On a quantitative piecewise rigidity result and Griffith-Euler-Bernoulli functionals for thin brittle beams

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Variational Models of Fracture,

Banff, May 12^{th}, 2016

Overview

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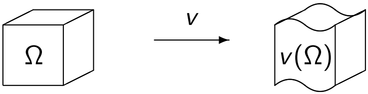
Nonlinear elasticity theory

Elastostatics: Understand stable deformations of a block Ω of elastic material, subject to boundary conditions and applied loads.



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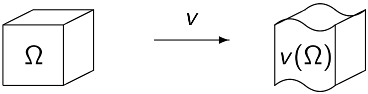

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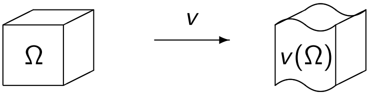

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with stored energy function $W : \mathbb{R}^{d \times d} \to \mathbb{R}$ which is

- frame indifferent, ≥ 0 with $W(F) = 0 \iff F \in \mathrm{SO}(d)$,
- sufficiently regular,
- non-degenerate: $W(F) \geq c\, \mathrm{dist}^2(F, SO(n))$.

Classical beam theory

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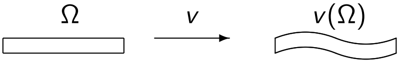

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Euler-Bernoulli theory: Energy functional for bending dominated configurations in terms of the mid-line deformation.

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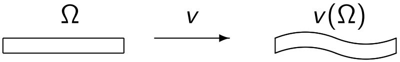

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Note: α is the 'Poisson effect relaxed' elastic modulus (cf. Friesecke/James/Müller '02).

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For rigorous Γ-convergence results (even 3D → 2D), see

- LeDret/Raoult '93: membranes
- Friesecke/James/Müller '02 & '06: hierarchy of plate theories

Variational fracture mechanics

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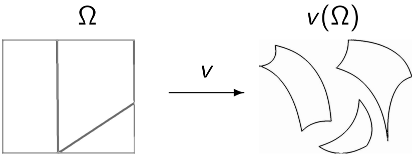

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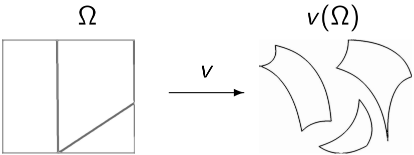

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Griffith-type energy functional (cf. Francfort/Marigo):

$$E_{\mathrm{Griff}}(v) = \underbrace{\int_{\Omega \setminus J_v} W(\nabla v)}_{\text{elastic energy}} + \underbrace{\beta \mathcal{H}^{d-1}(J_v)}_{\text{crack energy}},$$

W: stored energy function,
β: crack energy / surface area

Overview

The model and the main goal

Goal: Find effective theory for thin brittle beams for bending dominated configurations: a Griffith-Euler-Bernoulli theory.

$$E^h_{\mathrm{Griff}}(v) = \int_{\Omega_h \setminus J_v} W(\nabla v) + \beta_h \mathcal{H}^{d-1}(J_v), \quad \begin{array}{l} \Omega_h = (0,L) \times (-\frac{h}{2}, \frac{h}{2}),\ h \ll L, \\ v \in SBV(\Omega_h; \mathbb{R}^2). \end{array}$$

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Consequence: To model materials which

- respond elastically to small (e.g. infinitesimal) deflection,
- may fracture at large (finite) bending,

we assume that $\beta_h = h^2 \beta$.

Goal: Determine the Γ-limit of $h^{-3} E^h_{\mathrm{Griff}}$ as $h \to 0$.

Main result

More precise setup:
Rescale to common domain $\Omega = \Omega_1$ via $y(x_1, x_2) = v(x_1, hx_2)$ and for a large fixed $M \gg 1$ let

$$I^h(y) = h^{-3} \int_{\Omega_h} W(\nabla v)\, dx + h^{-1}\beta\mathcal{H}^1(J_v)$$

if $v \in SBV(\Omega; \mathbb{R}^2), \max\{\|v\|_{L^\infty}, \|\nabla v\|_{L^\infty}\} \leq M$. (Extend to all of $SBV(\Omega; \mathbb{R}^2)$ by $+\infty$.)

