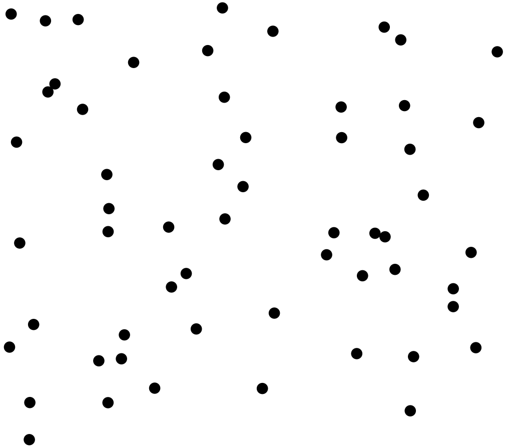
Limit theorems for edge length statistics of random geometric graphs

Matthias Schulte (University of Bern)

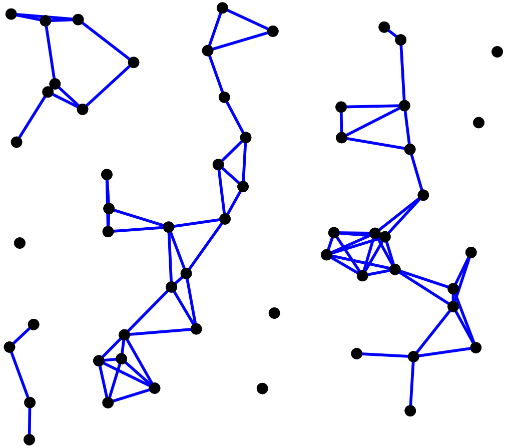
joint work with Laurent Decreusefond (Paris), Matthias Reitzner (Osnabrück) and Christoph Thäle (Bochum)

November 10, 2016

Random geometric graph



Random geometric graph



Framework

- η_t homogeneous Poisson process of intensity $t \geq 1$ on a compact convex set $W \subset \mathbb{R}^d$ with $\text{Vol}(W) = 1$

Framework

- η_t homogeneous Poisson process of intensity $t \geq 1$ on a compact convex set $W \subset \mathbb{R}^d$ with $\text{Vol}(W) = 1$
- For $\theta_t > 0$ construct the random geometric graph $G(\eta_t, \theta_t)$ by connecting two points $x_1, x_2 \in \eta_t$ by an edge if $\|x_1 - x_2\| \leq \theta_t$.

Framework

- η_t homogeneous Poisson process of intensity $t \geq 1$ on a compact convex set $W \subset \mathbb{R}^d$ with $\text{Vol}(W) = 1$
- For $\theta_t > 0$ construct the random geometric graph $G(\eta_t, \theta_t)$ by connecting two points $x_1, x_2 \in \eta_t$ by an edge if $\|x_1 - x_2\| \leq \theta_t$.
- $(\theta_t)_{t \geq 1}$ family of positive real numbers with $\theta_t \to 0$ as $t \to \infty$

Framework

- η_t homogeneous Poisson process of intensity $t \geq 1$ on a compact convex set $W \subset \mathbb{R}^d$ with $\mathrm{Vol}(W) = 1$
- For $\theta_t > 0$ construct the random geometric graph $G(\eta_t, \theta_t)$ by connecting two points $x_1, x_2 \in \eta_t$ by an edge if $\|x_1 - x_2\| \leq \theta_t$.
- $(\theta_t)_{t \geq 1}$ family of positive real numbers with $\theta_t \to 0$ as $t \to \infty$
- Throughout this talk, we consider the functionals

$$L_t^{(\tau)} = \frac{1}{2} \sum_{(x_1,x_2) \in \eta_{t,\neq}^2} \mathbf{1}\{\|x_1 - x_2\| \leq \theta_t\} \, \|x_1 - x_2\|^\tau, \quad t \geq 1, \quad \tau \in \mathbb{R}.$$

Expectation

$B^d(x,r)$ ball with centre x and radius $r>0$ in $\mathbb{R}^d$, $\kappa_d := \mathrm{Vol}(B^d(0,1))$

