FEM for stationary elliptic interface problems

Marcus Sarkis

collaborators: J. Guzmań, M. A. Sánchez and E. Burman

Outline

Outline

Interface Problem

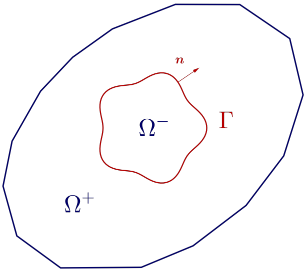


Figure: Domain $\Omega = \Omega^+ \cup \Omega^- \cup \Gamma$.
Smooth Interface Γ.

Consider the problem

$$
\begin{aligned}
\rho^\pm \Delta u^\pm &= f^\pm && \text{in } \Omega^\pm \\
u &= 0 && \text{on } \partial\Omega \\
[\rho \nabla u \cdot \boldsymbol{n}] &= \beta && \text{on } \Gamma \\
[u] &= \alpha && \text{on } \Gamma
\end{aligned}
$$

We denote

$$[\rho \nabla u \cdot n] = \rho^+ \nabla u^+ \cdot \boldsymbol{n}^+ + \rho^- \nabla u^- \cdot \boldsymbol{n}^-, \quad [u] = u^+ - u^-$$

$\boldsymbol{n}^\pm$: normal vector pointing outwards $\Omega^\pm$

Interface Problem

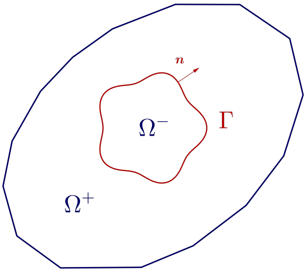


Figure: Domain $\Omega = \Omega^+ \cup \Omega^- \cup \Gamma$.
Smooth Interface Γ.

Consider the problem

$$\begin{aligned} \Delta u^\pm &= f^\pm && \text{in } \Omega^\pm \\ u &= 0 && \text{on } \partial\Omega \\ [\nabla u \cdot \boldsymbol{n}] &= \beta && \text{on } \Gamma \\ [u] &= 0 && \text{on } \Gamma \end{aligned}$$

We denote

$$[\nabla u \cdot n] = \nabla u^+ \cdot \boldsymbol{n}^+ + \nabla u^- \cdot \boldsymbol{n}^-, \quad [u] = u^+ - u^-$$

$\boldsymbol{n}^\pm$: normal vector pointing outwards $\Omega^\pm$

Stationary Immersed Boundary Problem

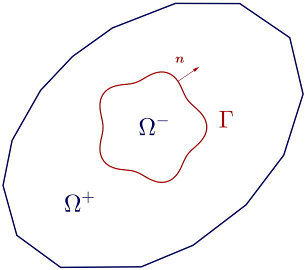


Figure: Domain $\Omega = \Omega^+ \cup \Omega^- \cup \Gamma$.
Smooth Interface Γ.

Consider the problem

$$\Delta u = f + g \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

$$g(x) = \int_0^A \beta(s)\delta(x - X(s))ds, \quad x \in \Omega$$

$\delta(x)$: Dirac delta function
$\boldsymbol{X}$: parametrization over $[0, A]$ of curve Γ

J. Thomas Beale and Anita T. Layton.
On the accuracy of finite difference methods for elliptic problems with interfaces.
Commun. Appl. Math. Comput. Sci., 1:91–119 (electronic), 2006

Yoichiro Mori.
Convergence proof of the velocity field for a Stokes flow immersed boundary method.
Comm. Pure Appl. Math., 61(9):1213–1263, 2008.

Immersed Boundary Method (IBM)

IBM is a finite difference method developed by Peskin (1977), as a computational tool to investigate blood flow in the presence of cardiac valves

Some properties:

- Discrete approximation $\delta_h(x)$ of the Dirac delta function $\delta(x)$
- Application in problems with fluid-structure interaction
- Beale and Layton (2006) proved error estimates for the Poisson equations
- Mori (2007) proved error estimates for the stationary Stokes equations
- Boffi and Gastaldi (2004) introduced a finite element approach to IBM
- IBM and FE-IBM are only first order accurate near the interface

Charles S. Peskin.
Numerical analysis of blood flow in the heart.
J. Computational Phys., 25(3):220–252, 1977.

Finite element approach to IBM

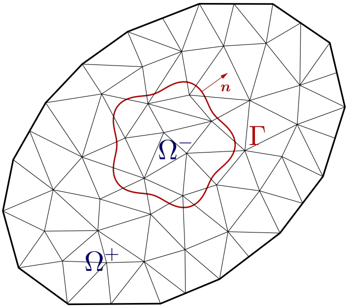


Figure: Domain $\Omega = \Omega^+ \cup \Omega^- \cup \Gamma$.
Smooth Interface Γ.

Difficulties:

- Mesh non aligned with the interface
- Regularity of the solution

Daniele Boffi and Lucia Gastaldi.
A finite element approach for the immersed boundary method.
Comput. & Structures, 81(8-11):491–501, 2003.
In honour of Klaus-Jürgen Bathe.

Finite element formulation

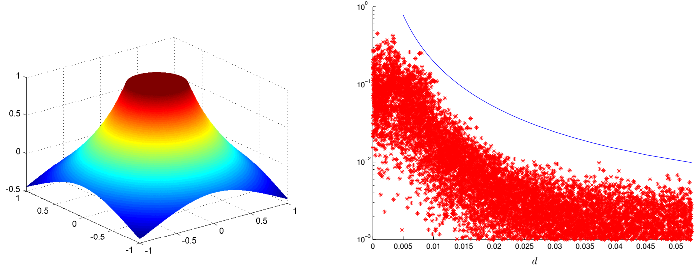
Find $u_h \in V_h$ such that

$$\int_\Omega \nabla u_h \cdot \nabla v dx = \int_\Omega f v dx + \int_\Gamma \beta(s) v(\boldsymbol{X}(s)) ds, \quad \forall v \in V_h$$

V_h: finite element space of continuous piecewise polynomials

- This can be seen as the finite element approach of IBM for Poisson equations
- By means of the variational formulation the delta function is replaced by an integral along the interface
- The method is formulated for a mesh independent of the interface (not matching)
- An important feature is that the left-hand side is not affected by the interface

Numerical test FE-IBM



h	$\|e_h^N\|_{L^2}$	r	$\|\nabla e_h^N\|_{L^2}$	r	$\|e_h^N\|_{L^\infty}$	r	$\|\nabla e_h^N\|_{L^\infty}$	r
1.8e-1	1.02e-1		4.71e-1		1.63e-1		7.01e-1	
8.8e-2	1.57e-2	2.70	1.38e-1	1.78	4.09e-2	2.00	3.26e-1	1.10
4.4e-2	6.72e-3	1.22	1.30e-1	0.09	2.85e-2	0.52	5.48e-1	-0.75
2.2e-2	2.02e-3	1.74	7.88e-2	0.72	1.07e-2	1.42	5.87e-1	-0.10
1.1e-2	7.65e-4	1.40	6.16e-2	0.36	7.24e-3	0.56	6.24e-1	-0.09
5.5e-3	2.71e-4	1.50	4.27e-2	0.53	4.39e-3	0.72	6.24e-1	0.00
2.8e-3	9.09e-5	1.58	2.83e-2	0.59	2.04e-3	1.11	7.80e-1	-0.32
1.4e-3	3.53e-5	1.36	2.24e-2	0.34	1.38e-3	0.57	8.78e-1	-0.17

Table: L^2 and L^∞ errors of the approximate solution by FE-IBM, on a non-uniform grid.