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If $y(x) = \bar{y}(x_1)$ for a.e. $x \in \Omega$ we also set

$$I^0(y) = \frac{\alpha}{24} \int_0^L |\kappa(t)|^2\, dt + \beta \#(J_{\bar{y}} \cup J_{\bar{y}'}),$$

if $\bar{y} \in \mathrm{PW}\text{-}W^{2,2}((0, L); \mathbb{R}^2)$, $|\bar{y}| \leq M$ and $|\bar{y}'| = 1$ a.e., $\kappa = \bar{y}'' \cdot (\bar{y}')^\perp$. (Extend to all of $SBV(\Omega; \mathbb{R}^2)$ by $+\infty$.)

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Theorem. (Gamma-convergence) [S. '16]
The I^h Γ-converge to I^0 on $SBV(\Omega;\mathbb{R}^2)$ (w.r.t. L^1) as $h \to 0$, i.e.,

(i) lim inf inequality:
whenever $y^h \to y$ in L^1,
$\liminf_{h\to 0} I^h(y^h) \geq I^0(y)$;

(ii) recovery sequences:
$\forall\, y\ \exists\, y^h$ with $y^h \to y$ in L^1 s.t.
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Remarks.

- In fact, $y^h \to y$, $\partial_1 y^h \to y'$, $h^{-1}\partial_2 y^h \to y'^{\perp}$ in L^p strongly for all $p < \infty$ and $D^s y^h \overset{*}{\rightharpoonup} D^s y$ weakly* as Radon measures.
- Body forces and clamped boundary conditions can be included.
- Entails a convergence theorem for (almost) minimizers (subject to suitable body forces and boundary conditions).

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  Should instead look at $E_{\mathrm{Griff}} \sim h\beta_h$:

  infinitesimal deflection $\sim$ vertical crack

  $\implies$ Griffith type small deflection beam theory
  (in analogy to the results presented).

  Applies to 'very brittle' materials.

Overview

Geometric rigidity: known results

Basic ingredient in the derivation of effective theories for elastic plates (cf. Friesecke/James/Müller '02 & '06): a quantitative geometric rigidity estimate.

Theorem. [Friesecke/James/Müller '02] Let $\Omega \subset \mathbb{R}^d$ a (connected) Lipschitz domain. For all $y \in W^{1,2}(\Omega, \mathbb{R}^d)$ there is $R \in SO(d)$ s.t.

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But there is a qualitative version:

Theorem. [Chambolle/Giacomini/Ponsiglione '07] Suppose $y \in SBV(\Omega; \mathbb{R}^d)$, $\mathcal{H}^1(J_y) < \infty$ and $\nabla y \in \mathrm{SO}(d)$ a.e. Then there exists a (Caccioppoli) partition (P_i) and $R_i \in \mathrm{SO}(d)$, $c_i \in \mathbb{R}^d$ such that

$$y(x) = \sum_i (R_i x + c_i)\chi_{P_i}(x).$$

I.e.: y is a collection of an at most countable family of rigid deformations

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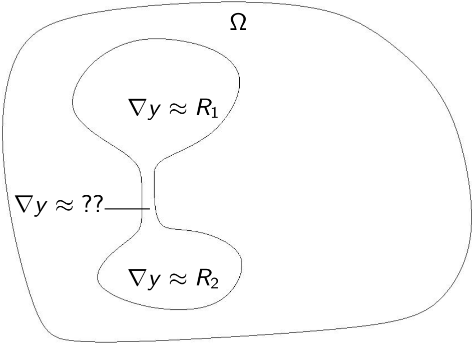
Problem: We need a quantitative version! $\Longrightarrow$ serious difficulties, e.g.:

- Thin 'tunnels' connecting large regions close to different rotations.



