Stable high-order finite difference methods for nonlinear wave-structure interaction in a moving reference frame

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Theoretical and Compuational Aspects of Nonlinear Surface Waves
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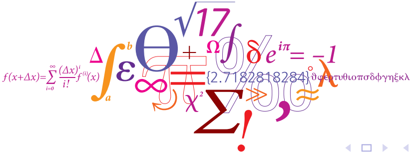


Motivation, Goals, Basic Strategy

Motivation: Green Ocean Energy-Green Shipping

- Offshore windmills
- Wave Power Devices
- Highly Efficient Ships

Goal: Efficient Inviscid Flow Solvers

1. To predict the nonlinear wave climate at a nearshore installation site
2. To predict the response of the structure(s) to the waves

Strategy: High-order Finite Difference + Fast Iterative Solvers

1. Solve the 3D nonlinear potential flow wave problem
2. Two paths to convergence *h*-type and *p*-type
3. Geometric discretization:
    - Overlapping curvilinear structured blocks (non-breaking waves)
    - Immersed boundary methods
    - Domain decomposition, Potential flow/Navier-Stokes

Nonlinear wave solver development at DTU

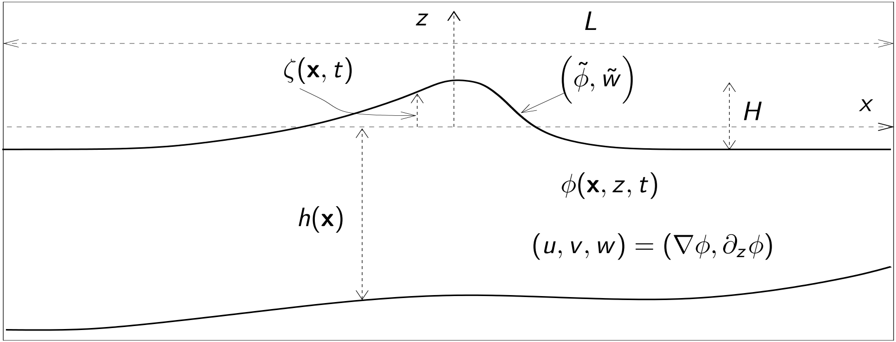
Fully nonlinear, extremely dispersive Boussinesq-type equations:

- Agnon, Madsen & Schäffer *J. Fluid Mech.* **399** (1999)
- Madsen, Bingham & Schäffer *Proc. Roy. Soy. Lond.* **359** (2003)
- Madsen, Bingham & Liu *J. Fluid Mech.* **462** (2002)
- Fuhrman & Bingham *Int. J. Numer. Meth. Fluids* **44** (2004)
- Fuhrman, Bingham & Madsen *Coastal Engineering* **52** (2005)
- Madsen, Fuhrman, Wang *Coastal Engineering* **53** (2006)
- Fuhrman & Madsen *Coastal Engineering* **55** (2008)
- Bingham, Madsen & Fuhrman *Coastal Engineering* **56** (2009)

Fully nonlinear potential flow solver (OceanWave3D):

- Bingham & Zhang *J. Eng. Math.* **58** (2007)
- Engsig-Karup, Bingham & Lindberg *J. Comp. Phys.* **228** (2009)
- Engsig-Karup, Madsen & Glimberg *Int. J. Numer. Meth. Fluids* **70** (2012)
- Kontos, Bingham & Lindberg *J. Hydrodynamics* **28** (2016)

The Basic Solution Strategy

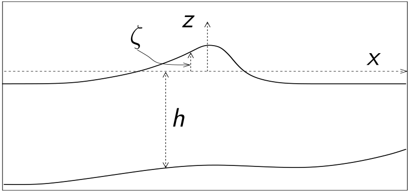


$$\partial_t \zeta = \quad -\nabla\zeta\cdot\nabla\tilde{\phi} + \tilde{w}(1+\nabla\zeta\cdot\nabla\zeta) \qquad \text{KFSBC}$$

$$\partial_t \tilde{\phi} = \quad -g\,\zeta - \frac{1}{2}\nabla\tilde{\phi}\cdot\nabla\tilde{\phi} + \frac{1}{2}\tilde{w}^2(1+\nabla\zeta\cdot\nabla\zeta) \qquad \text{DFSBC}$$

