Convergence in time of discrete evolutions generated by alternate minimizing schemes

Matteo Negri

Department of Mathematics - University of Pavia

http://matematica.unipv.it/negri/

Phase-field energy

Energy functional $\boxed{\mathcal{F}(t,u,v) = \mathcal{E}(t,u,v) + G_c \mathcal{L}(v)}$

$$\mathcal{U} = \{u \in H^1_0(\Omega)\} \qquad \mathcal{V} = \{v \in H^1(\Omega),\, 0 \le v \le 1\}$$

Linear elastic energy $\mathcal{E}(t,u,v) = \frac{1}{2}\int_\Omega (v^2+\eta)\, W(\boldsymbol{\varepsilon})\, dx - \int_\Omega f(t)\cdot u \, dx$

Dissipated energy $\mathcal{L}(v) = \frac{1}{2}\int_\Omega (v-1)^2 + |\nabla v|^2 \, dx$

[Ambrosio-Tortorelli (90), Bourdin-Francfort-Marigo (00), etc.]

$\mathcal{F}(t,\cdot,\cdot)$ is non-convex but separately quadratic

Which quasi-static evolution ?

$i)$ alternate minimization

$ii)$ vanishing viscosity

Time discrete evolution by alternate minimization

Time discretization $t_k = k\triangle t$. Set $u(t_0)$ and $v(t_0)$

Define by recursion, $u_m = u(t_{k-1})$ and $v_m = v(t_{k-1})$ for $m = 0$

$$\text{for } m \geq 1 \qquad \begin{cases} u_m \in \text{argmin}\left\{\mathcal{F}\big(t_k\,, \bullet\,, v_{m-1}\big) \text{ in } \mathcal{U}\,\right\} \\ v_m \in \text{argmin}\left\{\mathcal{F}(t_k\,, u_m\,, \bullet\,) \text{ in } \mathcal{V} \text{ with } v \leq v_{m-1}\right\} \end{cases}$$

minimization of quadratic functionals

Let $u(t_k) = \lim_{m\to+\infty} u_m$ and $v(t_k) = \lim_{m\to+\infty} v_m$

long/short increment

Then $(u(t_k), v(t_k))$ is equilibrium point for $\mathcal{F}(t_k, \cdot, \cdot)$ and separate minimizer

Time discrete evolution by alternate minimization

Time discretization $t_k = k\triangle t$. Set $u(t_0)$ and $v(t_0)$

Define by recursion, $u_m = u(t_{k-1})$ and $v_m = v(t_{k-1})$ for $m = 0$

$$\text{for } m \geq 1 \qquad \begin{cases} u_m \in \text{argmin}\left\{\mathcal{F}\big(t_k\,,\bullet\,,v_{m-1}\big) \text{ in } \mathcal{U}\,\right\} \\ v_m \in \text{argmin}\left\{\mathcal{F}(t_k\,,u_m\,,\bullet\,) \text{ in } \mathcal{V} \text{ with } v \leq v_{m-1}\right\} \end{cases}$$

minimization of quadratic functionals

Let $u(t_k) = \lim_{m\to+\infty} u_m$ and $v(t_k) = \lim_{m\to+\infty} v_m$

long/short increment

Then $(u(t_k), v(t_k))$ is equilibrium point for $\mathcal{F}(t_k, \cdot, \cdot)$ and separate minimizer

---

Piecewise-affine (stair like) interpolation $u_{\triangle t}$ and $v_{\triangle t}$

For $\triangle t \to 0$. Do $u_{\triangle t}$ and $v_{\triangle t}$ converge and, if so, to which evolution ?

