Regularity and symplectic properties of traceless SU(2) character varieties of tangles

Christopher Herald

Department of Mathematics and Statistics
University of Nevada, Reno

March 21, 2016

Kronheimer-Mrowka Singular Instanton Knot Homology $I^\natural(K)$

$K \subset S^3$ Set $X = S^3 \setminus (K \cup H \cup W)$.

$\mathcal{A}^\natural = \{$singular $SU(2)$ connections on X, traceless along K and H, giving nontrivial $SO(3)$ bundle with w_2 dual to $W\}$

Take Morse homology of $cs : \mathcal{A}^\natural/\mathcal{G} \to \mathbb{R}/\mathbb{Z}$.

Critical pts $\leftrightarrow$ $\{$traceless singular flat $SU(2)$ connections on $X\}/\mathcal{G}$

Conditions:

$hol_{\mu_K}(A)$ traceless

$hol_{\mu_H}(A)$ traceless

$hol_{\mu_W}(A) = -1$

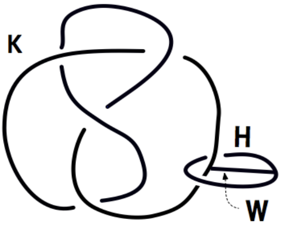


Figure: Add an earring to knot K.

$cs : \mathcal{A}^\natural/\mathcal{G} \to \mathbb{R}$ is, at best, Bott-Morse with many critical circles. (An incompressible T^2 separates knot complement from $K \cup H \cup W$; most flat connections are irreducible on both sides.) Thus, one must perturb to get a Morse function.

- $I^\natural(K)$ is $\mathbb{Z}_4$ graded.
- $I^\natural(K)$ is isomorphic to sutured Floer theory, which categorifies $\Delta(K) = \sum c_i t^i$. Thus $\sum |c_i| \leq \text{Rank} \;\; I^\natural(K) \leq \text{Rank} \;\; CI^\natural(K)$.
- There is a spectral sequence with E_2 page $Kh^{red}(\overline{K})$ abutting to $I^\natural(K)$, so $\text{Rank} \;\; I^\natural(K) \leq \text{Rank} \;\; KH^{red}(\overline{K})$.

There is no combinatorial definition of $I^\natural(K)$. Calculations have only been possible where these bounds determine $I^\natural(K)$.

flat connections $\leftrightarrow$ representations $\rho : \pi_1 \to SU(2)$

This talk is about work of $H^r K$, $r = 0, 1, 2, 3$, exploring a Lagrangian Floer homology related to the $I^\natural(K)$. {Hedden, –, Hogancamp, Kirk}

Identify the critical set of $cs : \mathcal{A}^\natural/\mathcal{G} \to \mathbb{R}/\mathbb{Z}$ with

$$R^\natural(X) = \{SU(2) \text{ reps} \mid Tr(\rho(\mu_K)) = Tr(\rho(\mu_H)) = 0, \rho(\mu_W) = -1\}/\text{conj}$$

which can be calculated from a π_1 presentation.

Overall goal: get a more tractable, topological definition of boundary operators defining $I^\natural(K)$, without instantons.

The Pillowcase and a Tangle Decomposition of K

Let a 2-sphere split K into two 2-tangles:

- T_0 =trivial 2-tangle with earring.
- T_1=the rest of K.

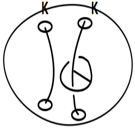


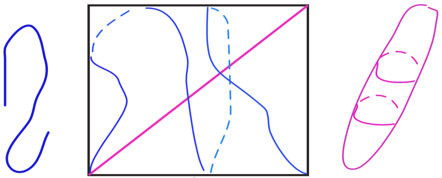
Figure: T_0

$$R(S^2 \setminus \ 4 \text{ points}) = \{\text{homomorphisms } \rho : \pi_1(S^2 \setminus \ 4 \text{ points}) \to SU(2) \mid \text{all 4 generators go to traceless elements}\}/\text{conjugation}$$

The four π_1 elements linking K in S^2 are sent to $\mathbf{i}$, $e^{\gamma \mathbf{k}}\mathbf{i}$, $e^{\theta \mathbf{k}}\mathbf{i}$, $e^{(\theta-\gamma)\mathbf{k}}\mathbf{i}$.