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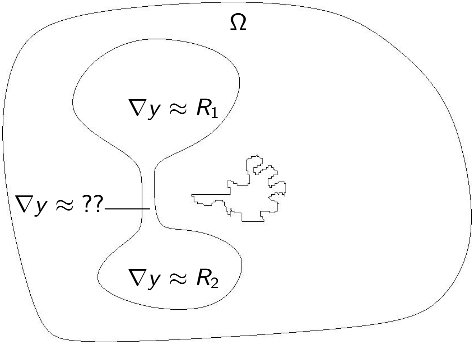

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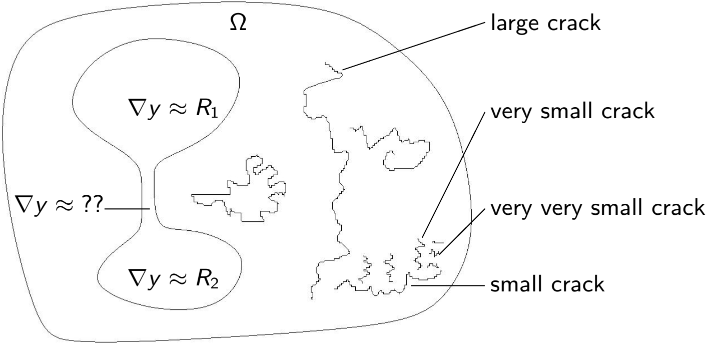

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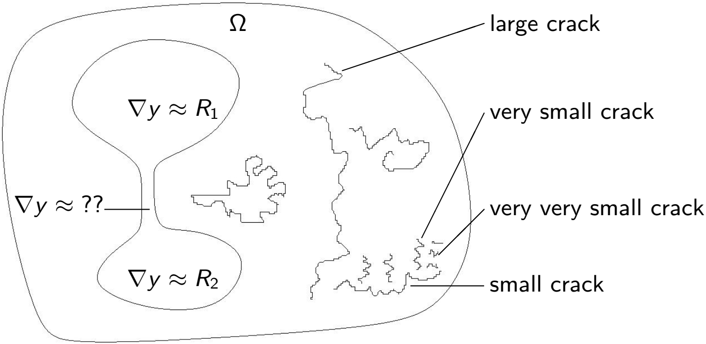

- Could even have a dense crack set.

Quantitative piecewise geometric rigidity . . . morally

Theorem (cheating version). [Friedrich/S. '15]
Let $\Omega \subset \mathbb{R}^2$ a Lipschitz domain, $M > 0$ and $0 < \eta < 1$.
$\exists\ C = C(\Omega, M, \eta)$, $\hat{C} = \hat{C}(\Omega, M, \eta, \dots)$ such that $\forall\ \varepsilon > 0$:

Suppose $y \in SBV(\Omega; \mathbb{R}^2)$ with $|y|, |\nabla y| \leq M$ a.e. satisfies

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and, for each P_j, a corresponding rigid motion $R_j \cdot + c_j$, $R_j \in \mathrm{SO}(2)$ and $c_j \in \mathbb{R}^2$, such that

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satisfies the estimates

$$\|u\|^2_{L^2(\Omega)} + \sum\nolimits_j \|\mathrm{sym}(R_j^T \nabla u)\|^2_{L^2(P_j)} + \varepsilon^\eta \|\nabla u\|^2_{L^2(\Omega)} \leq \hat{C}\varepsilon.$$

Modifications

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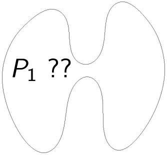

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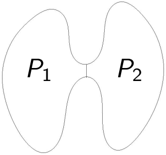

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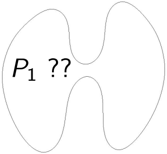

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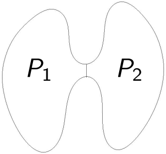

$\rightarrow$ We need to

- introduce a little bit of extra crack,
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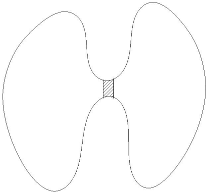
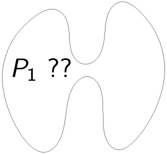
- modify y slightly: $y \rightsquigarrow \hat{y}$ (interpolate on neglected regions).