Parametrization

Arc-length parametrization of the "stair-like" path w.r.t. $\| \cdot \|_u$ and $\| \cdot \|_v$

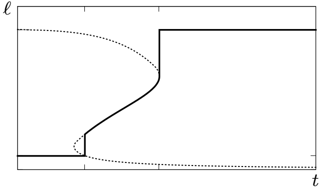
$$[0, S] \ni s \mapsto (t_{\triangle t}(s), u_{\triangle t}(s), v_{\triangle t}(s)) \in [0, T] \times H^1_0 \times H^1$$

Then $(t_{\triangle t}, u_{\triangle t}, v_{\triangle t})$ is bounded in $W^{1,\infty}([0,S]; [0, T] \times H^1_0 \times H^1)$

There exists a limit evolution for $\triangle t \to 0$. Properties and characterization?

A simple sharp crack example by Griffth's criterion

cf. "arc-length methods" [Verhoosel-Remmers-Gutierrez (09)]



dotted: critical $G(t, \ell) = G_c$
stable $G(t, \ell) < G_c$ (left)
unstable $G(t, \ell) > G_c$ (right)

bold: evolution $\ell(t)$

For $t'(s) = 0$ unstable propagation (discontinuity point in time) [long increment]

For $t'(s) > 0$ stable or critical propagation (continuity point in time) [short increment]

Continuity points: equilibrium

Equilibrium for $t'(s) > 0$.

In terms of energy variations: [Knees-N.]

$$\begin{cases} \partial_u \mathcal{F}(t(s), u(s), v(s))[\phi] = 0 & \text{for all } \phi \\ \partial_v \mathcal{F}(t(s), u(s), v(s))[\xi] \geq 0 & \text{for all } \xi \leq 0 \end{cases}$$

Equivalently $u(s)$ and $v(s)$ are separate minimizers.

Continuity points: equilibrium

Equilibrium for $t'(s) > 0$.

In terms of energy variations: [Knees-N.]

$$\begin{cases} \partial_u \mathcal{F}(t(s), u(s), v(s))[\phi] = 0 & \text{for all } \phi \\ \partial_v \mathcal{F}(t(s), u(s), v(s))[\xi] \geq 0 & \text{for all } \xi \leq 0 \end{cases}$$

Equivalently $u(s)$ and $v(s)$ are separate minimizers.

In terms of PDEs:

$$\begin{cases} -\text{div}\left(\boldsymbol{\sigma}_{v(s)}(u(s))\right) = f(s) \quad \text{for } \boldsymbol{\sigma}_v(u) = (v^2 + \eta)\,\boldsymbol{\sigma}(u) \\ \left[v(s)\,W(Du(s)) + (v(s) - 1) - \Delta v(s)\right]_+ = 0 \end{cases}$$

Technically $[...]$ is a negative finite Radon measure.

Not known if $v \in H^2$ cf. [Akagi-Kimura]

Continuity points: consistency with Griffith criterion

Phase field energy release? | Simultaneous evolution.

$$\widetilde{\mathcal{E}}(t,v) = \mathcal{E}(t,u_{t,v},v) \quad \text{for} \quad u_{t,v} \in \text{argmin}\,\{\mathcal{E}(t,\cdot,v) : u \in \mathcal{U}\}.$$

$$\partial_v \widetilde{\mathcal{E}}(t,v)[\xi] = \int_\Omega v\,\xi\, W(\boldsymbol{\varepsilon}_{t,v})\,dx = \partial_v \mathcal{E}(t,u_{t,v},v)[\xi]$$

Normalized admissible variations $\quad \widehat{\Xi} = \{\xi \in H^1 : \xi \le 0 \text{ and } d\mathcal{L}(v)[\xi] = 1\}$

$$\boxed{\mathcal{G}(t,v) = \inf\left\{ -\partial_v \widetilde{\mathcal{E}}(t,v)[\xi] : \xi \in \widehat{\Xi}\right\} \ge 0}$$

Continuity points: consistency with Griffith criterion

Phase field energy release? | Simultaneous evolution.

