On the Motion of the Free Boundary of a Self-Gravitating Incompressible Fluid

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Self-gravitating flow equation:

We consider the motion of a fluid body in two space dimensions subject to self-gravitational force.

We assume that

- the fluid density is 1,
- the fluid is inviscid, incompressible, irrotational, the surface tension is zero,
- the fluid is subject to self-gravitation,
- the fluid occupies a simply connected bounded domain,
- outside the fluid domain it is vacuum.

- Let $\Omega(t)$ be the fluid domain, $\partial\Omega(t)$ be the interface at time t.

The motion of the fluid is described by

$$\begin{cases} \mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla P - \nabla\phi & \text{in } \Omega(t) \\ \operatorname{div}\mathbf{v} = 0, \qquad \operatorname{curl}\mathbf{v} = 0, & \text{in } \Omega(t) \\ P = 0, & \text{on } \partial\Omega(t) \\ \Delta\phi = 2\pi\chi_{\Omega(t)} \end{cases} \tag{1}$$

$\mathbf{v}$ is the fluid velocity, P is the fluid pressure.

$$\nabla\phi(x,t) = \int_{\Omega(t)} \frac{x-y}{|x-y|^2}\,dy$$

The Taylor sign condition

When surface tension is zero, the motion can be subject to the Taylor instability.

- Taylor sign condition:
$$-\frac{\partial P}{\partial \mathbf{n}} \geq c_0 > 0 \tag{2}$$
on the interface $\partial\Omega(t)$. $\mathbf{n}$ is the unit outward normal to $\partial\Omega(t)$.

Fact: The Taylor sign condition always holds for our problem.

Proof: $\Delta P = -\Delta\phi - |\nabla \mathbf{v}|^2 = -2\pi - |\nabla \mathbf{v}|^2 < 0$ in $\Omega(t)$.
Hopf Maximal principle implies

$$-\frac{\partial P}{\partial \mathbf{n}} \geq c_0 > 0.$$

Surface equation in Lagrangian coordinates

- Notation: we identify $(x, y) = x + iy$;
- Let $\partial\Omega(t) : z = z(\alpha, t)$, $\alpha \in [0, 2\pi]$; α is the Lagrangian coordinate.
- $z_t = z_t(\alpha, t)$ is the fluid velocity;
- $z_{tt} = z_{tt}(\alpha, t)$ is the acceleration.
- $P = 0$ on $\partial\Omega(t)$ implies: $\nabla P \perp \partial\Omega(t)$
- $\nabla P = i\mathfrak{a} z_\alpha$, where $\mathfrak{a} = -\frac{1}{|z_\alpha|}\frac{\partial P}{\partial \mathbf{n}}$

the first and third equations gives:

$$z_{tt} + i\mathfrak{a} z_\alpha = -\nabla\phi$$

Basic facts on Cauchy integral:

- Let $\Omega \subset \mathbb{C}$ be a bounded domain with chord-arc boundary $\partial\Omega : z = z(\alpha), \alpha \in [0, 2\pi]$, oriented counterclockwise. Define

$$\mathfrak{H} f(\alpha) := \frac{1}{\pi i} \int_0^{2\pi} \frac{z_\beta(\beta)}{z(\beta) - z(\alpha)} f(\beta)\, d\beta.$$

  then

- g is the boundary value of a holomorphic function G on Ω iff $g = \mathfrak{H} g$;

Let

$$\mathbb{P}_H := \frac{1}{2}(I + \mathfrak{H}), \quad \mathbb{P}_A := \frac{1}{2}(I - \mathfrak{H})$$

Then $\mathbb{P}_H$ is the projection onto the space of holomorphic functions in Ω, $\mathbb{P}_A$ is the projection onto the space of holomorphic functions in Ω^c.

The surface equation in Lagrangian coordinates

Equation of the free surface:

$$\begin{cases} z_{tt} + i\mathfrak{a} z_\alpha = -\nabla\phi \\ \bar{z}_t = \mathfrak{H}\bar{z}_t \end{cases} \tag{3}$$

where $\mathfrak{H}$ is the Hilbert transform.

$$-\nabla\phi := -2\partial_{\bar{z}}\phi = -\frac{\pi}{2}(I - \bar{\mathfrak{H}})z. \tag{4}$$

Is the initial value problem for system (3)-(13) locally wellposed?