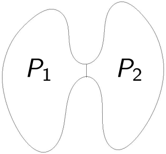


Modifications

As stated, the theorem cannot be true.



should rather be



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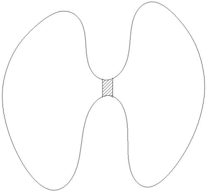

Caveat: Do not introduce artificial energy! We still want that

$$\int_\Omega W(\nabla \hat{y})\,dx \approx \int_\Omega W(\nabla y)\,dx \quad \text{and} \quad \mathcal{H}^1(J_{\hat{y}}) \approx \mathcal{H}^1(J_y).$$

Quantitative piecewise geometric rigidity . . . the full story

Theorem. [Friedrich/S. '15]
Let $\Omega \subset \mathbb{R}^2$ a Lipschitz domain, $M > 0$ and $0 < \eta, \rho < 1$.
There are constants $C = C(\Omega, M, \eta)$, $\hat{C} = \hat{C}(\Omega, M, \eta, \rho)$ and $c > 0$ such that for $h > 0$ small enough:

Suppose $y \in SBV(\Omega; \mathbb{R}^2)$ with $|y|, |\nabla y| \leq M$ a.e. satisfies

$$h^{-1}\varepsilon := h^{-1} \int_\Omega \mathrm{dist}^2(\nabla y, \mathrm{SO}(2)) + \mathcal{H}^1(J_y) \leq M,$$

and set $\Omega_\rho = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > c\rho\}$.

Then there is an open Ω_y with $|\Omega_\rho \setminus \Omega_y| \leq C\rho h^{-1}\varepsilon$, a modification $\hat{y} \in SBV(\Omega)$ with $|y|, |\nabla y| \leq cM$ and

- $\|\hat{y} - y\|^2_{L^2(\Omega_y)} + \|\nabla\hat{y} - \nabla y\|^2_{L^2(\Omega_y)} \leq C\rho\varepsilon$,
- $\mathcal{H}^1(J_{\hat{y}} \cap \Omega_\rho) \leq Ch^{-1}\varepsilon$,
- $h^{-1} \int_{\Omega_\rho} W(\nabla\hat{y})\, dx \leq h^{-1} \int_\Omega W(\nabla y)\, dx + C\rho h^{-1}\varepsilon$,

with the following properties:

Quantitative piecewise geometric rigidity . . . the full story

There is a Caccioppoli partition $\mathcal{P} = (P_j)_j$ of Ω_ρ with

$$\sum_j \tfrac{1}{2}\mathrm{Per}(P_j, \Omega_\rho) \le \mathcal{H}^1(J_y) + C\rho h^{-1}\varepsilon$$

and, for each P_j, a corresponding rigid motion $R_j \cdot + c_j$, $R_j \in \mathrm{SO}(2)$ and $c_j \in \mathbb{R}^2$, such that the modified displacement $\hat{u} : \Omega \to \mathbb{R}^2$ defined by

$$\hat{u}(x) := \begin{cases} \hat{y}(x) - (R_j\, x + c_j) & \text{for } x \in P_j \\ 0 & \text{for } x \in \Omega \setminus \Omega_\rho \end{cases}$$

satisfies the estimates

(i) $\mathcal{H}^1(J_{\hat{u}}) \le Ch^{-1}\varepsilon$, (ii) $\|\hat{u}\|^2_{L^2(\Omega_\rho)} \le \hat{C}\varepsilon$,

(iii) $\sum_j \|\mathrm{sym}(R_j^T \nabla \hat{u})\|^2_{L^2(P_j)} \le \hat{C}\varepsilon$, (iv) $\|\nabla \hat{u}\|^2_{L^2(\Omega_\rho)} \le \hat{C}\varepsilon^{1-\eta}$.

Proof strategy

The proof is very long and involved.
Basic (oversimplified) idea:

- Start with very very small cracks (1^{st} generation).
  $\rightarrow$ Either heal them, if surrounded by a region with small energy,
  $\rightarrow$ or enlarge them to very small cracks by using the (large) energy of the surrounding region.
- Consider now very small cracks (2^{nd} generation).
- And so on ...
- Caveat: The (elastic + crack) energy of a region is 'used' to
  - heal cracks or
  - enlarge cracks.

  But: Possibly infinitely many generations of scales. Must make sure that energy is 'used' not too often.

A Korn-Poincaré inequality in *SBD*

Important ingredient: a novel Korn-Poincaré inequality in *SBD* obtained by Friedrich '15.