$$\widetilde{\mathcal{E}}(t,v) = \mathcal{E}(t,u_{t,v},v) \quad \text{for} \quad u_{t,v} \in \text{argmin}\,\{\mathcal{E}(t,\cdot,v) : u \in \mathcal{U}\}.$$

$$\partial_v \widetilde{\mathcal{E}}(t,v)[\xi] = \int_\Omega v\,\xi\, W(\boldsymbol{\varepsilon}_{t,v})\,dx = \partial_v \mathcal{E}(t,u_{t,v},v)[\xi]$$

Normalized admissible variations $\quad \widehat{\Xi} = \{\xi \in H^1 : \xi \le 0 \text{ and } d\mathcal{L}(v)[\xi] = 1\}$

$$\boxed{\mathcal{G}(t,v) = \inf\left\{-\partial_v \widetilde{\mathcal{E}}(t,v)[\xi] : \xi \in \widehat{\Xi}\right\} \ge 0}$$

Griffith's criterion in KT form. | If $t'(s) > 0$ then $d\mathcal{L}'(v(s)) \ge 0$

- $\mathcal{G}(t(s),v(s)) \le G_c$
- $(\mathcal{G}(t(s),v(s)) - G_c)\,\mathcal{L}'(v(s)) = 0$

Behind simultaneous $(\boldsymbol{u}, \boldsymbol{v})$ evolution

A couple of **quantitative estimates**

- If $u \in \text{argmin}\,\{\mathcal{E}(t,\cdot,v) : u \in \mathcal{U}\}$ then $u \in W^{1,p}(\Omega)$ for $p \gtrsim 2$ and $\Omega \subset \mathbb{R}^2$

$$\|u_1 - u_2\|_{W^{1,p}} \le C\|v_0 - v_1\|_{L^q} \le C'\|v_0 - v_1\|_{H^1} \qquad \text{for } q \gg 2$$

[Herzog-Meyer-Wachsmuth (11) cf. also Knees-Rossi-Zanini (13)]

- If $v_k \in \text{argmin}\,\{\mathcal{F}(t,u_k,\cdot) : v \le v_{k-1}\}$ then

$$\|v_1 - v_2\|_{H^1} \le C\|u_1 - u_2\|_{W^{1,p}}\|u_1 + u_2\|_{W^{1,p}}$$

Behind simultaneous (u, v) evolution

A couple of quantitative estimates

- If $u \in \text{argmin}\,\{\mathcal{E}(t,\cdot,v) : u \in \mathcal{U}\}$ then $u \in W^{1,p}(\Omega)$ for $p \gtrsim 2$ and $\Omega \subset \mathbb{R}^2$

$$\|u_1 - u_2\|_{W^{1,p}} \le C\|v_0 - v_1\|_{L^q} \le C'\|v_0 - v_1\|_{H^1} \qquad \text{for } q \gg 2$$

[Herzog-Meyer-Wachsmuth (11) cf. also Knees-Rossi-Zanini (13)]

- If $v_k \in \text{argmin}\,\{\mathcal{F}(t,u_k,\cdot) : v \le v_{k-1}\}$ then

$$\|v_1 - v_2\|_{H^1} \le C\|u_1 - u_2\|_{W^{1,p}}\|u_1 + u_2\|_{W^{1,p}}$$

Alternate minimization as a "simultaneous quasi-Newton" method

$$-\begin{pmatrix} \partial_v \mathcal{F}(\dots)_{k-1} \\ \partial_u \mathcal{F}(\dots)_{k-1} \end{pmatrix} = \left(\begin{array}{c|c} \partial^2_{vv}\mathcal{F}(\dots)_{k-1} & 0 \\ \hline 0 & \partial^2_{uu}\mathcal{F}(\dots)_{k-1} \end{array}\right) \begin{pmatrix} v_k - v_{k-1} \\ u_k - u_{k-1} \end{pmatrix}$$

Discontinuity points: a "normalized gradient flow"