Laplace Problem for $\tilde{w}$ (Dirichlet to Neumann Operator)[1]

$$
\begin{aligned}
\nabla^2\phi + \partial_{zz}\phi &= 0, \qquad -h < z < \zeta \\
\partial_z\phi + \nabla h \cdot \nabla\phi &= 0, \qquad z = -h
\end{aligned}
$$



[1] Bingham & Zhang (2007) J. Eng. Math. **58**, (Li & Flemming (1997) Coastal Eng. **30**)

Laplace Problem for $\tilde{w}$ (Dirichlet to Neumann Operator)[1]

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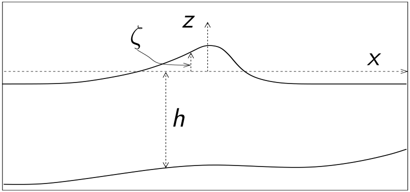

$\Downarrow$

Sigma transform the vertical coordinate:

$$\sigma(\mathbf{x}, z, t) = \frac{z + h(\mathbf{x})}{\zeta(\mathbf{x}, t) + h(\mathbf{x})}$$

[1] Bingham & Zhang (2007) J. Eng. Math. **58**, (Li & Flemming (1997) Coastal Eng. **30**)

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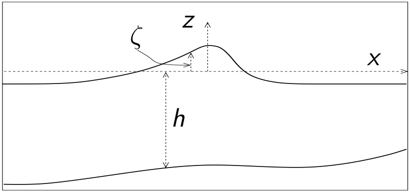


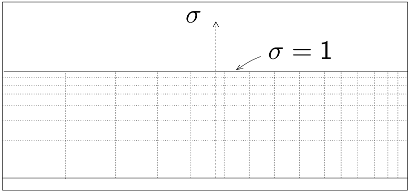
Sigma transform the vertical coordinate:

$$\sigma(\mathbf{x}, z, t) = \frac{z + h(\mathbf{x})}{\zeta(\mathbf{x}, t) + h(\mathbf{x})}$$

$\Downarrow$

$$\begin{aligned} \Phi &= \tilde{\phi}, \qquad \sigma = 1 \\ \nabla^2\Phi + \nabla^2\sigma(\partial_\sigma\Phi) + 2\nabla\sigma.\nabla(\partial_\sigma\Phi) &+ \\ \left(\nabla\sigma.\nabla\sigma + \partial_z\sigma^2\right)(\partial_{\sigma\sigma}\Phi) &= 0, \qquad 0 \le \sigma < 1 \\ (\partial_z\sigma + \nabla h.\boldsymbol{\nabla}\sigma)(\partial_\sigma\Phi) + \nabla h.\nabla\Phi &= 0, \qquad \sigma = 0 \end{aligned}$$

with $\Phi(\mathbf{x}, \sigma, t) = \phi(\mathbf{x}, z, t)$



- Gives a fixed computational geometry, no need to re-grid

[1] Bingham & Zhang (2007) J. Eng. Math. **58**, (Li & Flemming (1997) Coastal Eng. **30**)

Solution by Arbitrary-Order Finite Differences[2]

Fortran 90 open source code (https://github.com/apengsigkarup/OceanWave3D-Fortran90)

- Structured, but non-uniform grid.
- Choose $p+1$ neighbors to develop 1D, p order FD schemes.
- Leads to a linear system

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

  $\mathbf{A}$ is sparse with at most $(p+1)^d$, non-zeros per row, in $d = 2, 3$ dimensions.
- GMRES iterative solution preconditioned by the linearized, 2nd-order version of the matrix:

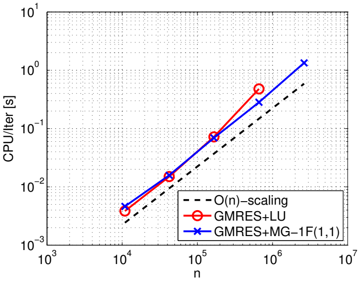
$$\mathbf{A}_2^{-1} \{\mathbf{A}(t)\,\mathbf{x} = \mathbf{b}\}$$

- One multigrid cycle for the preconditioning step.
- Solution in $O(10)$ iterations, independent of physics and # of grid points N.
- Time stepping by the classical 4th-order Runge-Kutta scheme.