Convergence

Theorem Suppose that Ω is a rectangle and assume that u solves the interface problem with the Dirichlet boundary conditions replaced with periodic boundary conditions. Let u_h be its FE-IBM approximation. Let $z \in \Omega$ and let $d = dist(z, \Gamma) \geq \kappa h$ for a sufficiently large fixed constant κ. Furthermore, suppose $dist(\Gamma, \partial\Omega) > d$. Then, we have

$$|\nabla(I_h u - u_h)(z)| \leq Ch(\log(1/h)\frac{h}{d^2} + 1)(\|u\|_{C^2(\Omega^-)} + \|u\|_{C^2(\Omega^+)}),$$

where C depends only on the quasi-uniformity of the mesh.

Outline

Interface Problem

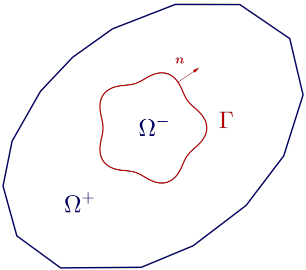


Figure: Domain $\Omega = \Omega^+ \cup \Omega^- \cup \Gamma$. Smooth Interface Γ.

Consider the problem

$$\begin{aligned} \Delta u^{\pm} &= f^{\pm} && \text{in } \Omega^{\pm} \\ u &= 0 && \text{on } \partial\Omega \\ [\nabla u \cdot \boldsymbol{n}] &= \beta && \text{on } \Gamma \\ [u] &= 0 && \text{on } \Gamma \end{aligned}$$

Motivation: Implementation of more general problems as Stokes (Navier-Stokes) equations

Objective: Develop finite element methods with piecewise polynomials of order k

Finite element method

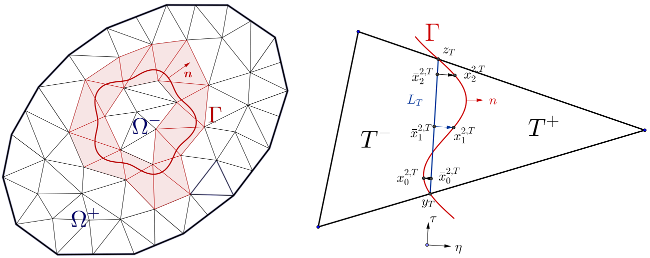


Figure: Domain $\Omega = \Omega^+ \cup \Omega^- \cup \Gamma$. Smooth Interface Γ.

Find $u_h \in V_h$, such that

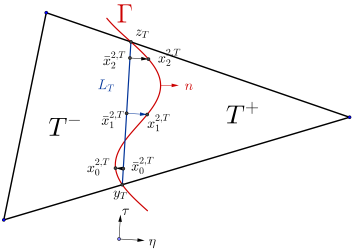
$$\int_\Omega \nabla u_h \cdot \nabla v \, dx = \int_\Omega f v \, dx + \int_\Gamma \beta v \, ds - \sum_{T \in \mathcal{T}_h^\Gamma} \left(\int_{T^-} \nabla w_T^u \cdot \nabla v \, dx + \int_{T^+} \nabla w_T^u \cdot \nabla v \, dx \right)$$

for all $v \in V_h$, where w_T^u is a correction function

Construction correction function

Define the local space for $T \in \mathcal{T}_h^\Gamma$

$$S^k(T) = \left\{ w \in L^2(T) : w|_{T^\pm} \in \mathbb{P}^k(T^\pm) \right\}.$$



$$\left[D_{\boldsymbol{\eta}}^{k-\ell} w_T^u(x_i^{\ell,T}) \right] = \left[D_{\boldsymbol{\eta}}^{k-\ell} u(x_i^{\ell,T}) \right] \quad \text{for } 0 \le i \le \ell \text{ and } 0 \le \ell \le k,$$

$$\begin{cases} (w_T^u)^-(\theta) = 0, & \text{if } \theta \in \Omega^- \cup \Gamma; \\ (w_T^u)^+(\theta) = 0, & \text{if } \theta \in \Omega^+. \end{cases} \quad \text{for all degree } k \text{ Lagrange points } \theta \text{ of } T.$$

Remark

Define the bilinear form and the functional

$$a(u,\, v) \;=\; \sum_{T\in\mathcal{T}_h} \int_T \nabla u\cdot\nabla v\,dx, \qquad F(v) \;=\; \int_\Omega f\,v\,dx.$$

Define the global correction function

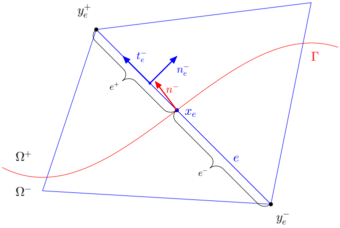
$$w^u \;=\; \begin{cases} 0 & \text{if } T\in\mathcal{T}_h\backslash\mathcal{T}_h^\Gamma, \\ w_T^u & \text{if } T\in\mathcal{T}_h^\Gamma. \end{cases}$$

We defined our scheme as; Find $u_h \in V_h$ such that

$$a(u_h,\, v) \;=\; F(v) - a(w^u,\, v) \qquad \forall v\in V_h.$$

$$\Longrightarrow \quad a(u-u_h,\, v) \;=\; a(w^u,\, v) \quad \forall v\in V_h.$$

Linear case: edge-based correction method



- Let $\mathcal{E}_h^\Gamma$ be the edges intersecting Γ
- $a_e = \boldsymbol{n}^- \cdot \boldsymbol{t}_e^-$, $b_e = \boldsymbol{t}^- \cdot \boldsymbol{t}_e^-$

Find $u_h \in V_h$ such that:

$$\int_\Omega \nabla u_h \cdot \nabla v \, dx = E_h(v) \quad \forall v \in V_h$$

$$E_h(v) = \int_\Omega f\, v dx + \int_\Gamma \beta v ds - \sum_{e \in \mathcal{E}_h^\Gamma} \frac{h_{e-} h_{e+}}{2} a_e \beta(x_e) [\nabla v \cdot \boldsymbol{n}]|_e$$

A priori error estimate

Define Lagrange interpolation operator I_h onto the finite element space V_h

Main result:

Theorem

Suppose that Ω is convex and that the family of meshes $\{\mathcal{T}_h\}_{h>0}$ are quasi-uniform and shape regular, then

$$\|\nabla(I_h u - u_h)\|_{L^\infty(\Omega)} \leq C\, h^k \left(\|u^+\|_{C^{k+1}(\Omega^+)} + \|u^-\|_{C^{k+1}(\Omega^-)}\right),$$

$$\|I_h u - u_h\|_{L^\infty(\Omega)} \leq C h^{k+1} \log(1/h) \left(\|u^+\|_{C^{k+1}(\Omega^+)} + \|u^-\|_{C^{k+1}(\Omega^-)}\right),$$

where $C > 0$ are constant independent of h.

What do we need?