Theorem: Reitzner/S./Thäle (2016+)

Let $g_W(y) := \mathrm{Vol}(W \cap (W+y))$, $y \in \mathbb{R}^d$. For $\tau > -d$,

$$\mathbb{E} L_t^{(\tau)} = \frac{t^2}{2} \int_{B^d(0,\theta_t)} \|y\|^\tau g_W(y) \, \mathrm{d}y = \frac{d\kappa_d}{2(\tau+d)} t^2 \theta_t^{\tau+d} (1 + O(\theta_t)).$$

Expectation

$B^d(x,r)$ ball with centre x and radius $r>0$ in $\mathbb{R}^d$, $\kappa_d := \mathrm{Vol}(B^d(0,1))$

Theorem: Reitzner/S./Thäle (2016+)

Let $g_W(y) := \mathrm{Vol}(W \cap (W+y))$, $y \in \mathbb{R}^d$. For $\tau > -d$,

$$\mathbb{E}L_t^{(\tau)} = \frac{t^2}{2} \int_{B^d(0,\theta_t)} \|y\|^\tau g_W(y) \, \mathrm{d}y = \frac{d\kappa_d}{2(\tau+d)} t^2 \theta_t^{\tau+d}(1+O(\theta_t)).$$

There are two asymptotic regimes:

- $\lim_{t\to\infty} t^2\theta_t^d = \lambda \in [0,\infty)$: $\quad \lim_{t\to\infty} \mathbb{E}L_t^{(0)} = \frac{\kappa_d}{2}\lambda$
- $\lim_{t\to\infty} t^2\theta_t^d = \infty$: $\quad \lim_{t\to\infty} \mathbb{E}L_t^{(0)} = \infty$

Covariances

Theorem: Reitzner/S./Thäle (2016+)

For $\tau_1, \tau_2 > -d$ such that $\tau_1 + \tau_2 > -d$,

$$\operatorname{Cov}(L_t^{(\tau_1)}, L_t^{(\tau_2)}) = \left(\frac{d\kappa_d}{2(\tau_1 + \tau_2 + d)} t^2 \theta_t^{\tau_1+\tau_2+d} + \frac{d^2\kappa_d^2}{(\tau_1 + d)(\tau_2 + d)} t^3 \theta_t^{\tau_1+\tau_2+2d} \right)(1 + O(\theta_t)).$$

Covariances

Theorem: Reitzner/S./Thäle (2016+)

For $\tau_1, \tau_2 > -d$ such that $\tau_1 + \tau_2 > -d$,

$$\mathrm{Cov}(L_t^{(\tau_1)}, L_t^{(\tau_2)})$$
$$= \left(\frac{d\kappa_d}{2(\tau_1+\tau_2+d)} t^2 \theta_t^{\tau_1+\tau_2+d} + \frac{d^2\kappa_d^2}{(\tau_1+d)(\tau_2+d)} t^3 \theta_t^{\tau_1+\tau_2+2d} \right)(1+O(\theta_t)).$$

Remark:

The typical vertex has the expected degree $\kappa_d t\theta_t^d$.

- $\lim_{t\to\infty} t\theta_t^d = 0$ sparse regime
- $\lim_{t\to\infty} t\theta_t^d = c \in (0,\infty)$ thermodynamic regime
- $\lim_{t\to\infty} t\theta_t^d = \infty$ dense regime

Asymptotic covariances

Theorem: Reitzner/S./Thäle (2016+)

For distinct $\tau_1, \dots, \tau_m > -d/2$ define

$$\tilde{L}_t^{(\tau_i)} = (L_t^{(\tau_i)} - \mathbb{E} L_t^{(\tau_i)}) / \max\{t\theta_t^{\tau_i + d/2}, t^{3/2}\theta_t^{\tau_i + d}\}, \quad i \in \{1, \dots, m\}.$$