The water wave equation

$$\begin{cases} \mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla P - (0,-1) & \text{in } \Omega(t) \\ \operatorname{div}\mathbf{v} = 0, \qquad \operatorname{curl}\mathbf{v} = 0, & \text{in } \Omega(t) \\ P = 0, & \text{on } \partial\Omega(t) \end{cases} \tag{5}$$

$\mathbf{v}$ is the fluid velocity, P is the fluid pressure.

The domain $\Omega(t)$ is unbounded and occupies the lower part of the space $\mathbb{R}^n$.

The surface equation in Lagrangian coordinates

Let $z = z(\alpha, t)$, $\alpha \in \mathbb{R}$ be the equation for the free surface in Lagrangian coordinate α.

Equation of the free surface for the 2d water waves:

$$\begin{cases} z_{tt} = i\mathfrak{a} z_\alpha - i \\ \bar{z}_t = \mathfrak{H}\bar{z}_t \end{cases} \tag{6}$$

where $\mathfrak{H}$ is the Hilbert transform,

$$\mathfrak{a} = -\frac{1}{|z_\alpha|}\frac{\partial P}{\partial \mathbf{n}}.$$

Local wellposedness of the water waves:

Local wellposedness of (6) in Sobolev spaces:

- S. Wu (1997):
- Key idea: derived a quasilinear equation by taking one time derivative to the first equation in (6). We showed that the Taylor sign condition always holds, this implies that the quasilinear equation is of the hyperbolic type, and consequently locally well-posed was proved.

Local wellposedness of the self-gravitating flow

By taking one time derivative to the first equation in (3) we get a quasilinear equation

$$\begin{cases} \bar{z}_{ttt} + i\mathfrak{a}\bar{z}_{t\alpha} = -i\mathfrak{a}_t\bar{z}_\alpha - \partial_t\overline{\nabla\phi} \\ \bar{z}_t = \mathfrak{H}\bar{z}_t \end{cases} \tag{7}$$

with the right hand side the lower order terms. Recall

$$\mathfrak{a} = -\frac{1}{|z_\alpha|}\frac{\partial P}{\partial \mathbf{n}} \geq c_o > 0$$

for our problem. The equation is of the hyperbolic type. So yes it is locally well-posed.

Steady states of the self-gravitating flow:

A disk moving with constant velocity v_0:

$$z(\alpha, t) := e^{i\alpha} + v_0 t$$

is a equilibrium solution.

The center of mass

We can show that the center of mass of the fluid body

$$C(t) := \frac{1}{|\Omega|} \int_{\Omega(t)} \mathbf{x} \, dx \, dy$$

always moves along a straight line with constant velocity, that is

$$\frac{d^2}{dt^2} C(t) = 0$$

By working in a frame moving with constant velocity, we can assume that the center of mass is fixed. That is $v_0 = 0$.

The question

Question: for a perturbation of size ϵ to the equilibrium state at $t = 0$, what is the lower bound of the life span of the solution?

- Local wellposedness of the problem gives a lower bound on the life span $O(\epsilon^{-1})$.
- Our main focus is to understand if the lower bound can be bigger: $O(\epsilon^{-2})$.

Main Theorem (Bieri, Miao, Shahshahani, and Wu)

Theorem (A lower bound for the life span)

Suppose $z(0,\alpha) = e^{i\alpha} + \epsilon f(\alpha)$ and $z_t(0,\alpha) = \epsilon g(\alpha)$ where f and g are smooth. Then if $\epsilon > 0$ is sufficiently small the local-in-time solution from the above theorem can be extended at least to $T^* = \frac{c}{\epsilon^2}$, where the constant c is independent of ϵ.

Key ideas

For the longer life span:

We find a transformation of the unknown and a new coordinate system, so that the new unknown in the new coordinate system satisfies an equation with no quadratic nonlinearity. That is, the equation is more linear when we view it in the right unknown in the right coordinate system.

History for self-gravitating flow

Assume that the Taylor sign condition holds:

$$-\frac{\partial P}{\partial \mathbf{n}} \geq c_0 > 0.$$

Lindblad & Nordgren (2009): A priori estimate for the 2d problem, without the irrotationality assumption.
Nordgren (2010): local in time wellposedness for the 3d problem in Sobolev spaces.

History for the water wave problem

- Local wellposedness in Sobolev spaces:
- Nalimov, Yosihara: small data, 2d
- S. Wu: arbitrary data, 2d and 3d
- with vorticity, surface tension, bottom, assuming the Taylor sign condition:
- Christodoulou & Lindblad, Ambrose & Masmoudi, Lindblad, D. Lannes, Coutand & Shkoller, P. Zhang & Z. Zhang, Shatah & Zeng, Alazard, Burq & Zuily .....

