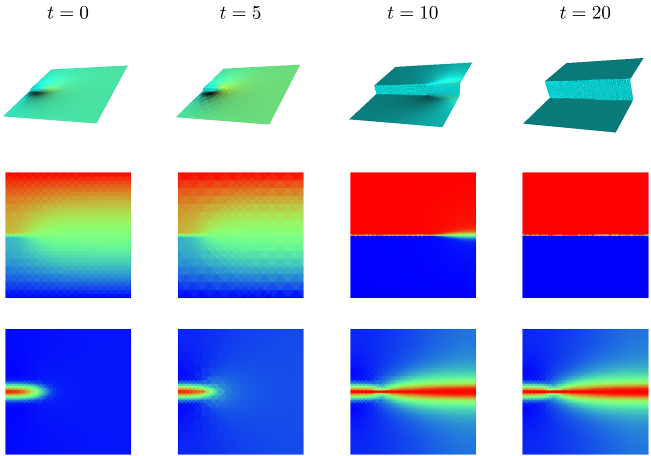


Figure: u and $|\nabla u|$ (top), u (middle), z (bottom)

Merging two cracks

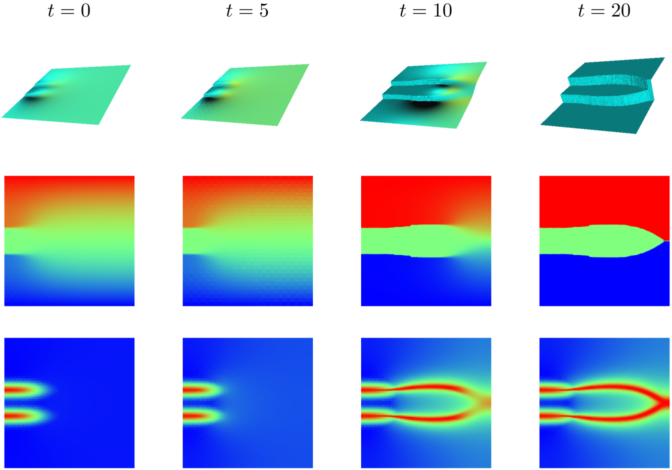


Figure: u and $|\nabla u|$ (top), u (middle), z (bottom)

Two straight cracks

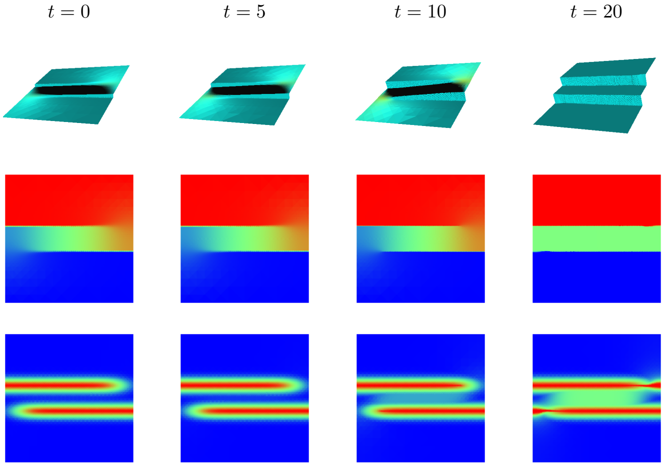


Figure: u and $|\nabla u|$ (top), u (middle), z (bottom)

Subcrack beyween two straight cracks

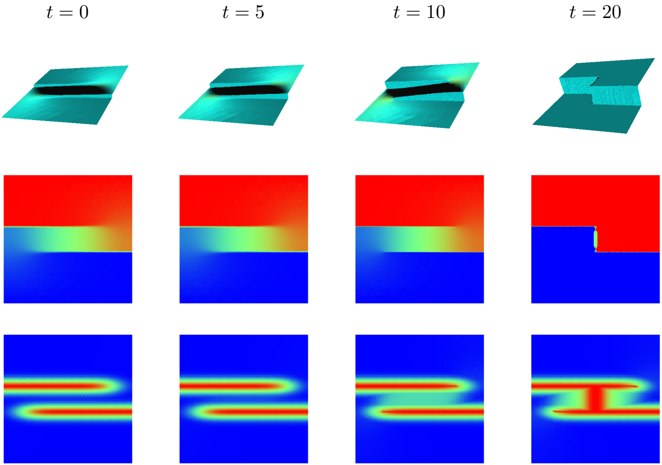


Figure: u and $|\nabla u|$ (top), u (middle), z (bottom)