A normalized gradient flow if $t'(s) = 0$

Steepest descent (normalized gradient flow): for a.e. $s \in [0, S]$ s.t. $t'(s) = 0$

$$\begin{cases} \hat{u}'(s) \in \text{argmin}\,\{\partial_u \mathcal{F}(t(s), u(s), v(s))[\phi] : \|\phi\|_{v(s)} \le 1\} \\ \hat{v}'(s) \in \text{argmin}\,\{\partial_v \mathcal{F}(t(s), u(s), v(s))[\xi] : \|\xi\|_{u(s)} \le 1,\ \xi \le 0\} \end{cases}$$

Discontinuity points: a "normalized gradient flow"

A normalized gradient flow if $t'(s) = 0$

Steepest descent (normalized gradient flow): for a.e. $s \in [0, S]$ s.t. $t'(s) = 0$

$$\begin{cases} \hat{u}'(s) \in \text{argmin} \{\partial_u \mathcal{F}(t(s), u(s), v(s))[\phi] : \|\phi\|_{v(s)} \leq 1\} \\ \hat{v}'(s) \in \text{argmin} \{\partial_v \mathcal{F}(t(s), u(s), v(s))[\xi] : \|\xi\|_{u(s)} \leq 1,\, \xi \leq 0\} \end{cases}$$

Which PDEs? Up to a normalization factor

$$\begin{cases} -\text{div}\left(\boldsymbol{\sigma}_{v(s)}(u(s) + u'(s))\right) = f(s) & \text{[viscoelastic phase-field flow]} \\ \quad \dots & \text{[variational inequality or a differential inclusion]} \end{cases}$$

Families of intrinsic norms

Write the energy

$$\begin{aligned}\mathcal{F}(t,u,v) &= \tfrac{1}{2}\int_\Omega (v^2+\eta)\, W(\boldsymbol{\varepsilon})\,dx - \int_\Omega f(t)\cdot u\,dx + G_c\mathcal{L}(v) \\ &= \qquad \tfrac{1}{2}\|u\|_v^2 \qquad\quad + \qquad \beta_t(u) \qquad + \quad c_v\end{aligned}$$

Note that $\|u\|_v$ and $\|u\|_{H^1_0}$ are equivalent

Hence $\partial_u\mathcal{F}(t,u,v)[\phi] = \langle u,\phi\rangle_v + \beta_t(\phi)$ and the slope is

$$|\partial_u\mathcal{F}(t,u,v)|_v = \max\big\{ -\partial_u\mathcal{F}(t,u,v)[\phi] \,:\, \|\phi\|_v \le 1\big\}$$

Similarly define the norm $\|v\|_u$ and the unilateral slope:

$$|\partial_v^-\mathcal{F}(t,u,v)|_u = \max\big\{ -\partial_v\mathcal{F}(t,u,v)[\xi] \,:\, \xi\le 0\,, \|\xi\|_u \le 1\big\}$$

Technically: a parametrized BV evolution for $\triangle t \to 0$

A Lipschitz parametrization $s \mapsto (t(s), u(s), v(s))$ with [Knees-N. (..)]

$$t'(s) \geq 0 \qquad v'(s) \leq 0 \qquad 0 < t'(s) + \|u'(s)\|_{v(s)} + \|v'(s)\|_{u(s)} \leq 1$$

- for every s with $t'(s) > 0$ **equilibrium**

$$|\partial_u \mathcal{F}(t(s), u(s), v(s))|_{v(s)} = |\partial_v^- \mathcal{F}(t(s), u(s), v(s))|_{u(s)} = 0$$

- for every s **energy balance**

$$\begin{aligned}
\mathcal{F}(t(s), u(s), v(s)) = \mathcal{F}(0, u_0, v_0) &+ \int_0^s \mathcal{P}_{ext}(t(r), u(r), v(r))\, t'(r)\, dr \\
&- \int_0^s |\partial_u \mathcal{F}(t(r), u(r), v(r))|_{v(r)}\ \|u'(r)\|_{v(r)}\, dr \\
&- \int_0^s |\partial_v^- \mathcal{F}(t(r), u(r), v(r))|_{u(r)}\ \|v'(r)\|_{u(r)}\, dr
\end{aligned}$$