Global behavior for small, smooth and localized data for the water waves

- S. Wu (2009): almost global well-posedness for 2-D,
- S. Wu (2011): global well-posedness for 3-D
- Germain, Masmoudi & Shatah (2012): global well-posedness for 3-D
- Ionescu & Pusateri (2013): 2-D water waves, global existence and modified scattering
- Alazard & Delort (2013): similar result
- Hunter, Ifrim & Tataru (2014): 2-D water wave, almost global, global existence.
- Deng, Ionescu, Pulsader & Pusateri (2016): 3-D gravity-capillary water waves, global existence

The main idea of Wu (2009) for the water wave equation:

Constructed a near identity two-step transformation that consists of a change of the unknown function and a change of the coordinate system, so that the new unknown in the new coordinate system satisfies an equation that do not have any quadratic nonlinear terms.

Question: for the self-gravitating fluid

Is there a new unknown and a new coordinate system, so that the new unknown in the new coordinate system satisfies an equation that do not contain any quadratic nonlinear terms?

The main differences

- The equilibrium of the water waves is the flat interface.
- The nonlinear near identity transformation for the water waves is for perturbations near the flat interface.
- the gravity of the water waves is a constant vector, it is an external force.

- The equilibrium for the self-gravitating flow is a disk.
- A nonlinear near identity transformation for the self-gravitating flow, if exist, is for perturbations near the disk.
- the self-gravity nonlinearly depend on the interface. It is an internal force.

A transformation of the unknown function.

- Let $z = x + iy = z(\alpha, t)$: equation of the interface at time t.

Proposition

The quantity

$$\delta = (I - \mathfrak{H})(z\bar{z} - 1)$$

satisfies the equation:

$$(\partial_t^2 + i\mathfrak{a}\partial_\alpha - \pi)\delta = G \tag{8}$$

where G contains only cubic or higher order terms.

The left hand side still contains quadratic nonlinear terms. The right hand side is coordinate invariant.

Idea:

Look for an appropriate change of coordinates k.

- Chain rule:

$$U_{k^{-1}}(\partial_t^2 + i\mathfrak{a}\partial_\alpha - \pi)\delta = \{(\partial_t + b\partial_{\alpha'})^2 + iA\partial_{\alpha'} - \pi\}\delta \circ k^{-1}$$

- $U_{k^{-1}} f := f \circ k^{-1}$
- $$b = k_t \circ k^{-1}, \quad A = \mathfrak{a}k_\alpha \circ k^{-1}$$

- Goal: look for a coordinate change k so that

$$b = k_t \circ k^{-1}, \quad A - \pi = \mathfrak{a}k_\alpha \circ k^{-1} - \pi$$

  are quadratic.

Proposition

Let k be defined by

$$(I - \mathfrak{H})(\log \bar{z} + ik) = 0.$$

Then

$$b = k_t \circ k^{-1}, \quad A - \pi = \mathfrak{a} k_\alpha \circ k^{-1} - \pi$$

are quadratic.

- The equation: let $\delta \circ k^{-1} := \chi$.

$$\{(\partial_t + b\partial_{\alpha'})^2 + iA\partial_{\alpha'} - \pi\}\chi = \text{cubic}$$

- A new difficulty: is the operator $i\partial_{\alpha'} - 1$ positive?
- The term $-\pi\chi$ is from the self-gravity.

The energy estimate

- The energy for the equation

$$\{(\partial_t + b\partial_{\alpha'})^2 + iA\partial_{\alpha'} - \pi\}\Theta = \text{cubic}$$

where $\Theta = (I - \mathfrak{H})f$ is

-

$$E^{\Theta} := \int_0^{2\pi} \frac{|(\partial_t + b\partial_{\alpha'})\Theta|^2}{A} d\beta + \int_0^{2\pi} \left(i\Theta_\beta \overline{\Theta} - \frac{\pi}{A}|\Theta|^2 \right) d\beta. \tag{9}$$

- Main difficulty: The second term in (9) may be negative.
- we know for $\Theta = (I - \mathfrak{H})f$, the term

$$\int_0^{2\pi} i\Theta_\beta \overline{\Theta} d\beta$$

  is non-negative, by Green's identity.
- If $\partial\Omega(t)$ is the unit circle, Fourier expansion implies that

$$\int_0^{2\pi} \left(i\Theta_\beta \overline{\Theta} - |\Theta|^2 \right) d\beta \geq 0$$