Checker pattern fracture toughness $\gamma(x) = 0.5(1 + 0.2\cos 10x \cos 10y)$

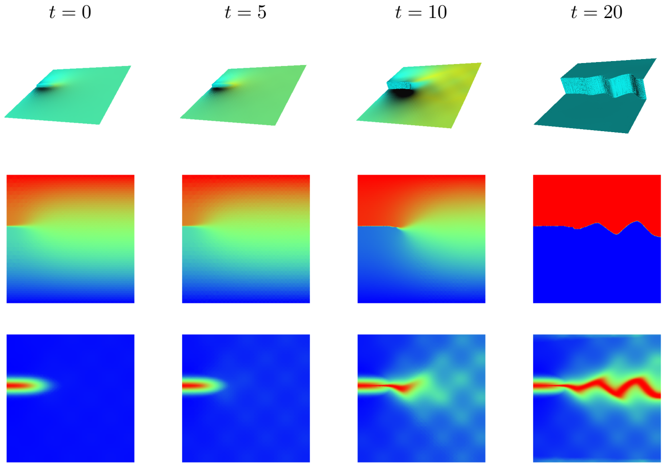


Figure: u and $|\nabla u|$ (top), u (middle), z (bottom)

Stripe pattern fracture toughness $\gamma(x) = 0.5(1 + 0.2\cos 10(x + y))$

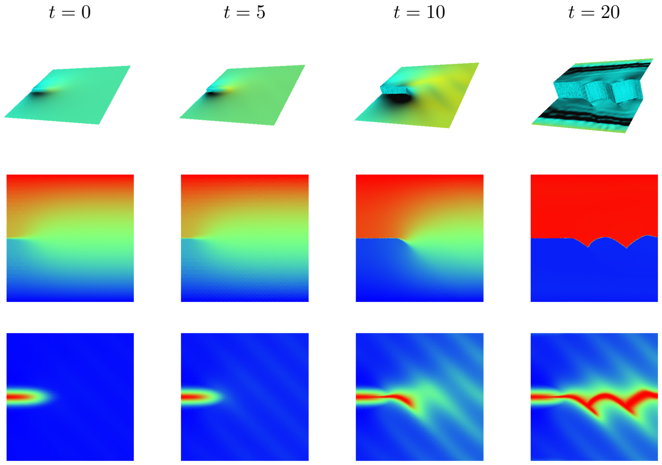


Figure: u and $|\nabla u|$ (top), u (middle), z (bottom)

crack propagation model in 3D

$\Omega : \mathbb{R}^3$: bdd domain, $\partial\Omega = \Gamma = \Gamma_D \cup \Gamma_N$: smooth
$u(x,t) \in \mathbb{R}^3$: displacement, $e[u] \in \mathbb{R}^{3\times 3}$: strain tensor,
$\sigma[u] = Ce[u] \in \mathbb{R}^{3\times 3}$: stress tensor,
$z(x,t) \in [0,1]$: damage variable,
$\gamma(x) > 0$: fracture toughness, $g(x,t)$, α, $\epsilon > 0$: given

$$
\begin{cases}
\operatorname{div}\left((1-z)^2\sigma[u]\right) = 0 & (x \in \Omega,\ t > 0) \\
\alpha z_t = \left(\epsilon \operatorname{div}(\gamma(x)\nabla z) - \dfrac{\gamma(x)}{\epsilon} z + \sigma[u] : e[u](1-z)\right)_+ & (x \in \Omega,\ t > 0) \\
u = g(x,t) & (x \in \Gamma_D,\ t > 0) \\
\sigma[u]n = 0 & (x \in \Gamma_N,\ t > 0) \\
\dfrac{\partial z}{\partial n} = 0 & (x \in \Gamma,\ t > 0) \\
z(x,0) = z_0(x) \in [0,1] & (x \in \Omega)
\end{cases}
$$