Then $(\tilde{L}_t^{(\tau_1)}, \dots, \tilde{L}_t^{(\tau_m)})$ has the asymptotic covariance matrix

$$\Sigma := \begin{cases} \Sigma_1, & \lim_{t\to\infty} t\theta_t^d = 0 \\ \Sigma_1 + c\Sigma_2, & \lim_{t\to\infty} t\theta_t^d = c \in (0,1] \\ \frac{1}{c}\Sigma_1 + \Sigma_2, & \lim_{t\to\infty} t\theta_t^d = c \in (1,\infty) \\ \Sigma_2, & \lim_{t\to\infty} t\theta_t^d = \infty \end{cases}$$

with

$$\Sigma_1 = \left(\frac{d\kappa_d}{2(\tau_i + \tau_j + d)} \right)_{i,j=1}^m \text{ and } \Sigma_2 = \left(\frac{d^2\kappa_d^2}{\tau_i + d)(\tau_j + d)} \right)_{i,j=1}^m.$$

For $m \geq 2$, Σ_1 is positive definite and Σ_2 is singular.

Compound Poisson approximation

Theorem: Decreusefond/S./Thäle (2016)

Assume that $\lim_{t\to\infty} t^2\theta_t^d := \lambda \in [0,\infty)$ and let $\tau \in \mathbb{R}$. Then,

$$t^{2\tau/d} L_t^{(\tau)} \xrightarrow{d} \sum_{i=1}^{N} \|X_i\|^\tau =: Z \quad \text{as} \quad t \to \infty$$

with independent $N \sim \text{Poisson}(\kappa_d \lambda/2)$ and $X_i \sim \text{Uniform}(B^d(0,\lambda^{1/d}))$, $i \in \mathbb{N}$. In particular, there is a constant $C > 0$ depending on W and $\sup_{t\geq 1} t^2\theta_t^d$ such that

$$d_{TV}(t^{2\tau/d} L_t^{(\tau)}, Z) \leq C(|t^2\theta_t^d - \lambda| + t^{-\min\{2/d,1\}}), \quad t \geq 1.$$

The total variation distance of two random variables X, Y is

$$d_{TV}(X,Y) := \sup_{A\in\mathcal{B}(\mathbb{R})} |\mathbb{P}(X\in A) - \mathbb{P}(Y \in A)|.$$

Central limit theorems

Theorem: Reitzner/S./Thäle (2016+)

Assume that $\lim_{t\to\infty} t^2\theta_t^d = \infty$ and let N be a standard Gaussian r.v.

Central limit theorems

Theorem: Reitzner/S./Thäle (2016+)

Assume that $\lim_{t\to\infty} t^2\theta_t^d = \infty$ and let N be a standard Gaussian r.v.

- If $\tau > -d/2$,

$$\frac{L_t^{(\tau)} - \mathbb{E}L_t^{(\tau)}}{\sqrt{\operatorname{Var} L_t^{(\tau)}}} \xrightarrow{d} N, \quad \text{as} \quad t \to \infty.$$

Central limit theorems

Theorem: Reitzner/S./Thäle (2016+)

Assume that $\lim_{t\to\infty} t^2\theta_t^d = \infty$ and let N be a standard Gaussian r.v.

- If $\tau > -d/2$,
$$\frac{L_t^{(\tau)} - \mathbb{E}L_t^{(\tau)}}{\sqrt{\operatorname{Var} L_t^{(\tau)}}} \xrightarrow{d} N, \quad \text{as} \quad t \to \infty.$$
- If $\tau = -d/2$,
$$\frac{L_t^{(\tau)} - \mathbb{E}L_t^{(\tau)}}{\sqrt{d\kappa_d t^2 \ln(t^{2/d}\theta_t)/2 + 4\kappa_d^2 t^3 \theta_t^d}} \xrightarrow{d} N, \quad \text{as} \quad t \to \infty.$$

Central limit theorems

Theorem: Reitzner/S./Thäle (2016+)

Assume that $\lim_{t\to\infty} t^2\theta_t^d = \infty$ and let N be a standard Gaussian r.v.

