Regularity and symplectic properties of traceless $\text{SU}(2)$ character varieties of tangles

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Kronheimer-Mrowka Singular Instanton Knot Homology $I^\flat(K)$

$K \subset S^3$ Set $X = S^3 \setminus (K \cup H \cup W)$.

$A^\flat = \{\text{singular } SU(2) \text{ connections on } X, \text{ traceless along } K \text{ and } H, \text{ giving nontrivial } SO(3) \text{ bundle with } w_2 \text{ dual to } W\}$

Take Morse homology of $cs : A^\flat / \mathcal{G} \to \mathbb{R}/\mathbb{Z}$.

Critical pts $\leftrightarrow \{\text{traceless singular flat } SU(2) \text{ connections on } X\} / \mathcal{G}$

Conditions:

$hol_{\mu_K}(A)$ traceless
$hol_{\mu_H}(A)$ traceless
$hol_{\mu_W}(A) = -1$

Figure: Add an earring to knot $K$. 

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$cs : \mathcal{A}^h / G \to \mathbb{R}$ is, at best, Bott-Morse with many critical circles. (An incompressible $T^2$ separates knot complement from $K \cup H \cup W$; most flat connections are irreducible on both sides.) Thus, one must perturb to get a Morse function.

- $I^h(K)$ is $\mathbb{Z}_4$ graded.
- $I^h(K)$ is isomorphic to sutured Floer theory, which categorifies $\Delta(K) = \sum c_i t^i$. Thus $\sum |c_i| \leq \text{Rank } I^h(K) \leq \text{Rank } CI^h(K)$.
- There is a spectral sequence with $E_2$ page $Kh^{red}(\overline{K})$ abutting to $I^h(K)$, so $\text{Rank } I^h(K) \leq \text{Rank } KH^{red}(\overline{K})$.

There is no combinatorial definition of $I^h(K)$. Calculations have only been possible where these bounds determine $I^h(K)$. 

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This talk is about work of $H^r K$, $r = 0, 1, 2, 3$, exploring a Lagrangian Floer homology related to the $I^\flat(K)$. {Hedden, –, Hogancamp, Kirk}

Identify the critical set of $cs: A^\flat/G \to \mathbb{R}/\mathbb{Z}$ with

$$R^\flat(X) = \{SU(2) \text{ reps} \mid Tr(\rho(\mu_K)) = Tr(\rho(\mu_H)) = 0, \rho(\mu_W) = -1\}/\text{conj}$$

which can be calculated from a $\pi_1$ presentation.

Overall goal: get a more tractable, topological definition of boundary operators defining $I^\flat(K)$, without instantons.
The Pillowcase and a Tangle Decomposition of $K$

Let a 2-sphere split $K$ into two 2-tangles:

- $T_0 =$ trivial 2-tangle with earring.
- $T_1 =$ the rest of $K$.

![Figure: $T_0$](image)

$$R(S^2 \setminus 4 \text{ points}) = \{\text{homomorphisms } \rho : \pi_1(S^2 \setminus 4 \text{ points}) \to SU(2) \mid \text{all } 4 \text{ generators go to traceless elements}\}/\text{conjugation}$$

The four $\pi_1$ elements linking $K$ in $S^2$ are sent to $i$, $e^{\gamma k}i$, $e^{\theta k}i$, $e^{(\theta - \gamma) k}i$.

$$R(S^2 \setminus 4 \text{ points}) = \{(\theta, \gamma) \in [0, \pi] \times [0, 2\pi]\}/\sim,$$

edges identified to make pillowcase. Only the four corners are abelian.
Circles arise here due to fibration from $R^\natural (B \setminus (\text{arcs} \cup \text{earring}))$ to its image in pillowcase. In this illustration, pink arc hits blue arc in three points, with preimage two circles and a point.
Transversality in Gauge Theory vs Topology

“Holonomy perturbations” in gauge theory definition can be interpreted as follows.

• Drill out more curves (adding more generators to $\pi_1$).

• Impose certain relations between the meridinal and longitudinal holonomies of these new link components.

Theorem (H,–,K)

Doing this with the curve $P$ in the standard tangle causes each circle to contribute two generators to the chain complex.

Theorem (–,K)

There are also curves in the outside tangle complement that make $R_\pi(\text{outside tangle})$ into a 1-manifold.
After these perturbations $\pi$, we obtain a pair of 1-manifolds in the pillowcase. The traceless perturbed character variety for the part with the earring misses the singular corner points.
Recent work by Abouzaid and de Silva-Robbin-Salamon simplifies Lagrangian Floer homology $FH(L_1, L_2)$ in 2D surface. Boundary operator is combinatorially defined, i.e., $\partial$ defined by counting immersed disks.

