Global Strong Well-Posedness of the 3D-Primitive Equations for Non Smooth Initial Data

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Isothermal Primitive Equations

Primitive equations are given by

$$\begin{aligned} \partial_t v + u \cdot \nabla v - \Delta v + \nabla_H \pi &= f \quad \text{in } \Omega \times (0,T), \\ \partial_z \pi &= 0 \quad \text{in } \Omega \times (0,T), \\ \operatorname{div} u &= 0 \quad \text{in } \Omega \times (0,T), \\ v(0) &= a. \end{aligned} \tag{1}$$

- $\Omega = G \times (-h, 0)$, where $G = (0,1)^2$, $h > 0$
- velocity u is written as $u = (v, w)$ with $v = (v_1, v_2)$
- v and w denote the horizontal and vertical components of u,
- π pressure, f external force
- $\nabla_H = (\partial_x, \partial_y)^T$, $\Delta, \nabla, \operatorname{div}$ three dimensional operators.

System is complemented by the set of boundary conditions

$$\begin{aligned} \partial_z v &= 0, \quad w = 0 && \text{on } \Gamma_u \times (0,T), \\ v &= 0, \quad w = 0 && \text{on } \Gamma_b \times (0,T), \\ & u, \pi \text{ are periodic} && \text{on } \Gamma_l \times (0,T). \end{aligned} \tag{2}$$

- $\Gamma_u := G \times \{0\}$, $\Gamma_b := G \times \{-h\}$, $\Gamma_l := \partial G \times [-h, 0]$

Some History of Primitive Equations

- '92-'95 : full primitive equations introduced by Lions, Temam and Wang, existence of a global weak solution for $a \in L^2$.
- Uniqueness question seems to be open
- '95-'97 : Ziane, H^2-regularity of linearized resolvent problem.
- '01 : Guillén-González, Masmoudi, Rodiguez-Bellido : existence of a unique, local, strong solution for $a \in H^1$
- '07, Cao and Titi : breakthrough result : existence of a unique, global strong solution for arbitrary initial data $a \in H^1$
- proof based on a priori H^1-bounds for the solution, obtained by $L^\infty(L^6)$ energy estimates, slighty different bc.
- '07, '08 : Kukavica and Ziane : global strong well-posedness for arbitrary H^1-data, our b.c.
- Cao, Titi '12, Cao, Li, Titi '14 : global well-posedness results for $a \in H^2$ with only horizontal viscosity and diffusion or
- with vapor : '14 : Coti-Zelati, Huang, Kukavica, Teman, Ziane
- '16 Li, Titi : survey, state of the art

Global well-posedness for data different from H^1

Find spaces with less differentiability properties as $H^1(\Omega)$, which nevertheless guarantee global well-posedness of these equations.

- '04 : Bresch, Kazhikhov and Lemoine : uniqueness of 2d weak solutions for data a with $\partial_z a \in L^2$.
- '14 : Kukavica, Pei, Rusin and Ziane : uniqueness of weak solutions for continuous data
- '15 : Li, Titi : uniqueness of weak solutions for data in L^∞, as long as discontinuity is small
- observation : all existing results concerning the well-posedness are within L^2-setting.

Aims of this talk :

- develop an L^p-approach
- show existence of a unique, global strong solution to primitive equations for data a having less differentiability properties than H^1.

Strategy of L^p-Approach

- solution of the linearized equation is governed by an analytic semigroup T_p on the space X_p
- X_p is defined as the range of the hydrostatic Helmholtz projection $P_p : L^p(\Omega)^2 \to L^p_{\overline{\sigma}}(\Omega)^2$
- This space corresponds to solenoidal space $L^p_\sigma(\Omega)$ for Navier-Stokes equations
- generator of T_p is $-A_p$ called the hydrostatic Stokes operator.
- rewrite primitive equations as

$$\begin{cases} v'(t) + A_p v(t) = P_p f(t) - P_p(v \cdot \nabla_H v + w\partial_z v), & t > 0, \\ v(0) = a. \end{cases}$$