3D numerical simulation

$$\begin{cases} \operatorname{div}\left((1-z)^2\sigma[u]\right) = 0 \\ \alpha z_t = \left(\epsilon \operatorname{div}\left(\gamma(x)\nabla z\right) - \dfrac{\gamma(x)}{\epsilon} z + \sigma[u] : e[u](1-z)\right)_+ \end{cases}$$

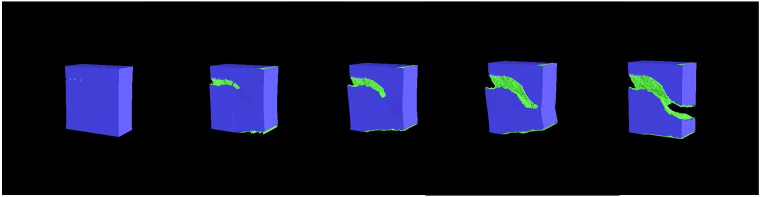


Figure: an example of 3D simulation, time increases from left to right

Irreversible diffusion equation

Irreversible diffusion equation and strong solution

$\Omega \subset \mathbb{R}^n$: bdd domain, $\Gamma = \partial\Omega$: smooth

$$\begin{cases} u_t = \big(\Delta u + f(x,t)\big)_+ & (x,t) \in Q := \Omega \times (0,T), \\ u(x,t) = 0 & (x,t) \in \Gamma \times (0,T), \\ u(x,0) = u_0(x) & x \in \Omega. \end{cases} \tag{1}$$

Definition 1 (strong solution)

Let $f \in L^2(Q)$, $u_0 \in L^2(\Omega)$. u is called a strong solution of (1) iff

(a) $u \in H^1(0,T; L^2(\Omega)) \cap L^2(0,T; H^2(\Omega) \cap H^1_0(\Omega))$

(b) $u_t = \big(\Delta u + f\big)_+ \quad \mathcal{H}^{n+1}\text{-a.e. in } Q$

(c) $u(0,\cdot) = u_0 \in L^2(\Omega)$

Remark) Definition of the weak solution (H^1 sol.) has problems.

Main results I

Theorem 2 (complementarity form)

u is a strong solution of (1) *iff*

(c1) $u \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)\cap H^1_0(\Omega))$,

(c2) $\partial_t u \geq 0$ *a.e. in* Q,

(c3) $\partial_t u - \Delta u - f \geq 0$ *a.e. in* Q,

(c4) $(\partial_t u - \Delta u - f)\,\partial_t u = 0$ *a.e. in* Q,

(c5) $u(0,\cdot) = u_0$.

Theorem 3 (uniqueness)

A strong solution of (1) *is unique, if it exists.*

Main results II

Theorem 4 (existence)

We suppose $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ *and* $f \in L^2(Q)$. *If there exists* $f^* \in L^2(\Omega)$ *with* $f(x,t) \le f^*(x)$ *a.e. in* Q, *then there is a strong solution of* (1).