Theorem (cheating version). [Friedrich '15]
Let $\varepsilon, h_* > 0$ (small), $\tilde{Q} \subset\subset Q = (-\frac{1}{2}, \frac{1}{2})^2$. There is a constant $C = C(h_*)$ and a universal constant $c > 0$ such that for all $u \in SBD^2(Q; \mathbb{R}^2)$ there is an exceptional set $E \subset \tilde{Q}$ with

$$\|u - (A \cdot + c)\|^2_{L^2(\tilde{Q} \setminus E)} \leq C\big(\|e(u)\|^2_{L^2(Q)} + \varepsilon \mathcal{H}^1(J_u)\big)$$

for some $A \in \mathbb{R}^{2\times 2}_{\mathrm{skew}}$, $c \in \mathbb{R}^2$, where

$$|E| \leq (1 + ch_*)\big(\mathcal{H}^1(J_u) + \varepsilon^{-1}\|e(u)\|^2_{L^2}\big)^2$$

and

$$\mathcal{H}^1(\partial E) \leq (1 + ch_*)\big(\mathcal{H}^1(J_u) + \varepsilon^{-1}\|e(u)\|^2_{L^2}\big).$$

Remark. A similar recent results by Chambolle/Conti/Francfort '15 even works in any dimension, but has no control on ∂E.

Other applications

Remark. The crack energy of $\hat{y}$ can be estimated more thoroughly. In fact:

$$\sum_{P\in\mathcal{P}} \tfrac{1}{2}\mathrm{Per}(P;\Omega) + \int_{J_{\hat{y}}\setminus\bigcup_{P\in\mathcal{P}}\partial P} \min\left\{\left|\frac{[\hat{y}]}{\sqrt{\varepsilon}\rho}\right|, 1\right\} d\mathcal{H}^1 \le \mathcal{H}^1(J_y) + c\rho.$$

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Example. Nonlinear-to-linear bulk Griffith models in the small strain limit (cf. Dal Maso, Negri, Percivale '02 for the elastic case).
In the presence of cracks Friedrich '15 obtains a Γ-convergence result with a limiting energy defined on triples $(u,\mathcal{P},T)$, where

- $u\in SBV(\Omega;\mathbb{R}^2)$, $\Omega\subset\mathbb{R}^2$, a deformation,
- $\mathcal{P}$ a Caccioppoli partition of Ω,
- T piecewise rigid motion subordinate to $\mathcal{P}$,

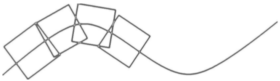
of the form

$$E(u,\mathcal{P},T) = \int_\Omega \tfrac{1}{2}Q\big(\mathrm{sym}(\nabla T^T\nabla u)\big) + \mathcal{H}^1(J_u\setminus\bigcup_{P\in\mathcal{P}}\partial P) + \sum_{P\in\mathcal{P}}\tfrac{1}{2}\mathrm{Per}(P;\Omega).$$

Overview

Pure elasticity

Warm up: Purely elastic case (cf. Friesecke/James/Müller '02)
. . . in a nutshell:



- Cover beam with small squares $Q_1, Q_2, \dots$ of side-length h.
- Geometric rigidity → approximating rigid motions $R_i \cdot + c_i$ on Q_i.

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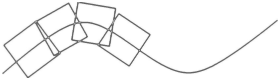
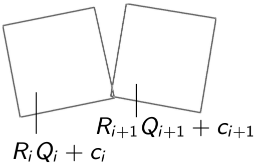
- Estimate on $|R_{i+1} - R_i|$ → $W^{2,2}$ compactness.





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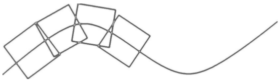

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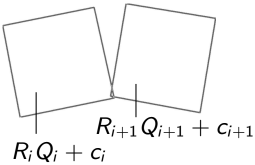
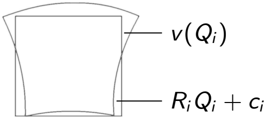
- Geometric rigidity → approximating rigid motions $R_i \cdot + c_i$ on Q_i.
- Estimate on $|R_{i+1} - R_i| \rightarrow W^{2,2}$ compactness.
- Fine estimate on $h^{-1}(\nabla v - R_i)$ and a weak convergence argument give
  - limiting infinitesimal strain
  - its x_2-linearity and
  - the Γ-lim inf inequality.