- consider integral equation

$$v(t) = e^{-tA_p} a + \int_0^t e^{-(t-s)A_p} \big(P_p f(s) + F_p v(s)\big)\, ds, \qquad t \geq 0,$$

where $F_p v := -P_p(v \cdot \nabla_H v + w\partial_z v)$

Strategy of L^p-approach

- show that v is unique, local, strong solution, i.e. $v \in C^1((0, T^*]; X_p) \cap C((0, T^*]; D(A_p)),\ p \in (1, \infty)$
- Note $D(A_p) \hookrightarrow W^{2,p}(\Omega)^2 \hookrightarrow H^1(\Omega)^2$ for $p \geq 6/5$
- Hence, one ontains existence of a unique, global, strong solution for arbirtrary $a \in [X_p, D(A_p)]_{1/p}$ for $1 < p < \infty$ provided
- $\sup_{0 \leq t \leq T} \|v(t)\|_{H^2(\Omega)}$ is bounded by some constant $B = B(\|a\|_{H^2(\Omega)}, T)$ for any $T > 0$.
- proof of global H^2-bound for v is based on $L^\infty(L^4)$-estimates for $\tilde{v}$
- in addition : $\|v(t)\|_{H^2(\Omega)}$ is decaying exponentially as $t \to \infty$.

Main Results

Theorem 1 :
Let $p \in (1,\infty)$, $a \in V_{1/p,p}$ and $f \equiv 0$. Then there exists a unique, strong global solution (v,π) to primitive equations within the regularity class

$$v \in C^1((0,\infty); L^p(\Omega)^2) \cap C((0,\infty); W^{2,p}(\Omega)^2), \ \pi \in C((0,\infty); W^{1,p}(G) \cap L^p_0(G)).$$

Moreover, the solution (v,π) decays exponentially, i.e. there exist constants $M, c, \tilde{c} > 0$ such that

$$\|\partial_t v(t)\|_{L^p(\Omega)} + \|v(t)\|_{W^{2,p}(\Omega)} + \|\pi\|_{W^{1,p}(G)} \leq M t^{-\tilde{c}} e^{-ct}, \quad t > 0.$$

Remarks :

- $V_{\theta,p} := [X_p, D(A_p)]_\theta$, $0 \leq \theta \leq 1$ and $1 < p < \infty$, is complex interpolation space between X_p and $D(A_p)$ of order θ
- note that $V_{1/p,p} \hookrightarrow H^{2/p,p}(\Omega)^2$ for all $p \in (1,\infty)$
- if $p = 2$, then $V_{1/2,2}$ coincides with H^1 subject to bc., i.e.

$$V_{1/2,2} = \{\varphi \in H^1_{\mathrm{per}}(\Omega)^2 : \mathrm{div}_H\, \bar{\varphi} = 0 \text{ in } G, \quad \varphi = 0 \text{ on } \Gamma_b\}.$$

Sketch of Proofs

Sketch of Proof of global well-posedness :

- resolvent estimates in L^p-setting
- hydrostatic Helmholtz projection and hydrostatic Stokes operator in L^p
- local well-posedness
- energy estimates and global well-posedness in L^2-setting
- exponential decay and bootstrap argument