Proof from [N. (14)] see also [Efendiev-Mielke (06), Sandier-Serfaty (04)]

Stopping criterion

Incremental problem: $u_m = u(t_{k-1})$ and $v_m = v(t_{k-1})$ for $m = 0$

for $m \geq 1$ $\quad \begin{cases} u_m \in \text{argmin}\left\{\mathcal{F}\big(t_k\,, \bullet\,, v_{m-1}\big) \text{ in } \mathcal{U}\,\right\} \\ v_m \in \text{argmin}\left\{\mathcal{F}(t_k\,, u_m\,, \bullet\,) \text{ in } \mathcal{V} \text{ with } v \leq v_{m-1}\right\} \end{cases}$

Set $u(t_k) = u_{\bar m}$ and $v(t_k) = v_{\bar m}$ for some $\bar m \geq 1$ according to ...

Stopping criterion

Incremental problem: $u_m = u(t_{k-1})$ and $v_m = v(t_{k-1})$ for $m = 0$

$$\text{for } m \geq 1 \qquad \begin{cases} u_m \in \text{argmin}\left\{\mathcal{F}\big(t_k\,, \bullet\,, v_{m-1}\big) \text{ in } \mathcal{U}\,\right\} \\ v_m \in \text{argmin}\left\{\mathcal{F}(t_k\,, u_m\,, \bullet\,) \text{ in } \mathcal{V} \text{ with } v \leq v_{m-1}\right\} \end{cases}$$

Set $u(t_k) = u_{\bar{m}}$ and $v(t_k) = v_{\bar{m}}$ for some $\bar{m} \geq 1$ according to ...

Piecewise-affine interpolation $u_{\triangle t}$ and $v_{\triangle t}$

$u_{\triangle t}$ and $v_{\triangle t}$ converge, as $\triangle t \to 0$, to a parametrized BV evolution

No information on uniqueness and speed of convergence.

Irreversibility constraint and time interpolation

Incremental problem: $u_m = u(t_{k-1})$ and $v_m = v(t_{k-1})$ for $m = 0$

for $m \geq 1$ $\quad \begin{cases} u_m \in \text{argmin}\,\big\{\mathcal{F}\big(t_k\,,\bullet\,,v_{m-1}\big) \text{ in } \mathcal{U}\,\big\} \\ v_m \in \text{argmin}\,\big\{\mathcal{F}(t_k\,,u_m\,,\bullet\,) \text{ in } \mathcal{V} \text{ with } v \leq v(t_{k-1})\big\} \end{cases}$

Set $u(t_k) = \lim_m u_m$ and $v(t_k) = \lim_m v_m$ [Bourdin (07)] [Burke-Ortner-Süli (10)]

Irreversibility constraint and time interpolation

Incremental problem: $u_m = u(t_{k-1})$ and $v_m = v(t_{k-1})$ for $m = 0$

$$\text{for } m \geq 1 \qquad \begin{cases} u_m \in \text{argmin}\,\big\{\mathcal{F}\big(t_k\,, \bullet\,, v_{m-1}\big) \text{ in } \mathcal{U}\,\big\} \\ v_m \in \text{argmin}\,\big\{\mathcal{F}(t_k\,, u_m\,, \bullet\,) \text{ in } \mathcal{V} \text{ with } v \leq v(t_{k-1})\big\} \end{cases}$$

Set $u(t_k) = \lim_m u_m$ and $v(t_k) = \lim_m v_m$ [Bourdin (07)] [Burke-Ortner-Süli (10)]

Piecewise-affine interpolation of $u(t_k)$ and $v(t_k)$ in the points t_k.