- If $\tau > -d/2$,
$$\frac{L_t^{(\tau)} - \mathbb{E}L_t^{(\tau)}}{\sqrt{\operatorname{Var} L_t^{(\tau)}}} \xrightarrow{d} N, \quad \text{as} \quad t\to\infty.$$
- If $\tau = -d/2$,
$$\frac{L_t^{(\tau)} - \mathbb{E}L_t^{(\tau)}}{\sqrt{d\kappa_d t^2 \ln(t^{2/d}\theta_t)/2 + 4\kappa_d^2 t^3 \theta_t^d}} \xrightarrow{d} N, \quad \text{as} \quad t\to\infty.$$
- If $\tau \in (-d, -d/2)$ and $\lim_{t\to\infty} t^{3+4\tau/d}\theta_t^{2(\tau+d)} = \infty$,
$$\frac{L_t^{(\tau)} - \mathbb{E}L_t^{(\tau)}}{d\kappa_d t^{3/2}\theta_t^{\tau+d}/(\tau+d)} \xrightarrow{d} N, \quad \text{as} \quad t\to\infty.$$

Quantitative central limit theorems

For two random variables X, Y define

$$d_K(X,Y) := \sup_{u\in\mathbb{R}} |\mathbb{P}(X \leq u) - \mathbb{P}(Y \leq u)|.$$

Theorem: Reitzner/S./Thäle (2016+)

Let $\tau > -d/4$ and let N be a standard Gaussian random variable. Then there is a constant $C > 0$ depending on τ and W such that

$$d_K\left(\frac{L_t^{(\tau)} - \mathbb{E}L_t^{(\tau)}}{\sqrt{\operatorname{Var} L_t^{(\tau)}}}, N\right) \leq Ct^{-1/2} \max\{1, (t\theta_t^d)^{-1/2}\}, \quad t \geq 1.$$

Stable limit theorems

Theorem: Decreusefond/S./Thäle (2016), Reitzner/S./Thäle (2016+)

Let $\lim_{t\to\infty} t^2\theta_t^d = \infty$ and ζ be unit-intensity Poisson process on $\mathbb{R}^+$.

Stable limit theorems

Theorem: Decreusefond/S./Thäle (2016), Reitzner/S./Thäle (2016+)

Let $\lim_{t\to\infty} t^2\theta_t^d = \infty$ and ζ be unit-intensity Poisson process on $\mathbb{R}^+$.

- If $\tau \in (-d, -d/2)$ and $\lim_{t\to\infty} t^{3+4\tau/d}\theta_t^{2(\tau+d)} = 0$,

$$t^{2\tau/d}(L_t^{(\tau)} - \mathbb{E}L_t^{(\tau)}) \xrightarrow{d} (2/\kappa_d)^{\tau/d} \lim_{a\to\infty} \sum_{x\in\zeta\cap[0,a]} x^{\tau/d} - \frac{a^{1+\tau/d}}{1+\tau/d}.$$

Stable limit theorems

Theorem: Decreusefond/S./Thäle (2016), Reitzner/S./Thäle (2016+)

Let $\lim_{t\to\infty} t^2\theta_t^d = \infty$ and ζ be unit-intensity Poisson process on $\mathbb{R}^+$.

- If $\tau \in (-d, -d/2)$ and $\lim_{t\to\infty} t^{3+4\tau/d}\theta_t^{2(\tau+d)} = 0$,

$$t^{2\tau/d}(L_t^{(\tau)} - \mathbb{E}L_t^{(\tau)}) \xrightarrow{d} (2/\kappa_d)^{\tau/d} \lim_{a\to\infty} \sum_{x\in\zeta\cap[0,a]} x^{\tau/d} - \frac{a^{1+\tau/d}}{1+\tau/d}.$$

- If $\tau = -d$,

$$t^{-2}(L_t^{(-d)} - \mathbb{E}\frac{1}{2}\sum_{(x_1,x_2)\in\eta_{t,\neq}^2} \mathbf{1}\{\left(\frac{2}{\kappa_d t^2}\right)^{1/d} \leq \|x_1 - x_2\| \leq \delta_t\}\|x_1 - x_2\|^{-d})$$

$$\xrightarrow{d} \frac{\kappa_d}{2} \lim_{a\to\infty} \sum_{x\in\zeta\cap[0,a]} x^{-1} - \log(a).$$

Stable limit theorems

Theorem: Decreusefond/S./Thäle (2016), Reitzner/S./Thäle (2016+)

Let $\lim_{t\to\infty} t^2\theta_t^d = \infty$ and ζ be unit-intensity Poisson process on $\mathbb{R}^+$.