Reformulation of Problem

- vertical component w of u is given by
  $$w(x,y,z) = \int_z^0 \mathrm{div}_H\, v(x,y,\zeta)\, d\zeta, \quad (x,y) \in G,\ -h < z < 0,$$
- let $\bar{v}$ be the average of v in the vertical direction, i.e.,
  $$\bar{v}(x,y) := \frac{1}{h} \int_{-h}^0 v(x,y,z)\, dz, \quad (x,y) \in G$$
- Hence our problem is equivalent to finding a function $v : \Omega \to \mathbb{R}^2$ and a function $\pi : G \to \mathbb{R}$ satisfying

$$\begin{aligned}
\partial_t v + v \cdot \nabla_H v + w \partial_z v - \Delta v + \nabla_H \pi &= f && \text{in } \Omega \times (0,T), \\
w &= \int_z^0 \mathrm{div}_H\, v\, d\zeta && \text{in } \Omega \times (0,T), \\
\mathrm{div}_H\, \bar{v} &= 0 && \text{in } G \times (0,T), \\
v(0) &= a,
\end{aligned}$$

as well as the boundary conditions

$$\begin{aligned}
\partial_z v &= 0 && \text{on } \Gamma_u \times (0,T), \\
v &= 0 && \text{on } \Gamma_b \times (0,T), \\
& v \text{ and } \pi \text{ are periodic} && \text{on } \Gamma_l \times (0,T).
\end{aligned}$$

L^p-Resolvent estimates

Let $\lambda \in \Sigma_{\pi-\varepsilon} \cup \{0\}$, $\varepsilon \in (0, \pi/2)$, let $p \in (1, \infty)$ and $f \in L^p(\Omega)^2$. Then the linear resolvent problem

$$\begin{aligned} \lambda v - \Delta v + \nabla_H \pi &= f \quad \text{in } \Omega, \\ \mathrm{div}_H \bar{v} &= 0 \quad \text{in } G, \end{aligned}$$

subject to the boundary conditions

$$\partial_z v = 0 \text{ on } \Gamma_u, \qquad v = 0 \text{ on } \Gamma_b, \qquad v \text{ and } \pi \text{ are periodic on } \Gamma_l.$$

admit a unique solution $(v, \pi) \in W^{2,p}_{\mathrm{per}}(\Omega)^2 \times W^{1,p}_{\mathrm{per}}(G) \cap L^p_0(G)$ and

$$|\lambda| \, \|v\|_{L^p(\Omega)} + \|v\|_{W^{2,p}(\Omega)} + \|\pi\|_{W^{1,p}(G)} \leq C \|f\|_{L^p(\Omega)}, \quad \lambda \in \Sigma_{\pi-\varepsilon} \cup \{0\}, f \in L^p(\Omega)^2.$$

Strategy :

- weak formulation and Babuska-Brezzi theory on mixed problems
- H^2-estimates via difference quotients
- rewrite problem as 2d Stokes equations and 3d Laplace problem with mixed boundary conditions
- use L^p-estimate for these problems

The hydrostatic Stokes Operator

- existence of classical Helmholtz projection (for NS) is equivalent to unique solvability of weak Neumann problem
- here : Neumann problem is replaced by

$$\Delta_H \pi = \operatorname{div}_H f \quad \text{in } G, \text{periodic bc}$$

- above problem admits unique weak solution $\pi \in W^{1,p}_{\mathrm{per}}(G) \cap L^p_0(G)$ satisfying $\|\pi\|_{W^{1,p}(G)} \leq C\|f\|_{L^p(G)}$
- For $f = \bar{v}$ let π unique solution and set

$$P_p v := v - \nabla_H \pi$$

- P_p is called hydrostatic Helmholtz projection.
- Set $X_p := \mathrm{Ran} P_p$
- analogous role as the solenoidal space $L^p_\sigma(\Omega)$ in theory of (NS).
- define hydrostatic Stokes operator A_p on X_p as

$$\begin{cases} A_p v := -P_p \Delta v \\ D(A_p) := \{v \in W^{2,p}_{\mathrm{per}}(\Omega)^2 : \mathrm{div}_H \bar{v} = 0 \text{ in } G, \partial_z v = 0 \text{ on } \Gamma_u, v = 0 \text{ on } \Gamma_b \end{cases}$$

- hydrostatic Stokes operator $-A_p$ generates a bounded analytic C_0-semigroup T_p on X_p