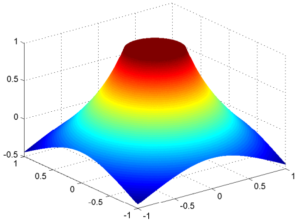
$$
\begin{aligned}
a(I_h u - u_h,\, v) \;&=\; a(I_h u - u,\, v) \,+\, a(u - u_h,\, v) \;=\; a(I_h u - u,\, v) \,+\, a(w^u,\, v) \\
&= \sum_{T \in \mathcal{T}_h \setminus \mathcal{T}_h^\Gamma} \int_T \nabla(I_h u - u) \cdot \nabla v \, dx \\
&\quad + \sum_{T \in \mathcal{T}_h^\Gamma} \int_T \nabla(I_h u + w_T^u - u) \cdot \nabla v \, dx
\end{aligned}
$$

Construction of w_T^u, for $T \in \mathcal{T}_h^\Gamma$, must satisfy

$$
\|\nabla(I_h u + w_T^u - u)\|_{L^\infty(T^\pm)} \;\leq\; C h^k
$$

where $T^\pm \;=\; T \cap \Omega^\pm$

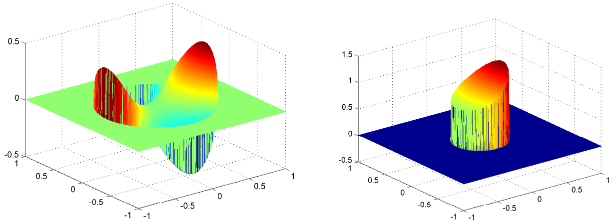
Numerical examples: $\beta \neq 0$ and $\alpha = 0$



h	$\|e_h^N\|_{L^2}$	r	$\|\nabla e_h^N\|_{L^2}$	r	$\|e_h^N\|_{L^\infty}$	r	$\|\nabla e_h^N\|_{L^\infty}$	r
1.8e-1	1.39e-1		4.44e-1		2.53e-1		5.20e-1	
8.8e-2	3.09e-2	2.17	1.72e-1	1.37	6.40e-2	1.98	3.84e-1	0.44
4.4e-2	7.32e-3	2.08	5.75e-2	1.58	1.58e-2	2.02	1.79e-1	1.10
2.2e-2	1.81e-3	2.02	2.18e-2	1.40	4.19e-3	1.91	1.20e-1	0.58
1.1e-2	4.50e-4	2.01	8.57e-3	1.35	8.92e-4	2.23	6.45e-2	0.89
5.5e-3	1.12e-4	2.01	3.57e-3	1.26	2.37e-4	1.91	3.17e-2	1.02
2.8e-3	2.68e-5	2.06	1.55e-3	1.21	6.23e-5	1.93	1.71e-2	0.90
1.4e-3	6.89e-6	1.96	7.68e-4	1.01	1.68e-5	1.90	8.33e-3	1.03

Table: Errors of the approximate solution by EBC-FEI, on a non-structure triangulation.

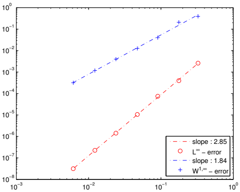
Numerical example: $\beta \neq 0$ and $\alpha \neq 0$



h	$\|e_h^N\|_{L^2}$	r	$\|\nabla e_h^N\|_{L^2}$	r	$\|e_h^N\|_{L^\infty}$	r	$\|\nabla e_h^N\|_{L^\infty}$	r
1.8e-1	9.28e-3		3.27e-2		1.42e-2		4.23e-2	
8.8e-2	5.41e-3	0.78	3.50e-2	-0.10	8.23e-3	0.79	6.61e-2	-0.64
4.4e-2	1.19e-3	2.18	1.18e-2	1.56	2.19e-3	1.91	3.18e-2	1.06
2.2e-2	2.89e-4	2.05	5.06e-3	1.23	7.41e-4	1.56	2.25e-2	0.50
1.1e-2	7.51e-5	1.94	2.42e-3	1.06	1.64e-4	2.17	1.15e-2	0.97
5.5e-3	1.89e-5	1.99	1.18e-3	1.04	4.45e-5	1.88	5.57e-3	1.04
2.8e-3	4.71e-6	2.00	5.74e-4	1.03	1.20e-5	1.89	2.68e-3	1.06
1.4e-3	1.18e-6	2.00	2.86e-4	1.01	3.03e-6	1.98	1.35e-3	0.98

Table: Errors of the approximate solution by EBC-FEI, on a non-structured triangulation.

Numerical examples



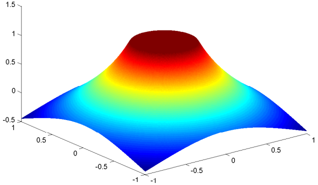


Figure: Plot of the L^∞ and $W^{1,\infty}$ errors (left) and the approximate solution (right) by the method, on non-structured meshes.

h	$\|e_h^N\|_{L^2}$	r	$\|\nabla e_h^N\|_{L^2}$	r	$\|e_h^N\|_{L^\infty}$	r	$\|\nabla e_h^N\|_{L^\infty}$	r
1.8e-1	8.87e-5		3.97e-4		3.80e-3		2.53e-2	
9.0e-2	9.73e-6	3.29	7.46e-5	2.49	9.04e-4	2.14	7.43e-3	1.82
4.7e-2	1.11e-6	3.33	1.06e-5	3.00	2.15e-4	2.21	2.58e-3	1.63
2.4e-2	1.30e-7	3.15	1.42e-6	2.95	5.06e-5	2.13	7.34e-4	1.84
1.2e-2	1.59e-8	3.14	2.24e-7	2.76	1.27e-5	2.07	2.16e-4	1.83
6.1e-3	1.96e-9	3.04	3.15e-8	2.85	3.15e-6	2.02	5.55e-5	1.98

Table: Errors of the approximate solution on a non-structured triangulation.

Numerical example

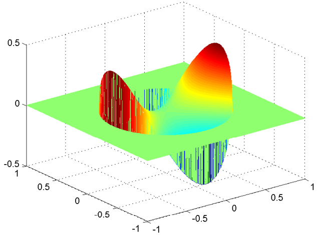


Figure: Plot of the approximate solution by the method, on non-structured triangulations.

h	$\Vert e_h^N \Vert_{L^2}$	r	$\Vert \nabla e_h^N \Vert_{L^2}$	r	$\Vert e_h^N \Vert_{L^\infty}$	r	$\Vert \nabla e_h^N \Vert_{L^\infty}$	r
2.5e-1	7.55e-4		2.19e-3		2.18e-2		1.05e-1	
1.2e-1	5.41e-5	3.80	2.22e-4	3.31	2.56e-3	3.09	1.96e-2	2.42
6.2e-2	4.37e-6	3.63	3.60e-5	2.62	4.83e-4	2.40	5.78e-3	1.76
3.1e-2	4.41e-7	3.31	5.11e-6	2.82	8.11e-5	2.57	1.53e-3	1.92
1.6e-2	3.38e-8	3.70	6.99e-7	3.07	1.45e-5	2.48	4.35e-4	1.90

Table: Errors of the approximate solution on non-structured triangulations.

Stokes Interface Problem

$$
\begin{aligned}
-\Delta \boldsymbol{u} + \nabla p &= \boldsymbol{f}, && \text{in } \Omega \\
\nabla \cdot \boldsymbol{u} &= 0, && \text{in } \Omega \\
\boldsymbol{u} &= 0, && \text{on } \partial\Omega \\
[u] &= \boldsymbol{\alpha}, && \text{on } \Gamma \\
[(\nabla \boldsymbol{u} - p\mathrm{I})\boldsymbol{n}] &= \boldsymbol{\beta}. && \text{on } \Gamma
\end{aligned}
$$

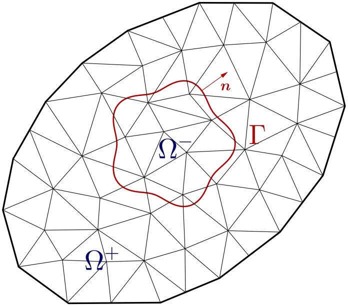


Figure: Domain $\Omega = \Omega^+ \cup \Omega^- \cup \Gamma$. Smooth Interface Γ.

Finite Element Method

Find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times M_h$ such that

$$\begin{aligned}
\int_\Omega \nabla \boldsymbol{u}_h : \nabla \boldsymbol{v}\, dx - \int_\Omega p_h \nabla \cdot \boldsymbol{v} dx &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} dx + \int_\Gamma \boldsymbol{\beta} \cdot \boldsymbol{v} ds \\
&\quad - \sum_{T \in \mathcal{T}_h^\Gamma} \left(\int_T w_T^p \nabla \cdot \boldsymbol{v} dx + \int_T \nabla \boldsymbol{w}_T^{\boldsymbol{u}} : \nabla \boldsymbol{v}\, dx \right) \\
\int_\Omega q \nabla \cdot \boldsymbol{u}_h dx &= - \sum_{T \in \mathcal{T}_h^\Gamma} \int_T q \nabla \cdot \boldsymbol{w}_T^{\boldsymbol{u}} dx
\end{aligned}$$

for all $(\boldsymbol{v}, q) \in \boldsymbol{V}_h \times M_h$.