Theorem 5 (comparison principle)

Let u^i $(i = 1, 2)$ *be a strong solution of* (1) *with* $u_0 = u_0^i \in H^2(\Omega) \cap H_0^1(\Omega)$, $f = f^i \in L^2(Q)$, *respectively. We suppose that there exists* $f^* \in L^2(\Omega)$ *with* $f^i(x,t) \le f^*(x)$ *a.e. in* Q $(i = 1, 2)$. *If* $u_0^1 \le u_0^2$ *a.e. in* Ω *and* $f^1 \le f^2$ *a.e. in* Q, *then* $u^1 \le u^2$ *a.e. in* Q *holds.*

Main results III

Theorem 6 (asymptotic behavior)

If $f \in L^2(\Omega)$, then there exists $\bar{u} \in H^2(\Omega) \cap H^1_0(\Omega)$ such that

$$\lim_{t \to \infty} \|u(\cdot, t) - \bar{u}\|_{H^1(\Omega)} = 0,$$

where $\bar{u}$ is given as a unique solution of the following variational inequality:

$$\bar{u} \in K := \{v \in H^1_0(\Omega);\ v \geq u_0 \ \textit{a.e. in } \Omega\},$$

$$\int_\Omega \nabla \bar{u} \cdot \nabla (v - \bar{u})\, dx \geq \langle f, v - \bar{u} \rangle \qquad ({}^\forall v \in K).$$

(*An obstacle problem with obstacle u_0*)

Main reulsts IV

Theorem 7 (gradient flow structure)

We suppose $f \in L^2(\Omega)$. We define

$$E(u) := \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega fu\,dx.$$

Then $[t \mapsto E(u(\cdot,t))] \in W^{1,1}(0,T)$ and

$$\frac{d}{dt}E(u(\cdot,t)) = -\int_\Omega |u_t|^2\,dx \le 0 \quad a.e.\ \ t \in (0,T)$$

holds.

Known reults

General theory of doubly nonlinear evolution equation:

$$\partial\Psi(u_t(t)) + \partial\Phi(u(t)) \ni f(t) \quad \text{in } H$$

including (1) has been studied in [V. Barbu, '75], [T. Arai, '79], [T. Senba, '86], [U. Gianazza and G. Savaré, '94]. The boundedness of $\partial\Psi$ is usually assumed and (1) is excluded in most studies.

Theorem 8 (T. Arai, 1979)

If $f \in W^{1,1}(0,T;L^2(\Omega))$, then there exists a strong solution u of (1) and $u_t,\ \Delta u \in L^\infty(0,T;L^2(\Omega))$ holds.

Remark: $f \in W^{1,1}(0,T;L^2(\Omega))$

$$\Longrightarrow \ |f(x,t)| \le f^*(x) := |f(0,x)| + \int_0^T |f_t(x,t)|\,dt$$

[U. Gianazza and G. Savaré, '94] also proved existence and uniqueness of a weak solution to (1) for $f = 0$.

Weak and strong solutions

1. [weak solution without uniqueness]
   - H^1-solution, $(\Delta u + f)$: Radon Measure (Gianazza-Savaré)
   - characterization as a constrained gradent flow in energy form (M. Negri)
   - H^1-limit of minimizing sequence
2. [not so strong solution] $u \in H^1(0, T; H^1(\Omega))$ (D. Knee et al)
   - unique existense of the solution
   - technical definition of the positive part $(\Delta u + f)_+$
   - not weaker than the strong solution
3. [strong solution]
   $u \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega) \cap H^1_0(\Omega))$ (Akagi-K.)
4. [viscosity solution]
   - unique existense of the solution
   - no energy gradient structure

Sketch of proof I (gradient flow structure, uniqueness)

Since a strong solution u of (1) satisfies $u \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)\cap H^1_0(\Omega))$, we can verify

$$\frac{d}{dt}E(u(\cdot,t)) = -\int_\Omega (\Delta u + f(x))u_t\,dx = -\int_\Omega |u_t|^2\,dx \le 0.$$

For strong solutions $u_1,\ u_2,\ w := u_1 - u_2$.

$$\begin{aligned}\frac{d}{dt}\int_\Omega |\nabla w(x,t)|^2\,dx &= -2\int_\Omega w_t(x,t)\,\Delta w(x,t)\,dx\\ &= -2\int_\Omega \{(\Delta u_1(x,t)+f(x,t))_+ - (\Delta u_2(x,t)+f(x,t))_+\}\\ &\qquad\qquad \cdot\{(\Delta u_1(x,t)+f(x,t)) - (\Delta u_2(x,t)+f(x,t))\}\,dx\\ &\le -2\int_\Omega |(\Delta u_1(x,t)+f(x,t))_+ - (\Delta u_2(x,t)+f(x,t))_+|^2\,dx \le 0.\end{aligned}$$