$R(S^2 \setminus \ 4 \text{ points}) = \{(\theta, \gamma) \in [0, \pi] \times [0, 2\pi]\}/\sim$, edges identified to make pillowcase. Only the four corners are abelian.

Fiber Product Structure



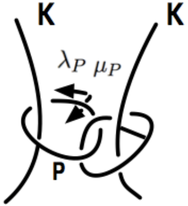
Circles arise here due to fibration from $R^\natural(B \setminus (\text{arcs} \cup \text{earring}))$ to its image in pillowcase. In this illustration, pink arc hits blue arc in three points, with preimage two circles and a point.

Transversality in Gauge Theory vs Topology

"Holonomy perturbations" in gauge theory definition can be interpreted as follows.

• Drill out more curves (adding more generators to π_1.

• Impose certain relations between the meridinal and longitudinal holonomies of these new link components.



Theorem (H,–,K)

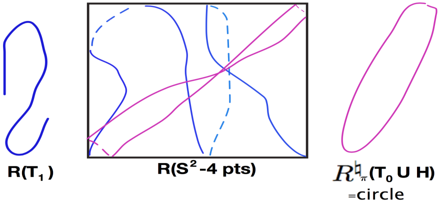
Doing this with the curve P in the standard tangle causes each circle to contribute two generators to the chain complex.

Theorem (–,K)

There are also curves in the outside tangle complement that make R_π(outside tangle) into a 1-manifold.

Effect in Pillowcase

After these perturbations π, we obtain a pair of 1-manifolds in the pillowcase. The traceless perturbed character variety for the part with the earring misses the singular corner points.



Lagrangian Floer Homology in Pillowcase

- Recent work by Abouzaid and de Silva-Robbin-Salamon simplifies Lagrangian Floer homology $FH(L_1, L_2)$ in 2D surface. Boundary operator is combinatorially defined, i.e., ∂ defined by counting immersed disks.

  Requirements:
  - Surface needs noncompact universal cover
  - Need immersed 1-manifolds with no fish tails (i.e., no double points creating null homotopic loop)
  - L_1, L_2 are homotopically essential.
  - Covers of L_1 and L_2 are not homotopic.

Theorem

$FH(L_1, L_2)$ depends only on the homotopy classes of L_1, L_2.

Defining Lagrangian Floer Homology in Pillowcase

We extend the definition to the pillowcase P=2-sphere with "corners".

• $L_1 = R_\pi^\natural(B \setminus T_0)$ (traceless representation variety for arcs with earring) misses corners.

• $L_2 = R(B \setminus T_1)$ hits corners, but with well-defined tangent direction.

Ultimately, we can extend combinatorial Lagrangian Floer theory to pillowcase with neighborhoods of corners deleted.

Using A_2, A_3 relation in this context we show, for an appropriate class of Lagrangians:

Theorem (H,–,K)

$FH(L_1, L_2)$ depends only on homotopy classes of L_1 and L_2 in $P \setminus \{corners\}$.

Gradings

$I^\natural(K)$ is $\mathbb{Z}_4$ graded. Adapting Seidel's *graded Lagrangians*, we define a relative $\mathbb{Z}_4$ grading $FH(L_1, C)$ when:

- $L_1 = R^\natural_\pi(T_0)$
- C=circle, or arc connecting corners of P, without fishtails

Theorem (H,–,K)

For all 2-bridge knots K, and all torus knots K checked so far, there is

- *a tangle decomposition $K = T_0 \cup T_1$,*
- *perturbations in T_0 and T_1 making $R^\natural(T_1)$ and $R(T_0)$ smooth,*

$$\bigoplus_{i=0}^{k} FH(L_1, C_i) \cong I^\natural(K).$$

More work is needed to show the traceless representation varieties never have fish tails, and that $FH(R^\natural(T_0), R(T_1))$ is not dependent on choice of perturbation or tangle decomposition.