Corrections

Velocity:

For each $T \in \mathcal{T}_h^\Gamma$, let $\boldsymbol{w}_T^{\boldsymbol{u}} \in S^k(T)$ such that

$$\left[D_{\boldsymbol{\eta}}^{k-\ell} \boldsymbol{w}_T^{\boldsymbol{u}}(x_i^{\ell,T})\right] = \left[D_{\boldsymbol{\eta}}^{k-\ell} \boldsymbol{u}(x_i^{\ell,T})\right] \qquad \text{for } 0 \le i \le \ell \text{ and } 0 \le \ell \le k,$$

$$I_h(\boldsymbol{w}_T^{\boldsymbol{u}}) = 0.$$

Pressure:

For each $T \in \mathcal{T}_h^\Gamma$, let $w_T^p \in S^{k-1}(T)$ such that

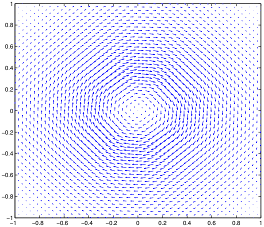
$$\left[D_{\boldsymbol{\eta}}^{k-\ell} w_T^p(x_i^{\ell,T})\right] = \left[D_{\boldsymbol{\eta}}^{k-\ell} p(x_i^{\ell,T})\right] \qquad \text{for } 0 \le i \le \ell \text{ and } 0 \le \ell \le k-1,$$

$$J_h(w_T^p) = 0.$$

Numerical example

Consider an exact solution of Stokes interface problem on $\Omega = (-1,1)^2$

$$\boldsymbol{u}(x,y) = \begin{pmatrix} u_1(x,y) = \begin{cases} 3y & \text{if } r \le 1/3, \\ \frac{4y}{3r} - y & \text{if } r > 1/3 \end{cases} \\ u_2(x,y) = \begin{cases} -3x, & \text{if } r \le 1/3, \\ x - \frac{4x}{3r} & \text{if } r > 1/3 \end{cases} \end{pmatrix}, \qquad p(x,y) = \begin{cases} 4 - \frac{\pi}{9} + 3x^2 - 3y^2 & \text{if } r \le 1/3, \\ \frac{\pi}{9} + 3x^2 - 3y^2 & \text{if } r > 1/3 \end{cases}$$



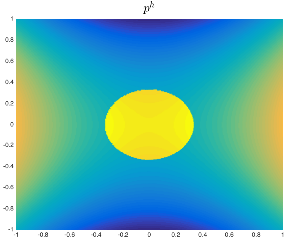


Figure: Plot of velocity (left) and discontinuous pressure (right).

Numerical example

h	$\|e_h^{\boldsymbol{u}}\|_{L^2}$	r	$\|e_h^{\boldsymbol{u}}\|_{L^\infty}$	r	$\|\nabla e_h^{\boldsymbol{u}}\|_{L^2}$	r	$\|\nabla e_h^{\boldsymbol{u}}\|_{L^\infty}$	r
2.5e-1	2.02e-3		3.47e-3		1.01e-1		3.86e-1	
1.3e-1	2.56e-4	2.98	4.23e-4	3.04	2.77e-2	1.87	1.48e-1	1.38
6.3e-2	3.06e-5	3.06	8.62e-5	2.30	7.21e-3	1.94	4.22e-2	1.81
3.1e-2	3.74e-6	3.03	1.14e-5	2.92	1.84e-3	1.97	1.33e-2	1.67
1.6e-2	4.55e-7	3.04	1.67e-6	2.77	4.63e-4	1.99	3.65e-3	1.87

Table: Errors and orders of convergence for velocity, on structured meshes.

h	$\|e_h^p\|_{L^2}$	r	$\|e_h^p\|_{L^\infty}$	r
2.5e-1	2.33e-2		6.32e-2	
1.3e-1	8.50e-3	1.46	3.31e-2	0.93
6.3e-2	2.65e-3	1.68	1.58e-2	1.07
3.1e-2	7.12e-4	1.89	5.09e-3	1.64
1.6e-2	1.84e-4	1.95	1.46e-3	1.80

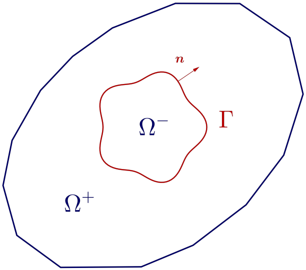
Table: Errors and orders of convergence for pressure, on structured meshes.

Outline

Model Problem

Consider the problem

$$
\begin{aligned}
-\rho^{\pm}\Delta u^{\pm} = f^{\pm} \quad & \text{in } \Omega^{\pm}, \\
u = 0 \quad & \text{on } \partial\Omega, \\
[u] = 0 \quad & \text{on } \Gamma, \\
[\rho D_{\boldsymbol{n}} u] = 0 \quad & \text{on } \Gamma.
\end{aligned}
$$



The jumps across the interface Γ are defined as

$$[\rho D_{\boldsymbol{n}} u] \;=\; \rho^- D_{\boldsymbol{n}^-} u^- + \rho^+ D_{\boldsymbol{n}^+} u^+, \quad [u] \;=\; u^+ - u^-$$

$\boldsymbol{n}^{\pm}$: unit normal vector pointing outwards $\Omega^{\pm}$

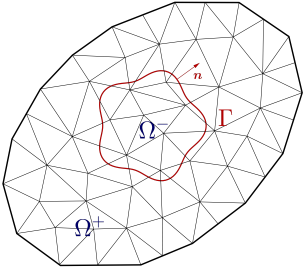
Objective

Develop a piecewise linear FEM, with error estimates independent of contrast $\frac{\rho^+}{\rho^-} \gg 1$

$$\begin{aligned} \|u - u_h\|_V &\leq C\,h\,\|f\|_{L^2(\Omega)} \\ \|u - u_h\|_{L^2(\Omega)} &\leq C\,h^2\,\|f\|_{L^2(\Omega)}. \end{aligned}$$

Difficulties

- Unfitted mesh (independent of the interface)
- Regularity: $u|_{\Omega^\pm} \in H^2(\Omega^\pm)$
- Convergence independent of contrast.



Two approaches

Interface Finite Element Method

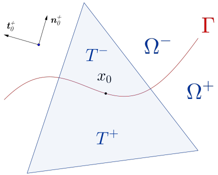
Define local piecewise polynomial finite element spaces on elements intersecting the interface. The basis functions are constructed to having them satisfy the jump conditions across the interface.

- FE version of the Immersed Interface Method of LeVeque and Li (1994)
- Adjerid, S. Ben-Romdhane M and Lin T.
- Lin T., Lin Y. and Zhang X.

Unfitted Nitsche's Method for Interface

Double the degrees of freedom on the elements intersecting the interface and add penalty terms to weakly enforce the continuity across the interface.

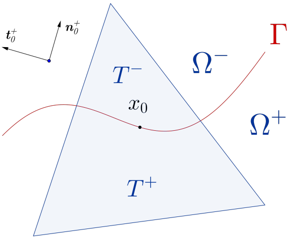
- Interface version of Nitsche's method
- Hansbo and Hansbo (2002), uniftted fem based on Nitsche's method
- Burman and Zunino, Stabilized unfitted Nitsche method.
- Burman and Hansbo, Unfitted Nitsche extended finite element method.