Since $|a_+ - b_+| \le |a-b|$,
$|a_+ - b_+|^2 \le |a_+ - b_+||a-b| = (a_+ - b_+)(a-b) \quad (a,b\in\mathbb{R})$

Sketch of proof II (existence (1))

- Implicit time discretization ($\tau > 0$ time increment)

$$\frac{u^k(x) - u^{k-1}(x)}{\tau} = \left(\Delta u^k(x) + f^k(x)\right)_+ \quad \text{a.e.} \quad x \in \Omega$$

- Piecewise linear interpolation $u_\tau \in C^0([0, T]; H_0^1(\Omega))$,
  Piecewise constant interpolation $\bar{u}_\tau \in L^\infty(0, T; H_0^1(\Omega))$,
- $\{u_\tau\}_\tau$: bdd in $H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; H_0^1(\Omega))$,
  $\{\bar{u}_\tau\}_\tau$: bdd in $L^\infty(0, T; H_0^1(\Omega))$.
- Subsequences of $\{u_\tau\}_\tau$ and $\{\bar{u}_\tau\}_\tau$ converge to
  ${}^\exists u \in H^1(0, T; L^2(\Omega)) \cap L^\infty(0, T; H_0^1(\Omega))$.

$$u_\tau \to u \quad \text{in } C^0([0, T]; L^2(\Omega)), \qquad u_\tau, \bar{u}_\tau \to u \quad \text{weakly star in } L^\infty(0, T; H_0^1(\Omega)),$$
$$u_\tau \to u \quad \text{weakly in } H^1(0, T; L^2(\Omega)), \quad u_\tau, \bar{u}_\tau \to u \quad \text{weakly in } L^2(0, T; H_0^1(\Omega)),$$

This H^1 estimate is not sufficient for strong solution.

Sketch of proof II (existence (2))

Lemma 9

Fix $k \in \mathbb{N}$. If $f^k \in L^2(\Omega)$, $u^{k-1} \in H_0^1(\Omega) \cap H^2(\Omega)$, then there uniquely exists $u^k \in H_0^1(\Omega) \cap H^2(\Omega)$ such that

$$\frac{u^k(x) - u^{k-1}(x)}{\tau} = \left(\Delta u^k(x) + f^k(x)\right)_+ \quad a.e. \quad x \in \Omega,$$

where this u^k is given as a unique minimizer of

$$J_k(v) := \frac{1}{2\tau}\int_\Omega |v - u_{k-1}|^2\,dx + \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \int_\Omega f_k v\,dx$$

$$u^k := \mathop{\mathrm{arg\,min}}_{v \in K_0^k} J_k(v), \qquad K_0^k := \{v \in H_0^1(\Omega);\ v \geq u^{k-1}\}.$$

Furthermore, we have the following estimate:

$$-\Delta u^k(x) \leq \max(-\Delta u^{k-1}(x), f^k(x)) \quad a.e.\ x \in \Omega$$

Sketch of proof II (existence (3))

- For $f^* \in L^2(\Omega)$ with $f \in L^2(Q)$, $f(x,t) \leq f^*(x)$ a.e. $(x,t) \in Q$, we define $f^k := \frac{1}{\tau}\int_{(k-1)\tau}^{k\tau} f(x,t)dt$. Then

$$\begin{aligned} -\Delta u^k &\leq \max(-\Delta u_0,\ f^1, \cdots, f^k) \\ &\leq \max(-\Delta u_0,\ f^*) \quad \text{a.e. in } \Omega \quad (k = 1, \cdots, [T/\tau]) \end{aligned}$$

- $\{\Delta u_\tau\}_\tau$: bdd in $L^2(Q)$
- $\Delta u_\tau \to \Delta u \quad$ weakly in $L^2(Q)$
- u becomes a strong solution of (1).
- To prove Lemma 9, we need to improve the regularity estimate for variational inequality.