- If $\tau \in (-d, -d/2)$ and $\lim_{t\to\infty} t^{3+4\tau/d}\theta_t^{2(\tau+d)} = 0$,

$$t^{2\tau/d}(L_t^{(\tau)} - \mathbb{E}L_t^{(\tau)}) \xrightarrow{d} (2/\kappa_d)^{\tau/d} \lim_{a\to\infty} \sum_{x\in\zeta\cap[0,a]} x^{\tau/d} - \frac{a^{1+\tau/d}}{1+\tau/d}.$$

- If $\tau = -d$,

$$t^{-2}(L_t^{(-d)} - \mathbb{E}\frac{1}{2} \sum_{(x_1,x_2)\in\eta_{t,\neq}^2} \mathbf{1}\{\left(\frac{2}{\kappa_d t^2}\right)^{1/d} \leq \|x_1 - x_2\| \leq \delta_t\}\|x_1 - x_2\|^{-d})$$

$$\xrightarrow{d} \frac{\kappa_d}{2} \lim_{a\to\infty} \sum_{x\in\zeta\cap[0,a]} x^{-1} - \log(a).$$

- If $\tau < -d$,

$$t^{2\tau/d} L_t^{(\tau)} \xrightarrow{d} (2/\kappa_d)^{\tau/d} \sum_{x\in\zeta} x^{\tau/d}.$$

Summary

For $\lim_{t\to\infty} t^2\theta_t^d = \infty$ and $\gamma := \lim_{t\to\infty} t^{3+4\tau/d}\theta_t^{2(\tau+d)}$:

	$\gamma = 0$	$\gamma \in (0,\infty)$	$\gamma = \infty$
$\tau \geq -d/2$	Gaussian	Gaussian	Gaussian
$\tau \in (-d,-d/2)$	$\frac{d}{\lvert\tau\rvert}$-stable	$\frac{d}{\lvert\tau\rvert}$-stable + Gaussian (?)	Gaussian
$\tau \leq -d$	$\frac{d}{\lvert\tau\rvert}$-stable	$\frac{d}{\lvert\tau\rvert}$-stable	$\frac{d}{\lvert\tau\rvert}$-stable

Summary

For $\lim_{t\to\infty} t^2\theta_t^d = \infty$ and $\gamma := \lim_{t\to\infty} t^{3+4\tau/d}\theta_t^{2(\tau+d)}$:

	$\gamma = 0$	$\gamma \in (0,\infty)$	$\gamma = \infty$
$\tau \geq -d/2$	Gaussian	Gaussian	Gaussian
$\tau \in (-d, -d/2)$	$\frac{d}{\lvert\tau\rvert}$-stable	$\frac{d}{\lvert\tau\rvert}$-stable + Gaussian (?)	Gaussian
$\tau \leq -d$	$\frac{d}{\lvert\tau\rvert}$-stable	$\frac{d}{\lvert\tau\rvert}$-stable	$\frac{d}{\lvert\tau\rvert}$-stable

- There are multivariate Compound-Poisson, Gaussian and stable limit theorems.