Finite element space

Jump conditions of basis functions:
$v \in \mathbb{P}^1(T^+)$, $\exists!\, \mathsf{E}(v) \in \mathbb{P}^1(T^-)$ satisfying:

$$\begin{aligned} \mathsf{E}(v)(x_0) &:= v(x_0) \\ (D_{\boldsymbol{t}_0^+}\mathsf{E}(v))(x_0) &:= (D_{\boldsymbol{t}_0^+}v)(x_0) \\ \rho^-(D_{\boldsymbol{n}_0^+}\mathsf{E}(v))(x_0) &:= \rho^+(D_{\boldsymbol{n}_0^+}v)(x_0) \end{aligned}$$



Basis functions:

$$G(v) = \begin{cases} v & \text{in } T^+ \\ \mathsf{E}(v) & \text{in } T^- \end{cases}$$

Local space:

$$S^1(T) = \begin{cases} \text{span}\,\{G(v_1), G(v_2), G(v_3)\} & \text{if } T \in \mathcal{T}_h^\Gamma \\ \mathbb{P}^1(T) & \text{if } T \in \mathcal{T}_h \backslash \mathcal{T}_h^\Gamma \end{cases}$$

Finite element space

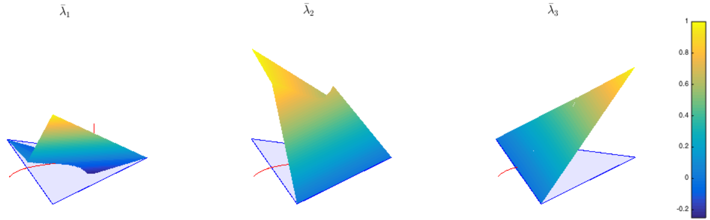
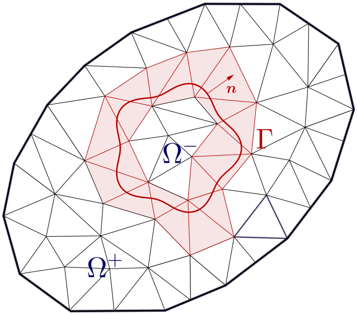


Figure: Illustration of basis function using barycentric coordinates on T^+ and $\rho^- = 1, \rho^+ = 10$.

Global finite element space:

$$V_h := \left\{ v \,:\, v|_T \in S^1(T),\, \forall T \in \mathcal{T}_h,\, v \text{ is continuous across all edges in } \mathcal{E}_h \backslash \mathcal{E}_h^\Gamma \right\}$$

Finite element space



$\mathcal{T}_h^\Gamma$, $\mathcal{E}_h^\Gamma$ –

Global finite element space:

$$V_h := \left\{ v : v|_T \in S^1(T),\ \forall T \in \mathcal{T}_h,\ v \text{ is continuous across all edges in } \mathcal{E}_h \backslash \mathcal{E}_h^\Gamma \right\}$$

Local approximation on $S^1(T)$

Local interpolation operator: Let $u^{\pm} \in H^2(\Omega^{\pm})$ with $[u] = 0$ and $[\rho D_{\boldsymbol{n}} u] = 0$. For each $T \in \mathcal{T}_h^{\Gamma}$, define $I_T u \in S^1(T)$

$$I_T u = \begin{cases} I_T^{+} u & \text{on } T^{E,+} \\ I_T^{-} u & \text{on } T^{E,-} \end{cases}$$

If $T \in \mathcal{T}_h \backslash \mathcal{T}_h^{\Gamma}$ define $I_T(u)|_T$ to be the Scott-Zhang interpolation operator of u on T.

$I_T^{\pm}$ are defined satisfying the following conditions

$$\begin{aligned} (I_T^{-} u)(x_0) &:= (J_T u_E^{+})(x_0) &=: (I_T^{+} u)(x_0) \\ (D_{\boldsymbol{t}_0^{+}} I_T^{-} u)(x_0) &:= (D_{\boldsymbol{t}_0^{+}} (J_T u_E^{+}))(x_0) &=: (D_{\boldsymbol{t}_0^{+}} I_T^{+} u)(x_0) \\ \rho^{-} (D_{\boldsymbol{n}_0^{+}} I_T^{-} u)(x_0) &:= \rho^{-} (D_{\boldsymbol{n}_0^{+}} J_T u_E^{-})(x_0) &=: \rho^{+} (D_{\boldsymbol{n}_0^{+}} I_T^{+} u)(x_0). \end{aligned}$$

J_T: L^2 projection operator onto $\mathbb{P}^1(T^E)$.

Local approximation on $S^1(T)$

We define the interpolation operator I_h onto the finite element space V_h as the restriction of the local interpolation operator I_T, i.e.

$$I_h u|_T = I_T(u)|_T \qquad \text{for all } T \in \mathcal{T}_h.$$

Lemma: The following bounds hold, $j = 0, 1$:

$$h_T^j \|D^j(u - I_T u)\|_{L^2(T^{E,-})} \leq C\, h_T^2 \left(\|Du_E^-\|_{L^2(T^{E,-})} + \|D^2 u_E^-\|_{L^2(T^{E,-})} + \|D^2 u_E^+\|_{L^2(T^{E,+})} \right)$$

and

$$h_T^j \|D^j(u - I_T u)\|_{L^2(T^{E,+})} \leq C\, h_T^2 \Big(\|Du_E^+\|_{L^2(T^{E,+})} + \|D^2 u_E^+\|_{L^2(T^{E,+})} + \frac{\rho^-}{\rho^+} \|D^2 u_E^-\|_{L^2(T^{E,-})} \Big)$$

Finite element method

Find $u_h \in V_h$ such that: $\quad a_h(u_h, v) = (f, v)_\Omega \quad \forall v \in V_h$

$$
\begin{aligned}
a_h(w, v) \; := \; & \int_\Omega \rho \nabla_h w \cdot \nabla_h v - \sum_{e \in \mathcal{E}_h^\Gamma} \int_e \left(\{\rho \nabla_h v\} \cdot [\![w]\!] + \{\rho \nabla_h w\} \cdot [\![v]\!] \right) \\
& + \sum_{e \in \mathcal{E}_h^\Gamma} \left(\frac{\gamma}{|e^-|} \int_{e^-} \rho^- [\![w]\!] \cdot [\![v]\!] + \frac{\gamma}{|e^+|} \int_{e^+} \rho^+ [\![w]\!] \cdot [\![v]\!] \right) \\
& \sum_{e \in \mathcal{E}_h^\Gamma} \left(|e^-| \int_{e^-} \rho^- [\![\nabla_h v]\!] \, [\![\nabla_h w]\!] + |e^+| \int_{e^+} \rho^+ [\![\nabla_h v]\!] \, [\![\nabla_h w]\!] \right)
\end{aligned}
$$

Finite element method

Find $u_h \in V_h$ such that: $\quad a_h(u_h, v) = (f, v)_\Omega \quad \forall v \in V_h$

$$a_h(w,v) := \int_\Omega \rho \nabla_h w \cdot \nabla_h v - \sum_{e \in \mathcal{E}_h^\Gamma} \int_e \left(\{\rho \nabla_h v\} \cdot [\![w]\!] + \{\rho \nabla_h w\} \cdot [\![v]\!] \right)$$

$$+ \sum_{e \in \mathcal{E}_h^\Gamma} \left(\frac{\gamma}{|e^-|} \int_{e^-} \rho^- [\![w]\!] \cdot [\![v]\!] + \frac{\gamma}{|e^+|} \int_{e^+} \rho^+ [\![w]\!] \cdot [\![v]\!] \right)$$

$$\sum_{e \in \mathcal{E}_h^\Gamma} \left(|e^-| \int_{e^-} \rho^- [\![\nabla_h v]\!] \, [\![\nabla_h w]\!] + |e^+| \int_{e^+} \rho^+ [\![\nabla_h v]\!] \, [\![\nabla_h w]\!] \right)$$