- In fact we have

Lemma

Suppose $\Theta := (I - H)f$ *for some* 2π*-periodic function* f*. Then*

$$\int_0^{2\pi} \left(i\Theta_\beta \overline{\Theta} - |\Theta|^2\right) d\beta \geq$$
$$-\left(\|i\zeta_\beta\overline{\zeta} + 1\|_{L^\infty}\|\Theta\|_{L^2}^2 + \|\mu\|_{L^\infty}\|\Theta_\beta\|_{L^2}\|\Theta\|_{L^2}\right).$$

with $\zeta := z \circ k^{-1}$ *and* $\mu = |\zeta|^2 - 1$*.*

- $i\zeta_\beta\overline{\zeta} + 1$ and μ are small in the evolution. In fact for the static state $\zeta(t, \beta) = e^{i\beta}$, $i\zeta_\beta\overline{\zeta} + 1 = \mu = 0$

A comparison with the water wave case

Equation of the free surface for the 2d water waves:

$$\begin{cases} z_{tt} = i\mathfrak{a} z_\alpha - i \\ \bar{z}_t = \mathfrak{H} \bar{z}_t \end{cases} \tag{10}$$

- The new unknown is
$$\delta = (I - \mathfrak{H})(z - \bar{z})$$
- and the coordinate change k satisfies
$$(I - \mathfrak{H})(\bar{z} - k) = 0$$
- Let $\chi = \delta \circ k^{-1}$, then χ satisfies
-
$$\{(\partial_t + b\partial_{\alpha'})^2 - iA\partial_{\alpha'}\}\chi = \text{cubic}$$

  where b and $A - 1$ are quadratic.

Self-gravitating flow with constant vorticity

Ifrim & Tataru (2015): water waves with constant vorticity

The equation for self-gravitating flow with constant vorticity:
Let $\Omega(t)$ be the fluid domain, $\partial\Omega(t)$ be the interface at time t.

$$\begin{cases} \mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla P - \nabla\phi & \text{in } \Omega(t) \\ \operatorname{div}\mathbf{v} = 0, \quad \operatorname{curl}\mathbf{v} = 2\omega_0, & \text{in } \Omega(t) \\ P = 0, & \text{on } \partial\Omega(t) \\ \Delta\phi = 2\pi\chi_{\Omega(t)} & \end{cases} \tag{11}$$

here ω_0 is a constant.

$$\nabla\phi(x,t) = \int_{\Omega(t)} \frac{x-y}{|x-y|^2}\, dy$$

The Taylor sign condition

Proposition: The Taylor sign condition holds for our problem if

$$\omega_0^2 < \pi.$$

We derived an exact formula for

$$-\frac{\partial P}{\partial \mathbf{n}}$$

using the Riemann mapping, which shows the Proposition holds.

$$\Delta P = -\Delta\phi - |\nabla \mathbf{v}|^2 + |\mathrm{curl}\mathbf{v}|^2 = -2\pi + 4\omega_0^2 - |\nabla \mathbf{v}|^2.$$

Notice that for $\mathbf{v}_0 := \omega_0(y, -x)$, curl $\mathbf{v}_0 = 2\omega_0$, div $\mathbf{v}_0 = 0$, so

$$\operatorname{curl}(\mathbf{v} - \mathbf{v}_0) = 0, \quad \operatorname{div}(\mathbf{v} - \mathbf{v}_0) = 0$$

The surface equation is:

$$\begin{cases} z_{tt} + i\mathfrak{a} z_\alpha = -\nabla\phi \\ \bar{z}_t - i\omega_0 \bar{z} = \mathfrak{H}(\bar{z}_t - \imath\omega_0\bar{z}) \end{cases} \tag{12}$$

where $\mathfrak{H}$ is the Hilbert transform.

$$-\nabla\phi := -2\partial_{\bar{z}}\phi = -\frac{\pi}{2}(I - \bar{\mathfrak{H}})z. \tag{13}$$

Consider $\mathfrak{z} = e^{i\omega_0 t} z$.

The quantity

$$\tilde{\delta} = e^{-i\omega_0 t}\delta$$

where $\delta = (I - H)(z\bar{z} - 1)$ satisfies

$$(\partial_t^2 + i\mathfrak{a}\partial_\alpha - (\pi - w_0^2))\tilde{\delta} = G$$

where G contains only cubic and higher order terms. the coordinate change k satisfying

$$(I - H)(\log(\bar{z}e^{-i\omega_0 t + ik})) = 0$$

removes the quadratic nonlinear term on the left hand side.

Result: lifespan of $O(\epsilon^{-2})$ provided $\omega_0^2 < \pi$.

Thank you very much!