Elasticity + fracture: first steps

Idea: Try a similar approach. Preparations:

- Cover beam with small (overlapping) squares $Q_1, Q_2, \ldots$.
- If energy in Q_i large $\to$ Q_i 'bad',
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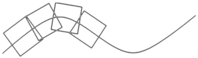

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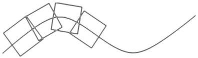

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Now apply the quantitative piecewise rigidity estimate on good squares:

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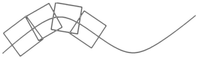

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- Fix $\eta = \frac{9}{10}$ and let $\rho > 0$. On each good Q_i we get

$$Q_{i,\rho},\ Q_{i,v},\ \hat{v}_i,\ (P_{i,j})_j,\ (R_{i,j})_j,\ (c_{i,j})_j \quad \text{s.t.} \ \ldots \text{ with } C,\ \hat{C}(\rho).$$

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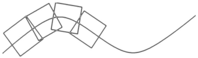

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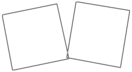
So . . . What's the problem?

New problems

Problem 1 (a bit severe ... more nasty)**:** We only have the estimate

$$\|\nabla \hat{v} - R_{i,1}\|_{L^2(P_{i,1})} \leq \|\mathrm{dist}(\nabla v, \mathrm{SO}(2))\|^{9/10}_{L^2(P_{i,1})}.$$



Still: E.g., estimating $R_{i+1} - R_i$ is still possible.

(Controlling of $\|\hat{v} - R_{i,1} \cdot - c_{i,1}\|_{L^2(P_{i,1})}$ and $\|\mathrm{sym}\big(R_{i,1}^T \nabla \hat{v}\big) - \mathrm{Id}\|_{L^2(P_{i,1})}$.)

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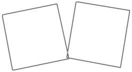

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Problem 2 (more severe)**:** Eventually, we must take the limit $\rho \to 0$.
But $Q_{i,\rho}$, $Q_{i,v}$, $\hat{v}_i$, $(P_{i,j})_j$, $(R_{i,j})_j$, $(c_{i,j})_j$ and $\hat{C}$ depend on ρ.

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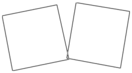

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Problem 3 (most severe)**:** To identify the limiting infinitesimal strain, we need to join the different $\hat{v}_i$ to one single beam deformation $\tilde{v}$.

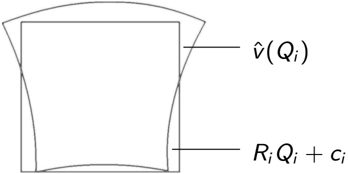
Note: Piecewise gluing $\to$ too much crack,
mollification thereof $\to$ too high energy near small cracks.

Instead: Blend smoothly with partitions of unity: $\tilde{v} = \sum_i \varphi_i \hat{v}_i$
(only cheating a bit).

Two main difficulties

Two challenges to overcome:

1. $\nabla \tilde{v} = \underbrace{\sum_i \varphi_i' \, \hat{v}_i \otimes \mathbf{e}_1}_{\text{must be very small!}} + \underbrace{\sum_i \varphi_i \, \nabla \hat{v}_i}_{\text{good for energy estimates}}.$



Must ensure that $\hat{v}_i$ is not 'too good'.

Need sharp estimates on $\hat{v}_{i+1} - \hat{v}_i$ on overlap $Q_{\rho,i} \cap Q_{\rho,i+1}$, also where $\hat{v}_i \not\approx v$. (In fact, will get only sufficiently strong L^p-estimates for $p < 2$.)

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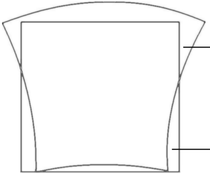

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2. Linearity of the limiting infinitesimal strain in x_2, morally

$$\partial_2 \Big(\lim_{h \to 0} h^{-1} (\tilde{R}^h)^T \nabla \tilde{v} \Big)_{11} \stackrel{!}{=} \bar{y}'' \cdot \bar{y}'^{\perp}.$$

Problem: $\nabla\tilde{v}$ is not a derivative.

Trick: Consider $(\tilde{R}^h)^T \tilde{v}$. Using a novel *GSBD* compactness argument due to Dal Maso '13, we get, morally, $\partial_2 \Big(\lim_{h \to 0} h^{-1} \nabla [(\tilde{R}^h)^T \tilde{v}] \Big)_{11} = 0$.

$\to$ Can move ∇ to $\tilde{R}^h$.

Thanks

Thank you for your attention!

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