Stabilization terms

A priori error estimates: Preliminaries

We prove:

- Coercivity: $c\|v\|_V \ \leq\ a_h(v,v), \quad \forall v \in V_h$
- Continuity: $a_h(w,v)\ \leq\ C\|w\|_W\|v\|_V, \quad \forall w \in H_h^2(\Omega^{\pm}),\ v \in V_h.$
- Interpolation error estimate: $\|I_h u - u\|_W$
- Inconsistency error estimate: $a_h(u-u_h, I_h u - u_h)$

$$\|v\|_V^2 \ = \|\sqrt{\rho}\nabla_h v\|_{L^2(\Omega)}^2 + \sum_{e\in\mathcal{E}_h^\Gamma}\left(\frac{\rho^-}{|e^-|}\|[\![v]\!]\|_{L^2(e^-)}^2 + \frac{\rho^+}{|e^+|}\|[\![v]\!]\|_{L^2(e^+)}^2\right)$$
$$+ \sum_{e\in\mathcal{E}_h^\Gamma}\left(\rho^-|e^-|\|[\![\nabla v]\!]\|_{L^2(e^-)}^2 + \rho^+|e^+|\|[\![\nabla v]\!]\|_{L^2(e^+)}^2\right).$$

$$\|v\|_W^2 \ = \ \|v\|_V^2 + \sum_{T\in\mathcal{E}_h^\Gamma}(\rho^-\|\{\nabla v\}\cdot\boldsymbol{n}\|_{L^2(e^-)}^2|e^-| + \rho^+\|\{\nabla v\}\cdot\boldsymbol{n}\|_{L^2(e^+)}^2|e^+|),$$

A priori error estimates: Energy estimate

Theorem: There exists $C > 0$, independent of h, ρ^- and ρ^+, such that

$$\|u - u_h\|_V \leq C\,h\Big(\sqrt{\rho^-}(||Du||_{L^2(\Omega^-)} + \|D^2u\|_{L^2(\Omega^-)} + \sqrt{\rho^+}(||Du||_{L^2(\Omega^+)} + \|D^2u\|_{L^2(\Omega^+)})\Big)$$

Energy estimate

Regularity estimate (convexity)

Energy error estimate

$$\|u - u_h\|_V \leq \frac{C}{\sqrt{\rho^-}}\|f\|_{L^2(\Omega)}$$

A priori error estimates: Energy estimate

Theorem: There exists $C > 0$, independent of h, ρ^- and ρ^+, such that

$$\|u - u_h\|_V \leq C\,h\Big(\sqrt{\rho^-}\big(\|Du\|_{L^2(\Omega-)} + \|D^2u\|_{L^2(\Omega-)} + \sqrt{\rho^+}\big(\|Du\|_{L^2(\Omega+)} + \|D^2u\|_{L^2(\Omega+)}\big)\Big)$$

Energy estimate

Regularity estimate (convexity)

Energy error estimate

$$\|u - u_h\|_V \leq \frac{C}{\sqrt{\rho^-}}\|f\|_{L^2(\Omega)}$$

A priori error estimates: Energy estimate

Theorem: There exists $C > 0$, independent of h, ρ^- and ρ^+, such that

$$\|u - u_h\|_V \leq C\,h\Big(\sqrt{\rho^-}\big(\,||Du||_{L^2(\Omega-)} + ||D^2u||_{L^2(\Omega-)}\,\big) + \sqrt{\rho^+}\big(\,||Du||_{L^2(\Omega+)} + ||D^2u||_{L^2(\Omega+)}\,\big)\Big)$$

Energy estimate

Regularity estimate (convexity)

Energy error estimate

$$\|u - u_h\|_V \leq \frac{C}{\sqrt{\rho^-}}\|f\|_{L^2(\Omega)}$$

A priori error estimates: Energy estimate

Theorem: There exists $C > 0$, independent of h, ρ^- and ρ^+, such that

$$\|u - u_h\|_V \leq Ch\Big(\sqrt{\rho^-}(\,||Du||_{L^2(\Omega-)} + ||D^2u||_{L^2(\Omega-)}\,)+ \sqrt{\rho^+}(\,||Du||_{L^2(\Omega+)} + ||D^2u||_{L^2(\Omega+)}\,)\Big)$$

Energy estimate | Regularity estimate (convexity)

Energy error estimate

$$\|u - u_h\|_V \leq \frac{C}{\sqrt{\rho^-}}\|f\|_{L^2(\Omega)}$$

A priori error estimates: L^2 estimate

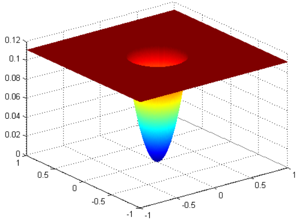
Theorem: Assuming Ω is convex, then it holds

$$\|u - u_h\|_{L^2(\Omega)} \;\leq\; \frac{C\,h^2}{\rho^-}\|f\|_{L^2(\Omega)}.$$

Numerical example

Exact solution: $\Omega = (-1,1)^2 \quad \Gamma = \{\boldsymbol{x} \in \Omega : \boldsymbol{x}_1^2 + \boldsymbol{x}_2^2 = R^2\}$
$\Omega^- := \{\boldsymbol{x} \in \Omega : \boldsymbol{x}_1^2 + \boldsymbol{x}_2^2 < R^2\}$

$$u(\boldsymbol{x}) = \begin{cases} \frac{r^\alpha}{\rho^-} & \text{if } \boldsymbol{x} \in \Omega^- \\ \frac{r^\alpha}{\rho^+} + R^\alpha\left(\frac{1}{\rho^-} - \frac{1}{\rho^+}\right) & \text{if } \boldsymbol{x} \in \Omega^+ \end{cases}$$



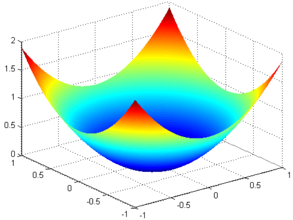


Figure: Approximate solution, case $\Gamma = \partial\Omega^-$ and for case $\Gamma = \partial\Omega^+$ (right).

Numerical example

Errors:

$$e_h^0 := \|u_h - u\|_{L^2(\Omega)} \qquad e_h^\infty := \|u_h - u\|_{L^\infty(\Omega)}$$

$$e_h^1 := \|\sqrt{\rho}(\nabla u_h - \nabla u)\|_{L^2(\Omega)} \qquad e_h^{1,\infty} := \|\sqrt{\rho}(\nabla u_h - \nabla u)\|_{L^\infty(\Omega)}$$

l	e_h^0	r	e_h^∞	r	e_h^1	r	$e_h^{1,\infty}$	r
1	8.2e-3		2.5e-2		1.1e-1		3.7e-1	
2	1.7e-3	2.28	5.7e-3	2.12	4.4e-2	1.30	2.1e-1	0.86
3	2.7e-4	2.63	1.3e-3	2.19	1.8e-2	1.29	9.7e-2	1.08
4	4.6e-5	2.57	3.2e-4	1.97	8.3e-3	1.12	5.2e-2	0.90
5	9.0e-6	2.34	7.2e-5	2.15	3.9e-3	1.07	2.5e-2	1.08
6	2.0e-6	2.19	1.8e-5	2.01	1.9e-3	1.03	1.3e-2	0.92
7	4.7e-7	2.08	4.6e-6	1.94	9.5e-4	1.02	6.8e-3	0.94

Table: Errors and convergence orders with $\rho^- = 1$ and $\rho^+ = 10^4$, stabilization parameters $\gamma = 10$ and $\gamma_F = 10$, and $h = 2^{-(l+3/2)}$.