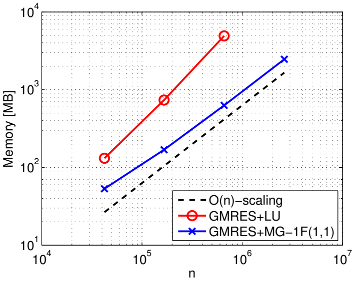
[2] Engsig-Karup, Bingham & Lindberg (2009) J. Comp. Phys. **228**

Scaling of the solution in 3D

Nonlinear test case, 6^{th}-order accurate operators



CPU time



RAM memory use

Massively parallel C++/CUDA GPU implementation [3]

- Critical to absolutely minimize memory use
- All FD coefficients and transformation weights are re-computed when needed (not stored!)
- GMRES is replaced by the defect-correction scheme (no extra vectors to save), iteration count is roughly doubled
- 40 - 100 times speed up for 1 GPU unit vs. 1 CPU

[3] Engsig-Karup et al (2012) *Int. J. Num. Meth. Fluids* **70**

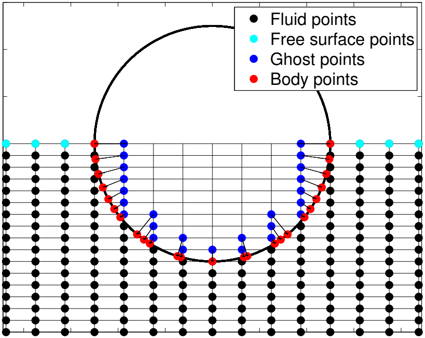
Wave-Body Interaction - Immersed Boundary Method[4]

The solution is built in CUDA/C++ on GPU architectures

Body boundary condition:

$$\mathbf{n} \cdot \nabla \phi = \mathbf{n} \cdot \mathbf{u}_B, \quad \mathbf{x} \in S_B,$$

- Identify fluid/ghost points.
- The Laplace equation is solved only on fluid points.
- Body points: Projection of ghost points onto the body.
- Form Weighted Least Square (WLS) stencil for each body point.
- Use the WLS method to approximate the normal derivative.



[4] Ole Lindberg post-doc (2012-2014) & Stavros Kontos, PhD project (2013-2016).

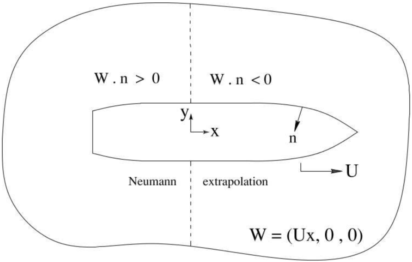
The Linear Forward Speed Problem

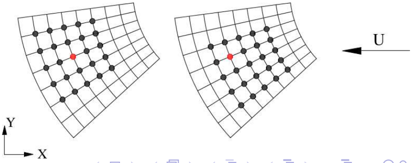
$\frac{\partial}{\partial t}\Big|_{\text{fixed}} = \frac{\partial}{\partial t}\Big|_{\text{moving}} - U\frac{\partial}{\partial x}$, simple upwinding is stable

$$\partial_t\zeta - U\,\partial_x\zeta = \partial_z\tilde{\phi} \quad \text{KFSBC}$$
$$\partial_t\tilde{\phi} - U\,\partial_x\tilde{\phi} = -g\zeta \quad \text{DFSBC}$$

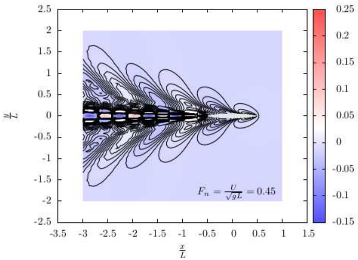
"Upwinding" on the ship boundary $\frac{\partial\phi}{\partial n}$ on S_b.

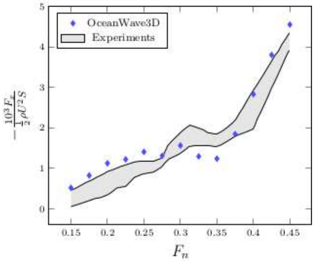
Upwind-biased convection $\frac{\partial}{\partial x}$ on $z = 0$.