Summary

For $\lim_{t\to\infty} t^2\theta_t^d = \infty$ and $\gamma := \lim_{t\to\infty} t^{3+4\tau/d}\theta_t^{2(\tau+d)}$:

	$\gamma = 0$	$\gamma \in (0,\infty)$	$\gamma = \infty$
$\tau \geq -d/2$	Gaussian	Gaussian	Gaussian
$\tau \in (-d,-d/2)$	$\frac{d}{\lvert\tau\rvert}$-stable	$\frac{d}{\lvert\tau\rvert}$-stable + Gaussian (?)	Gaussian
$\tau \leq -d$	$\frac{d}{\lvert\tau\rvert}$-stable	$\frac{d}{\lvert\tau\rvert}$-stable	$\frac{d}{\lvert\tau\rvert}$-stable

- There are multivariate Compound-Poisson, Gaussian and stable limit theorems.
- For the multivariate Gaussian case the covariance structure is known.

Summary

For $\lim_{t\to\infty} t^2\theta_t^d = \infty$ and $\gamma := \lim_{t\to\infty} t^{3+4\tau/d}\theta_t^{2(\tau+d)}$:

	$\gamma = 0$	$\gamma \in (0,\infty)$	$\gamma = \infty$
$\tau \geq -d/2$	Gaussian	Gaussian	Gaussian
$\tau \in (-d,-d/2)$	$\frac{d}{\lvert\tau\rvert}$-stable	$\frac{d}{\lvert\tau\rvert}$-stable + Gaussian (?)	Gaussian
$\tau \leq -d$	$\frac{d}{\lvert\tau\rvert}$-stable	$\frac{d}{\lvert\tau\rvert}$-stable	$\frac{d}{\lvert\tau\rvert}$-stable

- There are multivariate Compound-Poisson, Gaussian and stable limit theorems.
- For the multivariate Gaussian case the covariance structure is known.
- In some situations rates of convergence are available.

Poisson process approximation

Define the point process

$$\xi_t := \frac{1}{2} \sum_{(x_1,x_2)\in\eta^2_{t,\neq}} \mathbf{1}\{\|x_1 - x_2\| \leq \theta_t\} \, \delta_{\|x_1-x_2\|^d},$$

where δ_u denotes the unit Dirac measure concentrated at the point $u \in \mathbb{R}$.

Poisson process approximation

Define the point process

$$\xi_t := \frac{1}{2} \sum_{(x_1,x_2)\in\eta^2_{t,\neq}} \mathbf{1}\{\|x_1 - x_2\| \leq \theta_t\} \, \delta_{\|x_1-x_2\|^d},$$

where δ_u denotes the unit Dirac measure concentrated at the point $u \in \mathbb{R}$.

Theorem: S./Thäle (2012), Reitzner/S./Thäle (2016+)

Let ζ be a Poisson process on $\mathbb{R}^+$ with intensity $\kappa_d/2$.

- If $\lim_{t\to\infty} t^2\theta_t^d = \lambda, \quad t^2\xi_t \xrightarrow{d} \zeta \cap [0,\lambda] \quad \text{as} \quad t \to \infty.$
- If $\lim_{t\to\infty} t^2\theta_t^d = \infty, \quad t^2\xi_t \xrightarrow{d} \zeta \quad \text{as} \quad t \to \infty.$

Poisson process approximation

Define the point process

$$\xi_t := \frac{1}{2} \sum_{(x_1,x_2)\in\eta^2_{t,\neq}} \mathbf{1}\{\|x_1 - x_2\| \leq \theta_t\}\, \delta_{\|x_1-x_2\|^d},$$

where δ_u denotes the unit Dirac measure concentrated at the point $u \in \mathbb{R}$.

Theorem: Decreusefond/S./Thäle (2016)

There is a constant $C_W > 0$ depending on W such that for $0 \leq a \leq t^2\theta_t^d$ and $t \geq 1$,

$$d_{KR}\big((t^2\xi_t)|_{[0,a]}, \zeta|_{[0,a]}\big) \leq C_W(t^{-2/d}a^{1+1/d} + t^{-1}(a + a^2)).$$

The Kantorovich-Rubinstein distance of two finite point processes ϕ, ψ is

$$d_{KR}(\phi,\psi) := \sup_{\substack{h:\mathbf{N}\to\mathbb{R},\\ |h(x)-h(y)|\leq d_{TV}(x,y)}} \big|\mathbb{E}h(\phi) - \mathbb{E}h(\psi)\big|.$$