Numerical example

Errors:

$$\bar{e}_h^1 := \|\rho(\nabla u_h - \nabla u)\|_{L^2(\Omega)} \qquad \bar{e}_h^{1,\infty} := \|\rho(\nabla u_h - \nabla u)\|_{L^\infty(\Omega)}$$

$$e_h^{\boldsymbol{n},\infty} := \|\rho(D_{\boldsymbol{n}} u_h - D_{\boldsymbol{n}} u)\|_{L^\infty(\Gamma)} \qquad \tilde{e}_h^1 := \|\rho(\nabla u_h - \nabla u)\|_{L^2(\Omega)\setminus\mathcal{T}_h^\Gamma}$$

l	$\bar{e}_h^1$	r	$\bar{e}_h^{1,\infty}$	r	$\tilde{e}_h^{1,\infty}$	r	$e_h^{\boldsymbol{n},\infty}$	r
1	3.9e-1		7.0e-1		7.0e-1		3.7e-1	
2	1.6e-1	1.32	6.1e-1	0.19	4.0e-1	0.81	2.1e-1	0.86
3	6.4e-2	1.29	1.9e-1	1.68	1.9e-1	1.07	9.7e-2	1.08
4	2.9e-2	1.15	2.2e-1	-0.20	1.0e-1	0.91	4.9e-2	1.00
5	1.4e-2	1.09	1.4e-1	0.65	5.0e-2	1.01	2.5e-2	0.98
6	6.6e-3	1.04	6.6e-2	1.09	2.5e-2	0.99	1.2e-2	1.00
7	3.2e-3	1.02	5.1e-1	-2.95	1.3e-2	0.98	6.2e-3	1.00

Table: Errors and convergence orders with $\rho^- = 1$ and $\rho^+ = 10^4$, stabilization parameters $\gamma = 10$ and $\gamma_F = 10$.

Numerical example

Errors:

$$e_h^0 := \|u_h - u\|_{L^2(\Omega)}, \qquad e_h^1 := \|\sqrt{\rho}(\nabla u_h - \nabla u)\|_{L^2(\Omega)},$$
$$\bar{e}_h^1 := \|\rho(\nabla u_h - \nabla u)\|_{L^2(\Omega)},$$

ρ^+	e_h^0	$\bar{e}_h^{1,\infty}$	e_h^1
10^1	2.3e-6	6.5e-3	2.8e-3
10^2	2.0e-6	6.6e-3	2.0e-3
10^3	2.0e-6	6.6e-3	1.9e-3
10^4	2.0e-6	6.6e-3	1.9e-3
10^5	2.0e-6	6.6e-3	1.9e-3
10^6	2.0e-6	6.6e-3	1.9e-3

Table: Errors with $\rho^- = 1$ and $h = 2^{-(6+3/2)}$, stabilization parameters $\gamma = 10$ and $\gamma_F = 10$.

Finite element space

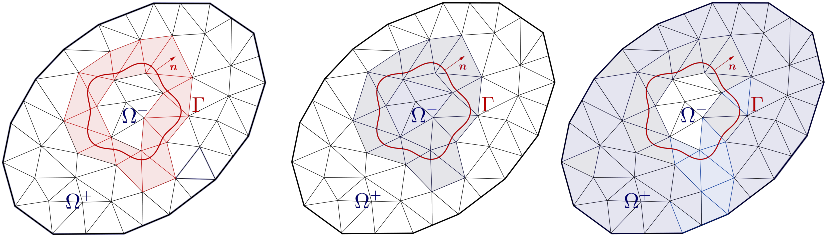


Figure: Illustration of definition of set $\mathcal{T}_h^\Gamma, \Omega_h^\pm$. Left figure: elements in $\mathcal{T}_h^\Gamma$ (red transparent). Center figure: set Ω_h^- (blue transparent). Right figure: set Ω_h^+ (blue transparent.

$$V_h^\pm = \{v \in \mathcal{C}(\Omega_h^\pm) : v|_T \in \mathbb{P}^1(T), \forall T \in \mathcal{T}_h^\pm \text{ and } v|_{\partial\Omega\cap\partial\Omega^\pm} \equiv 0\}.$$
$$V_h = V_h^- \times V_h^+.$$

Finite element method

Find $u_h = (u_h^-, u_h^+) \in V_h$, such that:

$$a_h(u_h, v) = (f, v), \qquad \text{for all } v \in V_h$$

$$
\begin{aligned}
a_h(w, v) = &\int_{\Omega^+} \rho^+ \nabla w^+ \cdot \nabla v^+ dx + \int_{\Omega^-} \rho^- \nabla w^- \cdot \nabla v^- dx \\
&+ \sum_{T \in \mathcal{T}_h^\Gamma} \int_{T_\Gamma} \left(\rho^- \nabla v^- \cdot \boldsymbol{n}^- [w] + \rho^- \nabla w^- \cdot \boldsymbol{n}^- [v] + \frac{\gamma}{h_T} \rho^- [w] \cdot [v] \right) ds \\
&+ \gamma_g^- \sum_{e \in \mathcal{E}_h^{\Gamma,-}} |e| \int_e \rho^- [\![\nabla v^-]\!] [\![\nabla w^-]\!] \, ds + \gamma_g^+ \sum_{e \in \mathcal{E}_h^{\Gamma,+}} |e| \int_e \rho^+ [\![\nabla v^+]\!] [\![\nabla w^+]\!] \, ds.
\end{aligned}
$$

γ, γ_g^-, and γ_g^+ are positive parameters to be chosen.

A priori error estimate

Define the energy norm $\|\cdot\|_V$

$$\|v\|_V^2 = \|\sqrt{\rho}\nabla v\|^2_{L^2(\Omega)} + \sum_{T\in\mathcal{T}_h^\Gamma} \frac{1}{h_T}\|\sqrt{\rho^-}[v]\|^2_{L^2(T_\Gamma)} + \sum_{e\in\mathcal{E}_h^{\Gamma,-}} |e|\|\sqrt{\rho^-}[\![\nabla v^-]\!]\|^2_{L^2(e)} + \sum_{e\in\mathcal{E}_h^{\Gamma,+}} |e|\|\sqrt{\rho^+}[\![\nabla v^+]\!]\|^2_{L^2(e)},$$

for $v = (v^-, v^+) \in H_h^2(\Omega_h^-) \times H_h^2(\Omega_h^+)$.

We prove:

- Coercivity (independent of contrast): $c\|v\|_V \leq a_h(v,v), \quad \forall v \in V_h$
- Continuity: $a_h(w,v) \leq C\|w\|_W\|v\|_V, \quad \forall w \in H_h^2(\Omega^\pm),\ v \in V_h.$
- Galerkin orthogonality: $a_h(u-u_h, v) = 0, \quad \forall v \in V_h$
- Interpolation error estimate:

$$\|u - I_h u\|_{V_A} \leq Ch(\sqrt{\rho^+}\|D^2u\|_{L^2(\Omega+)} + \sqrt{\rho^-}\|D^2u\|_{L^2(\Omega-)})$$

Energy estimate

Theorem: Suppose that Ω is convex. Then, there exist $C > 0$ independent of $\rho^{\pm}$, such that

$$\|u - u_h\|_V \leq \frac{C\,h}{\sqrt{\rho^-}}\|f\|_{L^2(\Omega)}.$$

Flux error estimate

Theorem: (Discrete Extension) Assume that $\mathcal{T}_h$ is quasi-uniform. Let $v_h \in V_h^+$. Then, there exists a function $Ev_h \in V_h^c = \{v \in C_0(\Omega) : v|_T \in \mathbb{P}^1(T) : \forall T \in \mathcal{T}_h\}$ such that $Ev_h = v_h$ in Ω_h^+, and

$$\|Ev_h\|_{H^1(\Omega)} \leq C\|v_h\|_{H^1(\Omega_h^+)},$$

with $C > 0$ independent of h.