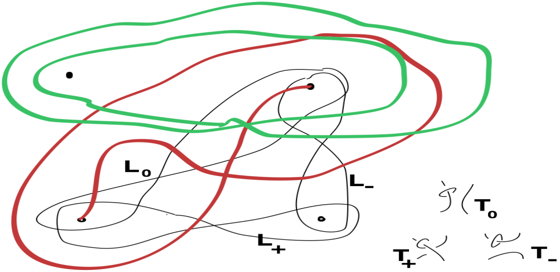
Further partial results

The Lagrangians in the pillowcase form an A^∞ category.

Theorem (H,–,Hogancamp,K)

Given an outer tangle T_1 with $R(T_1) = L$, for the three ways to put in the trivial tangle with earring $\{T_0, T_+, T_-\}$, set $L_0 = R^\natural(T_0)$, $L_+ = R^\natural(T_+)$, and $L_- = R^\natural(T_-)$. Then there is an exact triangle.

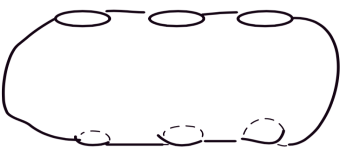
$$\begin{array}{ccc} & FH(L_0, L) & \\ \swarrow & & \nwarrow \\ FH(L_+, L) & \rightarrow & FH(L_-, L) \end{array}$$

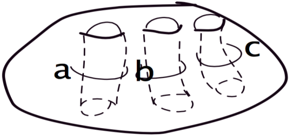


It appears that proving invariance of this Lagrangian Floer theory invariant will require more cut and paste techniques, for tangles with more strands or removing multiple balls (say, a ball around each crossing). Here's some progress on the former.

Symplectic properties of $R(S^2, 2n \text{ pts})$ following Goldman, Jeffrey-Weitsman

$$R(S^2, 2n \text{ pts}) \quad \text{vs} \quad \mathcal{M}(F_n) = Hom(\pi_1(F_n), SU(2))/conj$$

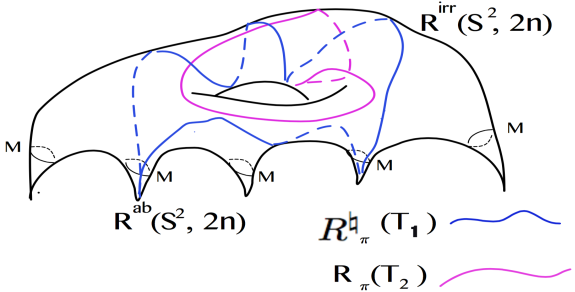




There is a Hamiltonian n-torus action on an open subset of $\mathcal{M}(F_n)$ with symplectic reduction $R(S^2, 2n)$. Essentially, $\mu = (tr(\rho(a)), tr(\rho(b)), tr(\rho(c)))$ is the moment map.

Generic Structure Theorem (–,K)

Assume S^2 splits a knot into n-strand tangles T_1 (with earring) and T_2 (without). After generic small holonomy perturbations w/ curves missing S^2, $R^\natural_{\pi_1}(T_1)$ and $R_{\pi_2}(T_2)$ are $(2n-3)$-dimensional smooth manifolds except 2^{n-1} points in $R_{\pi_2}(T_2)$ with $c(\mathbb{C}P^{n-2})$ neighborhoods. Restriction to the $2n$-punctured S^2 gives stratum preserving Lagrangian immersions with "cone embeddings" into the $(4n-6)$-dimensional $R(S^2, 2n)$ with its 2^{2n-2} singular points with $c(M)$ neighborhoods.



The 6-punctured 2-sphere (K)

In general, there is a double branched cover $p : F_{n-1} \to S^2$ branched along $2n$ points.
$p^* : R(S^2, 2n) \to R(F_{n-1}, 2n)_{-1} \cong R(F_{n-1}, 2n)_{+1} \cong \mathcal{M}(F_{n-1})$

Case $n = 3$

$$\begin{array}{ccc} R^{ab}(S^2,6) & \to & \{\text{nodal points}\} \\ \downarrow & & \downarrow \\ R^{bd}(S^2,6) & \to & \text{singular Kummer surface } T^4/\mathbb{Z}/2 \\ \downarrow & & \downarrow \\ R(S^2,6) & \to & R(F_2) = \mathbb{C}P^3 \end{array}$$

Singular points of $R(S^2, 6)$ have $c(S^2 \times S^3)$ neighborhoods.