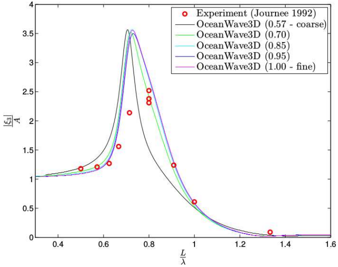


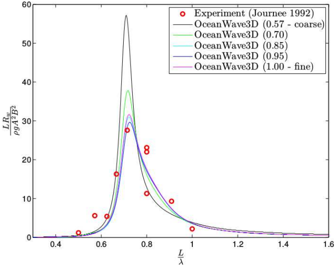


Linear Resistance, Seakeeping and Added Resistance



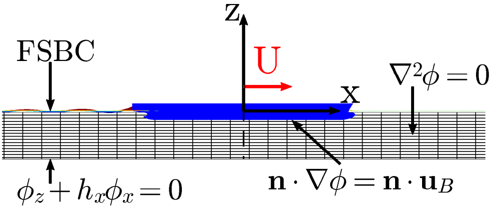






A WENO Scheme for Nonlinear Ship Motions[5]

The Non-Linear Forward Speed Problem



$$\partial_t\zeta + \partial_x\zeta\left(\partial_x\tilde{\phi} - \partial_z\tilde{\phi}\,\partial_x\zeta - U\right) = \partial_z\tilde{\phi}$$

$$\partial_t\tilde{\phi} + \partial_x\tilde{\phi}\left(\tfrac{1}{2}\partial_x\tilde{\phi} - U\right) - \tfrac{1}{2}(\partial_z\tilde{\phi})^2\,(1 + \partial_x\zeta\,\partial_x\zeta) = -g\zeta$$

- Simple upwinding of the convective terms is not stable.

[5]Stavros Kontos, PhD Thesis (2013-2016).

A WENO Scheme for Nonlinear Ship Motions[5]

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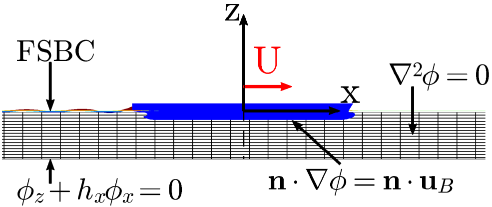

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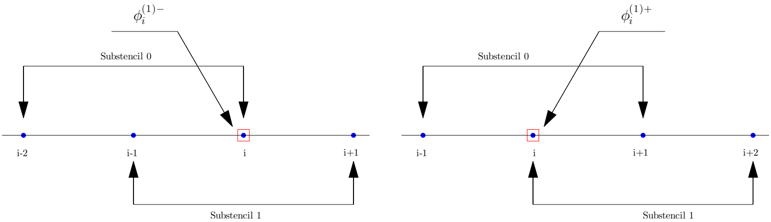
- Simple upwinding of the convective terms is not stable.
- Nonlinear convective problems require a nonlinear convective scheme!
- Motivated by the work of Osher & Shu et al, we have developed a nonlinear Weighted Essentially Non Oscillatory (WENO) scheme.

[5] Stavros Kontos, PhD Thesis (2013-2016).

The 1-D ENO Finite Difference scheme[6]

We need an approximation to the convective term $\frac{\partial \phi}{\partial x}$ at grid point i, $\phi_i^{(1)} \approx \left.\frac{\partial \phi}{\partial x}\right|_{x=x_i}$



Left-biased stencil ($p = 2$). Right-Biased stencil ($p = 2$).

Core Idea:

- Use only the smoothest sub-stencil to obtain a p^{th}-order approximation of $\phi_i^{(1)-/+}$.
- Combine the plus and minus approximation with an appropriate flux to obtain the final result.

[6] Osher & Shu (1991) *SIAM J. Numer. Anal.*

1-D WENO Finite Difference scheme[7]

Core Idea: Combine the approximations on all sub-stencils using non-linear weights ω_s:

$$\phi_i^{(1)-} = \sum_{s=0}^{p-1} \omega_s \phi_{s,i}^{(1)}$$

- If the solution is locally smooth, exploit the full extended stencil width.
  $\Rightarrow$ High Accuracy: $(2p-1)^{th}$-order.
- Any stencils that contain discontinuities are weighted to zero.
  $\Rightarrow$ Convergent approximation at p^{th} order accuracy.