Theorem: Assume that Ω is convex and the triangulation is quasi-uniform. Then, there exists a constant $C > 0$, independent of h and $\rho^{\pm}$, such that

$$\|\rho\nabla(u-u_h)\|^2_{L^2(\Omega)} = \|\rho\nabla(u-u_h)^-\|^2_{L^2(\Omega^-)} + \|\rho\nabla(u-u_h)^+\|^2_{L^2(\Omega^+)} \leq Ch\|f\|_{L^2(\Omega)}.$$

Flux error estimate

Theorem: (Discrete Extension) Assume that $\mathcal{T}_h$ is quasi-uniform. Let $v_h \in V_h^+$. Then, there exists a function $Ev_h \in V_h^c = \{v \in C_0(\Omega) : v|_T \in \mathbb{P}^1(T) : \forall T \in \mathcal{T}_h\}$ such that $Ev_h = v_h$ in Ω_h^+, and

$$\|Ev_h\|_{H^1(\Omega)} \leq C\|v_h\|_{H^1(\Omega_h^+)},$$

with $C > 0$ independent of h.

Theorem: Assume that Ω is convex and the triangulation is quasi-uniform. Then, there exists a constant $C > 0$, independent of h and $\rho^\pm$, such that

$$\|\rho\nabla(u-u_h)\|^2_{L^2(\Omega)} = \|\rho\nabla(u-u_h)^-\|^2_{L^2(\Omega-)} + \|\rho\nabla(u-u_h)^+\|^2_{L^2(\Omega+)} \leq Ch\|f\|_{L^2(\Omega)}.$$

Numerical example

Exact solution: $\Omega = (-1,1)^2 \quad \Gamma = \{\boldsymbol{x} \in \Omega : \boldsymbol{x}_1^2 + \boldsymbol{x}_2^2 = R^2\}$
$\Omega^- := \{\boldsymbol{x} \in \Omega : \boldsymbol{x}_1^2 + \boldsymbol{x}_2^2 < R^2\}$

$$u(\boldsymbol{x}) = \begin{cases} \frac{r^\alpha}{\rho^-} & \text{if } \boldsymbol{x} \in \Omega^- \\ \frac{r^\alpha}{\rho^+} + R^\alpha\left(\frac{1}{\rho^-} - \frac{1}{\rho^+}\right) & \text{if } \boldsymbol{x} \in \Omega^+ \end{cases}$$

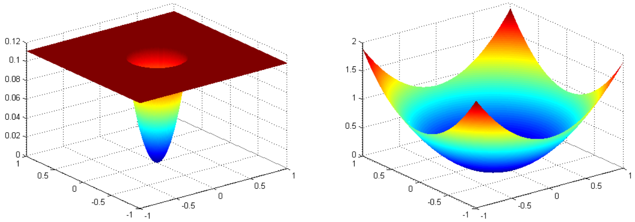


Figure: Approximate solution, case $\Gamma = \partial\Omega^-$ and for case $\Gamma = \partial\Omega^+$ (right).

Numerical example

Errors:

$$e_h^0 := \|u_h - u\|_{L^2(\Omega)} \qquad e_h^\infty := \|u_h - u\|_{L^\infty(\Omega)}$$

$$e_h^1 := \|\rho(\nabla u_h - \nabla u)\|_{L^2(\Omega)} \qquad e_h^{1,\infty} := \|\rho(\nabla u_h - \nabla u)\|_{L^\infty(\Omega)}$$

l	e_h^0	r	e_h^∞	r	e_h^1	r	$e_h^{1,\infty}$	r
1	1.9e-02		5.5e-02		3.7e-01		5.9e-01	
2	6.2e-03	1.64	1.5e-02	1.86	1.4e-01	1.38	2.7e-01	1.11
3	1.1e-03	2.46	2.8e-03	2.45	5.6e-02	1.31	1.2e-01	1.17
4	1.7e-04	2.69	5.0e-04	2.48	2.6e-02	1.11	5.4e-02	1.17
5	2.8e-05	2.66	9.8e-05	2.34	1.3e-02	1.03	2.4e-02	1.14
6	4.6e-06	2.59	1.9e-05	2.39	6.4e-03	1.01	1.2e-02	1.03
7	8.4e-07	2.45	4.2e-06	2.16	3.2e-03	1.00	6.0e-03	1.01

Table: Errors and orders of convergence with $\rho^- = 1$, $\rho^+ = 10^4$, stabilization parameters $\gamma = 10$ and $\gamma_G^\pm = 10$.

Numerical example $\beta \neq 0$

Smooth exact solution $u(\boldsymbol{x}) = \frac{(x_1^2+x_2^2)^{5/2}}{\rho^-}$ for $\boldsymbol{x} \in \Omega = (-1,1)^2$, and $\Gamma = \{\boldsymbol{x} : x_1^2 + x_2^2 = (1/3)^2\}$. Consider Ω^- as the inclusion. Let $\rho^+ = 10^4$ and $\rho^- = 1$. Note that the jump of the flux in nonhomogeneous. Stabilization parameters: $\gamma = 10$, $\gamma_g^+ = 10$ and $\gamma_g^- = 10$.

l	e_h^0	r	e_h^∞	r	e_h^1	r	$e_h^{1,\infty}$	r
1	4.2e-02		6.7e-02		7.9e+03		2.8e+04	
2	1.1e-02	2.00	1.8e-02	1.90	3.9e+03	1.00	1.4e+04	0.94
3	2.6e-03	2.00	4.6e-03	1.95	2.0e+03	1.00	7.4e+03	0.97
4	8.4e-04	3.30	2.3e-03	2.02	1.5e+03	0.74	6.8e+03	0.26
5	2.1e-04	2.00	6.0e-04	1.95	7.6e+02	1.00	3.4e+03	0.98

Table: $\Omega = (-1,1)^2$, $\Gamma = \{\mathbf{x} \in \Omega : x_1^2 + x_2^2 = (1/3)^2\}$, $\rho^- = 1$, $\rho^+ = 10^4$.

Outline

Summary

Immersed Interface approach:

- Piecewise linear FEM for an interface problem
- Optimal energy error estimate independent of contrast
- Optimal L^2 error estimate independent of contrast
- Extension to 3D

Nitsche's approach:

- Piecewise linear FEM for an interface problem
- Optimal energy error estimate independent of contrast
- Optimal L^2 error estimate independent of contrast
- Flux error estimate independent of contrast

Future projects

- Flux error estimate for interface finite element method
- 3D extension of Nitsche's finite element method
- Non-homogeneous jump conditions
- Higher-order method for interface problems
- Stokes interface problem
- Moving interface problems: Navier-Stokes equations

Future projects

- Flux error estimate for interface finite element method
- 3D extension of Nitsche's finite element method
- Non-homogeneous jump conditions
- Higher-order method for interface problems
- Stokes interface problem
- Moving interface problems: Navier-Stokes equations

Thanks for your attention!

References

Johnny Guzmán, Manuel A. Sánchez, and Marcus Sarkis.
On the accuracy of finite element approximations to a class of interface problems.
Math. Comp., In press.
(to appear).

Johnny Guzmán, Manuel A. Sánchez, and Marcus Sarkis.
Higher-order finite element methods for elliptic problems with interfaces.
Mathematical Modelling and Numerical Analysis, accepted. Arxiv preprint arXiv:1505.04347, November 2014.

Johnny Guzmán, Manuel A. Sánchez, and Marcus Sarkis.
A finite element method for high-contrast interface problems with error estimates independent of contrast.
Submitted. ArXiv preprint arXiv:1507.03873, July 2015.

E. Burman, J. Guzmán, M. A. Sánchez, and M. Sarkis.
Robust flux error estimation of Nitsche's method for high contrast interface problems.
Submitted. ArXiv e-prints, February 2016.