Proof of stable and compound Poisson limit theorems

Corollary: Reitzner/S./Thäle (2016+)

Let $\tau \in \mathbb{R}$ and $a \in \mathbb{R}$ with $0 \leq a \leq \lim_{t\to\infty} t^{2/d}\theta_t$ and let ζ be a Poisson process on $\mathbb{R}^+$ with intensity $\kappa_d/2$. Then,

$$\frac{1}{2} \sum_{(x_1,x_2)\in\eta^2_{t,\neq}} \mathbf{1}\{\|x_1 - x_2\| \leq \min\{t^{-2/d}a, \theta_t\}\}\|x_1 - x_2\|^\tau \xrightarrow{d} \sum_{x\in\zeta\cap[0,a^d]} x^{\tau/d}$$

as $t \to \infty$.

Proof of stable and compound Poisson limit theorems

Corollary: Reitzner/S./Thäle (2016+)

Let $\tau \in \mathbb{R}$ and $a \in \mathbb{R}$ with $0 \leq a \leq \lim_{t\to\infty} t^{2/d}\theta_t$ and let ζ be a Poisson process on $\mathbb{R}^+$ with intensity $\kappa_d/2$. Then,

$$\frac{1}{2} \sum_{(x_1,x_2)\in\eta^2_{t,\neq}} \mathbf{1}\{\|x_1 - x_2\| \leq \min\{t^{-2/d}a, \theta_t\}\}\|x_1 - x_2\|^\tau \xrightarrow{d} \sum_{x\in\zeta\cap[0,a^d]} x^{\tau/d}$$

as $t \to \infty$.

- This yields the compound Poisson limit theorem.

Proof of stable and compound Poisson limit theorems

Corollary: Reitzner/S./Thäle (2016+)

Let $\tau \in \mathbb{R}$ and $a \in \mathbb{R}$ with $0 \leq a \leq \lim_{t\to\infty} t^{2/d}\theta_t$ and let ζ be a Poisson process on $\mathbb{R}^+$ with intensity $\kappa_d/2$. Then,

$$\frac{1}{2} \sum_{(x_1,x_2)\in\eta^2_{t,\neq}} \mathbf{1}\{\|x_1 - x_2\| \leq \min\{t^{-2/d}a, \theta_t\}\}\|x_1 - x_2\|^\tau \xrightarrow{d} \sum_{x\in\zeta\cap[0,a^d]} x^{\tau/d}$$

as $t \to \infty$.

- This yields the compound Poisson limit theorem.
- Approximating $L_t^{(\tau)}$ by the left hand side and the stable random variable by the left hand side and letting $a \to \infty$ and $t \to \infty$ in the right way proves the limit theorem with the stable random variables.

Proof of the central limit theorem

- Consider the random variables

$$U_{t,a}^{(\tau)} := \frac{1}{2} \sum_{(x_1,x_2)\in\eta_{t,\neq}^2} \mathbf{1}\{t^{-2/d}a \leq \|x_1 - x_2\| \leq \theta_t\}\|x_1 - x_2\|^\tau,$$

  which have for $a > 0$ and $\tau < 0$ better moment properties than $L_t^{(\tau)}$.
- Central limit theorem with Berry-Esseen bounds for $U_{t,a}^{(\tau)}$ via Malliavin-Stein bounds for Poisson-U-statistics
- Approximate $L_t^{(\tau)}$ by $U_{t,a}^{(\tau)}$

Thank you!

L. Decreusefond, M. Schulte and Ch. Thäle (2016):
Functional Poisson approximation in Kantorovich-Rubinstein distance with applications to U-statistics and stochastic geometry, Ann. Probab. 44, 2147-2197

M. Reitzner, M. Schulte, Ch. Thäle (2016+):
Limit theory for the Gilbert graph, arXiv: 1312.4861

M. Schulte, Ch. Thäle (2012):
The scaling limit of Poisson-driven order statistics with applications in geometric probability, Stochastic Process. Appl. 122, 4096-4120