A strong solution is obtained by this H^2 estimate.

Regularity estimate for variational inequality I

For $V := H_0^1(\Omega)$, $\sigma \geq 0$, we define $a(u,v)$ and $A \in B(V,V')$ as

$$a(u,v) := \langle Au,\ v\rangle = \int_\Omega (\nabla u \cdot \nabla v + \sigma uv)\,dx \quad (u,v \in V).$$

For $f \in V'$ and $\psi \in V$, we define $\hat{f} := A\psi \in V'$.

$$K_0 := \{v \in V;\ v \geq \psi \text{ a.e. in } \Omega\}, \quad K_1 := \{v \in V;\ Av \geq f \text{ in } V'\}.$$

$$J(v) := \frac{1}{2}a(v,v) - \langle f, v\rangle, \quad \hat{J}(v) := \frac{1}{2}a(v,v) - \langle \hat{f}, v\rangle \quad (v \in V).$$

Regularity estimate for variational inequality II

Theorem 10

Problems (a)-(e) *are equiv. to each others, and they have a unique sol.*

- (a) $u \in K_0$, $J(u) \leq J(v)$ *for all* $v \in K_0$
- (b) $u \in K_0$, $a(u, v-u) \geq \langle f, v-u \rangle$ *for all* $v \in K_0$
- (c) $u \in K_0 \cap K_1$, $\langle Au - f, u - \psi \rangle = 0$
- (d) $u \in K_1$, $a(u, v-u) \geq \langle \hat{f}, v-u \rangle$ *for all* $v \in K_1$
- (e) $u \in K_1$, $\hat{J}(u) \leq \hat{J}(v)$ *for all* $v \in K_1$

Furthermore, if $f,\ \hat{f} = A\psi \in L^p(\Omega)$ $1 < p < \infty$, $p \geq 2n/(n+2)$, *then* (a)-(e) *are also equiv. to* (f)-(h). $K_2 := \{v \in V;\ f \leq Av \leq \max(f, \hat{f})\}$.

- (f) $u \in K_2$, $\hat{J}(u) \leq \hat{J}(v)$ *for all* $v \in K_2$
- (g) $u \in K_2$, $a(u, v-u) \geq \langle \hat{f}, v-u \rangle$ *for all* $v \in K_2$
- (h) $u \in K_0 \cap K_1 \cap W^{2,p}(\Omega)$, $(Au - f)(u - \psi) = 0$ *a.e. in* Ω

Regularity estimate for variational inequality III

- B.Gustafsson (1986): equivalence of (a)(b) and (f)(g)
- Estimates $\Delta u \in L^p(\Omega)$, $u \in W^{2,p}(\Omega)$ follow from $u \in K_2 := \{v \in V;\ f \leq Av \leq \max(f, \hat{f}) \text{ in } V'\}$.
- In the standard textbooks: D.Kinderlehrer-G.Stampacchia (1980) or A.Friedman (1982), the regularity estimate is shown by a penalty method.
- Gustafsson, Kinderlehrer-Stampacchia, Friedman assumed that $W^{2,p}(\Omega) \subset C^0(\overline{\Omega})$ (i.e. $p > n/2$) in order to use a maximum principle of subharmonic functions.
- We have improved the condition as $1 < p < \infty$, $p \geq 2n/(n+2)$, which enables us to choose $p = 2$ for any $n \in \mathbb{N}$.
- The condition of $(u \in K_2)$ for u^k gives

$$-\frac{u^k - u^{k-1}}{\tau} + f^k \leq -\Delta u^k \leq \max(-\Delta u^{k-1},\ f^k) \quad \text{a.e. in } \Omega.$$

Sketch of proof III(comparison principle, asymptotic behaviour)

Comparison principle

- uniqueness + comparison principle for VI
  $\Longrightarrow$ comparison principle for (1)