Local Existence

- rewrite primitive equations as

$$v'(t) + A_p v(t) = P_p f(t) + F_p v(t),\ t > 0, \quad v(0) = a,$$

  where $F_p v := -P_p(v \cdot \nabla_H v + w\partial_z v)$ and consider
- $v(t) = e^{-tA_p} a + \int_0^t e^{-(t-s)A_p} \big(P_p f(s) + F_p v(s)\big)\, ds, \quad t \geq 0$
- w is less regular than v with respect to (x, y), but w has good regularity properties with respect to z.
- major difficulty : nonlinear term $w\partial_z v$ is stronger as in Navier-Stokes
  - NS : $(u \cdot \nabla)u \quad \sim$ order 1
  - primitive : $w\partial_z \sim$ order 2
  - Kato-type iteration works only for nonlinear terms of order < 2
- anisotropic nature of nonlinear term is treated with function spaces

$$W_z^{r,q} W_{xy}^{s,p} := W^{r,q}((-h, 0); W^{s,p}(G)).$$

- Set $V_{\theta,p} := [X_p, D(A_p)]_\theta$
- There exists a unique local mild solution v provided $a \in V_{1/p,p}$
- parabolic theory implies : v is strong solution
- unfortunately : for $a \in V_{\delta,p}$ the dependency of life time T^* cannot be controled merely by $V_{\delta,p}$-norm of a
- however : $T^* = (C\|a\|_{V_{\delta+\varepsilon},p})^{-1}$

H^2- A Priori Bounds and Almost Global Existence

- local result : existence of a unique, strong solution on $(0, T^*]$
- for $t_1 \in (0, T^*)$ regard $v(t_1) \in D(A_p)$ as new initial data
- if $p = 2$, then unique local strong solution in $[0, T^*]$ may be extended to strong solution on $[0, T]$ for all $T \in (T^*, \infty)$ due to
- a priori bound : $\sup_{0 \le t \le T} \|v(t)\|_{H^2(\Omega)} \le B = B(T, \|a\|_{H^2(\Omega)})$.
- We show

$$\|Av\|_{L^2(\Omega)} \le C(\|\partial_t v\|_{L^2(\Omega)} + \|v\|^3_{H^1(\Omega)} + \|v\|_{H^1(\Omega)}\|v_z\|^3_{L^3(\Omega)}), \quad t \ge 0,$$

as well as the estimates

- $\|v(t)\|_{H^1(\Omega)} \le B_2(t, \|a\|_{H^1(\Omega)})$ for $t \ge 0$ and some function B_2.
- $\|v_z(t)\|_{L^3(\Omega)} \le B_3(t, \|a\|_{H^2(\Omega)})$ for $t \ge 0$ and some function B_3.
- $\|\partial_t v(t)\|_{L^2(\Omega)} \le B_4(t, \|a\|_{H^2(\Omega)})$ for $t \ge 0$ and some function B_4.
- choose $T_B > 0$, depending only on B, and extend v to a strong solution on $[0, T^* + T_B]$.
- If $T^* + T_B < T$, then $\|v(T^* + T_B)\|_{H^2(\Omega)} \le B$, so we extend v to $[0, T^* + 2T_B]$
- Repeating this argument yields a unique, strong solution on $[0, T]$.

Global existence for $1 < p < \infty$ and Exponential Decay

- let $v \in C^1((0,T^*];X_p) \cap C((0,T^*];D(A_p))$ be the local solution for $a \in V_{1/p,p}$
- for fixed $t_0 \in (0,T^*]$ regard $v(t_0) \in V_{1/2,2}$ as new initial value
- above steps imply global existence of v within the L^2-framework
- use spectral gap of A_2 to establish exponential decay of v for $p = 2$
- extend to the case $p \leq 3$ by a bootstrap argument, more precisely
- consider v as the solution of the linear primitive equations with force $f := -v \cdot \nabla_H v - w\partial_z v$, i.e.,

$$v(t) = e^{-tA_p}a + \int_0^t e^{-(t-s)A_p} P_p f(s)\, ds, \qquad t > 0.$$

- Since $\|f\|_{L^3(\Omega)} \leq C\|v\|^2_{H^2(\Omega)}$ is exponentially decaying as $t \to \infty$ we obtain the claim for $p \leq 3$
- Repeat argument, taking $p = 3$ for granted, and combine with embeddings yields the main assertion