Then $u_{\Delta t}$ and $v_{\Delta t}$ converge to u and v s.t. $v \in BV(0, T; L^1)$ with $\dot{v} \leq 0$

$$\mathcal{F}(t, u(t), v(t)) \leq \mathcal{F}(0, u_0, v_0) + \int_0^t \partial_t \mathcal{F}(r, u(r), v(r))\, dr$$

[local q.s. evolutions]

The transition v^- and v^+ in discontinuity points is not fully characterized.

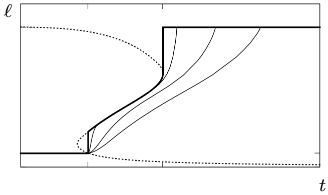
Evolution governed by a system of PDEs [Kuhn-Müller (10), N.]

$$\begin{cases} \varepsilon \dot{v}(t) = -\big[v(t)\, W(Du(t)) + (v(t) - 1) - \Delta v(t) \big]_+ \\ -\mathrm{div}\big(\boldsymbol{\sigma}_{v(t)}(u(t)) \big) = f(t) \end{cases}$$

We study:

- existence of solutions
  (by alternate minimizing movement)
- quasi-static limit for $\varepsilon \to 0$
  (by parametrized BV-evolutions)

  [Dal Maso-Toader (03)] [Mielke et. al.]



Let $t_k = k\triangle t$. We employ an alternate (1-step) minimizing movement:

[N.] cf. [Babadjain-Millot (14)]

$$\begin{cases} u_k \in \text{argmin}\,\{\mathcal{F}(t_k, u, v_{k-1})\}, \\ v_k \in \text{argmin}\,\{\mathcal{F}(t_k, u_k, v) + \frac{\varepsilon}{2\triangle t}\|v - v_{k-1}\|^2_{L^2} : v \le v_{k-1}\} \end{cases}$$

minimization of quadratic functionals and only "short" increments

Euler-Lagrange "equations"

$$\partial_u \mathcal{F}(t_k, u_k, v_{k-1}) = 0 \qquad \text{“}\partial_v \mathcal{F}(t_k, u_k, v_k) + \varepsilon \dot{v}_k = 0\text{”}$$

Piecewise affine interpolate $v_{\triangle t}$ and $u_{\triangle t}$ in the points t_k

Prove that $v_{\triangle t}$ and $u_{\triangle t}$ converge to a solution of the system of PDEs.

Compactness, energy balance and PDEs for $\boxed{\Delta t \to 0}$

1) Compactness

$$v_{\Delta t} \in W^{1,2}(0,T;L^2) \cap L^\infty(0,T;H^1) \qquad u_{\Delta t} \in L^\infty(0,T;W^{1,p}) \text{ for } p > 2$$

2) By l.s.c. of energy and slope and by a "upper gradient inequality"

separate convexity + argument by [Dal Maso-Francfort-Toader (05)]

$$\begin{aligned}\mathcal{F}(t,u(t),v(t)) &= \mathcal{F}(0,u_0,v_0) + \int_0^t \partial_t \mathcal{F}(r,u(r),v(r))\,dr + \\ &- \tfrac{1}{2}\int_0^t \|\varepsilon \dot v(r)\|^2_{L^2} + |\partial_v^- \mathcal{F}(r,u(r),v(r))|^2_{L^2}\,dr.\end{aligned}$$

(De Giorgi's representation of "gradient flows")

3) From the discrete Euler-Lagrange "equations"

$$\boxed{\begin{cases}\varepsilon \dot v(t) = -\big[v(t)\,W(Du(t)) + (v(t)-1) - \Delta v(t)\big]_+ \\ -\mathrm{div}\big(\boldsymbol{\sigma}_{v(t)}(u(t))\big) = f(t)\end{cases}}$$