Asymptotic behavior

- $^{\exists}u_\infty \in H^1_0(\Omega)\cap H^2(\Omega)$ s.t. $\lim_{t\to\infty}\|u(t,\cdot)-u_\infty\|_{H^1(\Omega)}=0$
- $u_k \le \bar{u}$ $(k\in\mathbb{N})$ follows from CP of sol. u_k of VI, and $u_\infty \le \bar{u}$ follows.
- $\bar{u} \le u_\infty$ follows from CP of sol. $\bar{u}\in H^1_0(\Omega)\cap H^2(\Omega)$ of VI, too.
- $u_\infty = \bar{u}$

Classical one phase Stefan problem

Stefan problem (melting ice in water)

$\Omega = \Omega_I(t) \cup \Gamma(t) \cup \Omega_W(t) \subset \mathbb{R}^n$
$q(x,t) \geq 0$: temparature
$q = 0$ on ice
$V(x,t)$: normal velocity of $\Gamma(t)$

$$\begin{cases} q_t = \Delta q & \text{in } \Omega_W(t) \\ q = 0 & \text{on } \Omega_I(t) \cup \Gamma(t) \\ q = h(x,t) \geq 0 & \text{on } \partial\Omega \\ q(x,0) = q_0(x) & \text{in } \Omega \\ \alpha V = -\partial_\nu q & \text{on } \Gamma(t) \end{cases}$$

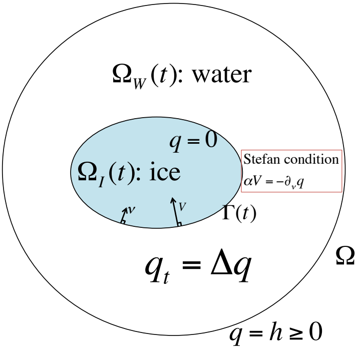


Figure: melting ice $\Omega_I(t)$ surrounded by water region $\Omega_W(t)$

Stefan problem $\Rightarrow$ irreversible diffusion equation

Stefan problem

$$\begin{cases} q_t = \Delta q & \text{in } \Omega_W(t) \\ q = 0 & \text{on } \Omega_I(t) \cup \Gamma(t) \\ q = h(x,t) \geq 0 & \text{on } \partial\Omega \\ q(x,0) = q_0(x) & \text{in } \Omega \\ \alpha V = -\partial_\nu q & \text{on } \Gamma(t) \end{cases}$$

$$\Longrightarrow \begin{cases} u_t = \left(\Delta u + f\right)_+ \\ u = g \quad \text{on } \partial\Omega \\ u(\cdot,0) = 0 \quad \text{in } \Omega \end{cases}$$

Baiocchi transformation

$$u(x,t) := \int_0^t q(x,s)ds,$$
$$g(x,t) := \int_0^t h(x,s)ds,$$
$$f(x) := q_0(x) - \alpha\chi_{\Omega_I(0)}(x)$$

Irreversible diffusion eq.!

This gives a new formulation of the Stefan problem.

$$\Omega_I(t) = \{x \in \Omega;\ u_t(x,t) = 0\}.$$

Conclusion and future problems

- The regularity estimate of the obstacle problem was improved.
- For the irreversible diffusion equation: $u_t = \big(\Delta u + f(x,t)\big)_+$, unique existence of a strong solution, gradient flow structure, comparison principle, and asymptotic behavior were shown.
- The results can be extended to the case of mixed boundary condition: $u = 0$ on Γ_D, $\partial_\nu u = 0$ on Γ_N, provided the H^2-regularity of the elliptic boundary value problem holds.
- Assumption on f : $f \in W^{1,1}(0,T;L^2(\Omega))$ was improved as $f \in L^2(Q)$, $f(x,t) \le f^*(x)$, $f^* \in L^2(\Omega)$.

- Well-posedness of the crack propagation model
- Abstract theory of the doubly nonlinear evolution equation including our irreverisble diffusion equation
- New approach to the Stefan problem