Characterization of Initial Data

Let $\theta \in [0,1]$ and $1 < p < \infty$. Then

$$V_{\theta,p} = D(A^\theta) =$$

$$\begin{cases} \{v \in H^{2\theta,p}_{per}(\Omega)^2 \cap L^p_{\overline{\sigma}}(\Omega) : \partial_z v\big|_{\Gamma_N} = 0,\ v\big|_{\Gamma_D} = 0\}, & 1/2 + 1/2p < \theta \leq 1, \\ \{v \in H^{2\theta,p}_{per}(\Omega)^2 \cap L^p_{\overline{\sigma}}(\Omega) : v\big|_{\Gamma_D} = 0\}, & 1/2p < \theta < 1/2 + 1/2p, \\ \{v \in H^{2\theta,p}_{per}(\Omega)^2 \cap L^p_{\overline{\sigma}}(\Omega)\}, & \theta < 1/2p. \end{cases}$$

The case $p = \infty$

Aim : extend above result within L^p-setting for $1 < p < \infty$ to $p = \infty$.

Strategy :

- show that for $a \in L^\infty$-type space like $L^\infty_{\overline{\sigma}}(\Omega)$ or $BUC_{\overline{\sigma}}(\Omega)$, there exists a unique, local mild solution v with $v(t_1) \in V_{\theta,p}$ for some (θ, p)
- apply previous L^p-result

- Step I : extend linear theory to L^∞-setting
- Step II : develop iteration scheme

Linear Theory : The Hydrostatic Stokes Operator via Perturbation

- resolvent equation : solve for pressure π : take average, apply div_H
- then $\nabla_H \pi = \nabla_H \Delta_H^{-1} \mathrm{div}_H \partial_z v \big|_{z=-1}$
- key observation : regard this as Kato-perturbation of Δ of order $1 + 1/p$
- A generates analytic semigroup S on $L^p_{\overline{\sigma}}(\Omega)$ and $L^\infty_{\overline{\sigma}}(\Omega)$
- A behaves like usual Stokes operator ; e.g. A admits
  - maximal $L^p - L^q$-regularity
  - $L^p - L^q$ smoothing

Consequences : Linear estimates for $p = \infty$

Let S be hydrostatic Stokes semigroup and P hydrostatic Helmholtz projection. Then

- $\|\nabla S(t) P f\|_\infty \leq C t^{-1/2} \|f\|_\infty, \quad t > 0$
- $\|\nabla S(t) P \nabla f\|_\infty \leq C t^{-1} \|f\|_\infty, \quad t > 0$
- $\|S(t) P \nabla f\|_\infty \leq C t^{-1/2} \|f\|_\infty, \quad t > 0$
- $\|\nabla S(t) P f\|_\infty \leq C \|P f\|_\infty, \quad t > 0$

Reference Solution and Iteration Scheme

- For $a \in BUC_{\overline{\sigma}}(\Omega) = \overline{C^{\infty}_{\overline{\sigma}}(\Omega)}^{\|\cdot\|_\infty}$ choose reference data $a_{ref} \in C^{\infty}_{\overline{\sigma}}(\Omega)$ such that $\|V_0\|_\infty$ is small, where $V_0 := a - a_{ref}$
- construct local reference solution v_{ref} with $v_{ref}(0) = a_{ref}$ as above
- define approximating sequence

$$V_{m+1}(t) := S(t)V_0 - \int_0^t S(t-s)P\nabla\cdot(U_m \otimes V_m)ds - \int_0^t S(t-s)P(U_m \cdot v_{ref} + u_{ref}\cdot\nabla V_m)ds$$