$[..]$ is a finite Radon measure with $[..]_+$ in L^2 [N.] cf. [Gianazza-Savaré (94)]

No chain rule and no compactness in $W^{1,2}(0,T;H^1)$ are needed

Parametrization, compactness etc. for $\boxed{\varepsilon \to 0}$

1) Arc-length $t \mapsto s(t) = \int_0^t 1 + \|\dot v_\varepsilon(r)\|_{L^2}\, dr$ with inverse $t_\varepsilon(s)$

- parametrized evolutions $s \mapsto (t_\varepsilon(s), v_\varepsilon(s))$ for $s \in [0, S_\varepsilon]$

2) Delicate technical point:

- length S is finite unif. w.r.t. ε

Discrete Gronwall estimate [Nocetto-Savaré-Verdi (00)] cf. [Knees-Rossi-Zanini (13)]

- compactness of v_ε in $W^{1,\infty}(0, S; L^2)$

3) Equilibrium and parametrized energy balance for the limit evolution

$$\mathcal{F}(t(s),\ u(s), v(s)) = \mathcal{F}(0, u_0, v_0) + \int_0^s \partial_t \mathcal{F}(t(r), u(r), v(r))\, t'(r)\, dr + \\ - \int_0^s |\partial_v^- \mathcal{F}(t(r), u(r), v(r))|_{L^2}\, dr$$

From energy balance to PDEs

From the energy balance (not from the PDE) get "two regimes"

- If $t'(s) > 0$ (continuity points)

$$\begin{cases} \big[v(s)\, W(Du(s)) + (v(s) - 1) - \Delta v(s) \big]_+ = 0 \\ -\mathrm{div}\left(\boldsymbol{\sigma}_{v(s)}(u(s)) \right) = f(t) \end{cases}$$

- If $t'(s) = 0$ (discontinuity points)

$$\begin{cases} \hat{v}'(s) = -\big[v(s)\, W(Du(s)) + (v(s) - 1) - \Delta v(s) \big]_+ \\ -\mathrm{div}\big(\boldsymbol{\sigma}_{v(s)}(u(s)) \big) = f(t) \end{cases}$$

Employ the chain rule and local compactness in $W^{1,2}(..., H^1)$

Not same as the q.s. alternate minimization (underlying norms are different)

From phase-field to sharp crack (I)

Γ-convergence of the energies

$$\mathcal{F}_\delta(t,u,v) = \int_\Omega (v^2 + \eta_\delta)\, W(\boldsymbol{\varepsilon})\, dx + G_c \int_\Omega v^2/(4\delta) + \delta|\nabla v|^2\, dx$$

$$\downarrow$$

$$\mathcal{F}_0(t,u) = \int_{\Omega\setminus J_u} W(\boldsymbol{\varepsilon})\, dx + G_c\, \mathcal{H}^{n-1}(J_u)$$

(convergence holds in a very general setting)

[Ambrosio-Tortorelli (90)] [Chambolle (03)] [Iurlano (12)]

Γ-convergence implies convergence of global minimizers (upon compactness)

What about critical points, slopes and energy release? [Francfort-Le-Serfaty (09)]

From phase-field to sharp crack (II)

Convergence of energies and derivatives imply convergence of evolutions

for gradient flows [Sandier-Serfaty (04)] for BV-evolutions [N. (14)]

At least we should have ...

✓ $\mathcal{F}_0(t, u) \le \liminf_{\delta \to 0} \mathcal{F}_\delta(t, u_\delta, v_\delta)$

? $\mathcal{G}_0(t, J_u) \le \liminf_{\delta \to 0} \mathcal{G}_\delta(t, v_\delta)$ or for a slope ...

Some partial results

- $G(t, \ell) = \lim_{\delta \to 0} \mathcal{G}_\delta(t, v_\delta)$ in a simplified geometry [Sicsic-Marigo (13)] [N. (13)]
- a different notion of evolution [Babadjian-Millot (14)]