Requirements:

- Surface needs noncompact universal cover
- Need immersed 1-manifolds with no fish tails (i.e., no double points creating null homotopic loop)
- $L_1, L_2$ are homotopically essential.
- Covers of $L_1$ and $L_2$ are not homotopic.

**Theorem**

$FH(L_1, L_2)$ depends only on the homotopy classes of $L_1, L_2$. 
We extend the definition to the pillowcase $P=2$-sphere with “corners”.

- $L_1 = R_{\pi}^f (B \setminus T_0)$ (traceless representation variety for arcs with earring) misses corners.
- $L_2 = R (B \setminus T_1)$ hits corners, but with well-defined tangent direction.

Ultimately, we can extend combinatorial Lagrangian Floer theory to pillowcase with neighborhoods of corners deleted.

Using $A_2$, $A_3$ relation in this context we show, for an appropriate class of Lagrangians:

**Theorem (H,–,K)**

$FH(L_1, L_2)$ depends only on homotopy classes of $L_1$ and $L_2$ in $P \setminus \{corners\}$. 
Gradings

$I^h(K)$ is $\mathbb{Z}_4$ graded. Adapting Seidel’s graded Lagrangians, we define a relative $\mathbb{Z}_4$ grading $FH(L_1, C)$ when:

- $L_1 = R^h_\pi(T_0)$
- $C =$ circle, or arc connecting corners of $P$, without fishtails

Theorem (H,–,K)

For all 2-bridge knots $K$, and all torus knots $K$ checked so far, there is

- a tangle decomposition $K = T_0 \cup T_1$,
- perturbations in $T_0$ and $T_1$ making $R^h(T_1)$ and $R(T_0)$ smooth,

$$\bigoplus_{i=0}^{k} FH(L_1, C_i) \cong I^h(K).$$

More work is needed to show the traceless representation varieties never have fish tails, and that $FH(R^h(T_0), R(T_1))$ is not dependent on choice of perturbation or tangle decomposition.
Further partial results

The Lagrangians in the pillowcase form an $A^\infty$ category.

**Theorem (H,−,Hogancamp,K)**

*Given an outer tangle $T_1$ with $R(T_1) = L$, for the three ways to put in the trivial tangle with earring $\{T_0, T_+, T_-\}$, set $L_0 = R^b(T_0)$, $L_+ = R^b(T_+)$, and $L_- = R^b(T_-)$. Then there is an exact triangle.***

\[ \begin{align*}
FH(L_0, L) \\
\downarrow \\
FH(L_+, L) \\
\downarrow \\
FH(L_-, L)
\end{align*} \]

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It appears that proving invariance of this Lagrangian Floer theory invariant will require more cut and paste techniques, for tangles with more strands or removing multiple balls (say, a ball around each crossing). Here’s some progress on the former.

Symplectic properties of $R(S^2, 2n \text{ pts})$

following Goldman, Jeffrey-Weitsman

There is a Hamiltonian $n$-torus action on an open subset of $\mathcal{M}(F_n)$ with symplectic reduction $R(S^2, 2n)$. Essentially, $\mu = (\text{tr}(\rho(a)), \text{tr}(\rho(b)), \text{tr}(\rho(c)))$ is the moment map.
Assume $S^2$ splits a knot into $n$-strand tangles $T_1$ (with earring) and $T_2$ (without). After generic small holonomy perturbations w/ curves missing $S^2$, $R^\h_{\pi_1}(T_1)$ and $R_{\pi_2}(T_2)$ are $(2n - 3)$-dimensional smooth manifolds except $2^{n-1}$ points in $R_{\pi_2}(T_2)$ with $c(CP^{n-2})$ neighborhoods. Restriction to the $2n$-punctured $S^2$ gives stratum preserving Lagrangian immersions with “cone embeddings” into the $(4n - 6)$-dimensional $R(S^2, 2n)$ with its $2^{2n-2}$ singular points with $c(M)$ neighborhoods.
The 6-punctured 2-sphere (K)

In general, there is a double branched cover \( p : F_{n-1} \to S^2 \) branched along \( 2n \) points.

\[
p^* : R(S^2, 2n) \to R(F_{n-1}, 2n)_{-1} \cong R(F_{n-1}, 2n)_{+1} \cong \mathcal{M}(F_{n-1})
\]

Case \( n = 3 \)

\[
\begin{align*}
R^{ab}(S^2, 6) & \to \{ \text{nodal points} \} \\
\downarrow & \\
R^{bd}(S^2, 6) & \to \text{singular Kummer surface } T^4 / \mathbb{Z}/2 \\
\downarrow & \\
R(S^2, 6) & \to R(F_2) = \mathbb{C}P^3
\end{align*}
\]

Singular points of \( R(S^2, 6) \) have \( c(S^2 \times S^3) \) neighborhoods.