- control $K_m(t) := \sup_{0<s<t} s^{1/2}\|\nabla V_m(s)\|_\infty$
- control $H_m(t) := \sup_{0<s<t} \|V_m(s)\|_\infty$
- setting $G_m(t) := \max\{K_m(t), H_m(t)\}$ we have

$$G_{m+1}(t) \le C_1\|V_0\|_\infty + C_2 G_m(t)^2 + C_3 G_m(t)$$

- if $a_{m+1} \le a_0 + c_1 a_m^2 + c_2 a_m$ and $c_2 < 1$ and $4c_1 a_0 < (1-c_2)^2$, then (a_m) is bounded
- Hence $G_m(t) \le C\|V_0\|$, i.e. (G_m) is bounded sequence
- Moreover, (V_m) is Cauchy sequence in $S := \{V \in C([0,T]; BUC_{\overline{\sigma}}(\Omega) : \|\nabla V(t)\|_\infty = o(t^{-1/2})\}$
- $v := v_{ref} + \lim_{m\to\infty} V_m$ is unique, local solution
- parabolicity : $v(t_1) \in \{u \in W^{1,\infty}(\Omega) : u|_{z=-1} = 0, div\overline{u} = 0\} \subset V_{1/2,2}$

Global well-posedness for Bounded Data

Theorem 2 :
Let $a \in BUC_{\overline{\sigma}}(\Omega)$ with $a = 0$ on $\Gamma_b = 0$. Then there exists a unique, global, strong solution to the 3D-primitive equations.

Periodic and Stationary Solutions for Large Forces

Theorem 3 (jointly with P. Galdi) :
Primitive equations admit a strong, periodic solution for non small periodic $f \in L^2(J, L^2)$

Corollary (jointly with P. Galdi) :
Primitive equations admit a stationary solution for non small periodic $f \in L^2(J, L^2)$

Remark :
Note that there is no smallness assumption on f as e.g. in Hsia, Shiue '13.

Periodic Solutions for large forces

v is called a weak T-periodic solution provided

- $v \in C(J; L^2(\Omega)) \cap L^2(J; H^1(\Omega))$ is a weak solution
- v satisfies strong energy inequality

$$\|v(t)\|_2^2 + 2\int_s^t \|\nabla v(\tau)\|_2^2 d\tau \leq \|v(s)\|_2^2 + 2\int_s^t (f(\tau), v(\tau))d\tau$$

- $v(t+T) = v(T)$ for all $t \geq 0$

A weak T-periodic solution v is called strong if in addition $v \in C(J; H^1(\Omega)) \cap L^2(J; H^2(\Omega))$

Proposition : Let $f \in L^2(J; L^2(\Omega))$ be T-periodic. Then there exists at least one weak T-periodic solution v

Proof : Galerkin procedure and Brouwer's fixed point theorem

Strong Periodic Solutions via Weak-Strong Uniqueness

- Let $f \in L^2(J; L^2(\Omega))$ be T-periodic. Then there exists unique global strong solution u for (arbitrary) $a \in H^1(\Omega)$
- weak-strong uniqueness theorem : $u = v$
- Idea of Proof : weak theory : there is $t_0 > 0$ with $v(t_0) \in H^1$
- take $v(t_0)$ as initial data for strong solution u
- take u as test function
- for $w = v - u$ one has

$$\|w(t)\|_2^2 + \int_{t_0}^{t} \|\nabla w(s)\|_2^2 ds \leq C \int_{t_0}^{t} [\|\nabla_H u(s)\|_2^4 + \|\nabla_H u(s)\|_2^2 \|D^2 u(s)\|_2^2] \|w(s)\|_2^2 ds$$

- blue term in $L^1(t_0, t)$ due to regularity of strong solutions u
- Gronwall : $w = 0$