[7] See *e.g.* C.-W. Shu (2009) *SIAM Review*

1-D WENO Finite Difference scheme

Non-Linear Weights

Non-Linear Weights ω_s

$$\omega_s = \frac{a_s}{\sum_{s=0}^{p-1} a_s}, \quad a_s = \frac{d_s}{(\epsilon + \beta_s)^2}, \quad s = 0, \ldots, p-1$$

The d_s: constant linear weights. Smooth solution $\Rightarrow$ $(2p-1)^{th}$-order.

The β_s: "smoothness indicators" which become large whenever discontinuities exist in the solution.

The $\epsilon = 10^{-6}$ avoids division by zero.

1-D WENO Finite Difference scheme[8]

Automated Derivation of the Linear Weights d_s

- The d_s have been derived using symbolic manipulation and tabulated in the litterature up to $p = 7$
- In fact they can be computed to arbitrary order by solving a simple Vandermonde-type system similar to the derivation of FD schemes

Conditions:

- $\phi_i^{(1)-} = \sum_{s=0}^{p-1} d_s \phi_{s,i}^{(1)} \Rightarrow (2p-1)^{th}$-order.
- $\sum_{s=0}^{p-1} d_s = 1$

We seek p coefficients which set to zero the first $p-1$ truncation error terms in the Taylor series expansion of the combined derivative approximation; and sum to one.

[8] Kontos et al (2016) J. Hydrodynamics

1-D WENO Finite Difference scheme

Automated Derivation of the Linear Weights d_s

An example

Consider the left-hand derivative approximation with $p = 2$:

$$\phi^{(1)}_{0,i} = \frac{1}{\Delta x}\left(\frac{1}{2}\phi_{i-2} - 2\phi_{i-1} + \frac{3}{2}\phi_i\right)$$
$$\phi^{(1)}_{1,i} = \frac{1}{\Delta x}\left(-\frac{1}{2}\phi_{i-1} + \frac{1}{2}\phi_{i+1}\right)$$

with leading truncation error terms

$$\phi^{(1)}_i = \phi^{(1)}_{0,i} - \frac{1}{3}\phi^{(3)}_i\,\Delta x^2 + \ldots$$
$$\phi^{(1)}_i = \phi^{(1)}_{1,i} + \frac{1}{6}\phi^{(3)}_i\,\Delta x^2 + \ldots$$

where $\phi^{(n)}_i$ indicates the exact nth derivative of ϕ at grid point i.

1-D WENO Finite Difference scheme

Automated Derivation of the Linear Weights d_s

An example

Set up a system of equations for the linear weights d_s:

$$\begin{bmatrix} -\frac{1}{3} & \frac{1}{6} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} d_0 \\ d_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

which gives: $d_0 = 1/3$, $d_1 = 2/3$, and the full order $2r - 1$ approximation:

$$\phi_i^{(1)-} \quad = \sum_{s=0}^{1} d_s \, \phi_{s,i}^{(1)} \quad = \frac{1}{\Delta x}\left(\frac{1}{6}\phi_{i-2} - \phi_{i-1} + \frac{1}{2}\phi_i + \frac{1}{3}\phi_{i+1}\right)$$

which is exactly the 4-point, 3$^{\mathrm{rd}}$-order approximation of $\phi_i^{(1)}$ which is obtained from a direct derivation of the coefficients.

- Recovers all tabulated d_s values in the literature
- Easy to code and implement arbitrarily high-order schemes
- Can be applied on non-uniform grids

1-D WENO Finite Difference scheme[9]

Smoothness Indicators

- Also here, integrated forms have been derived using symbolic manipulation and tabulated in the litterature up to $p = 7$

We propose a simplified smoothness indicator defined by:

$$\beta_s = \sum_{n=2}^{p} \left(\phi_{s,i}^{(n)} \, \Delta x^{n-1} \right)^2$$

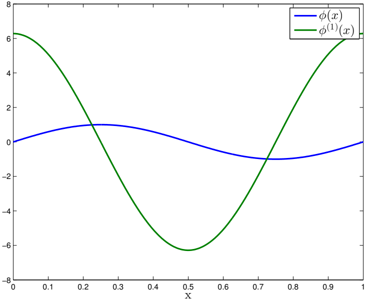
the sum of all possible higher derivatives on the stencil, scaled to have units of velocity (m/sec). Turns out to be a good measure of the smoothness of the velocity.

[9] Kontos et al (2016) J. Hydrodynamics

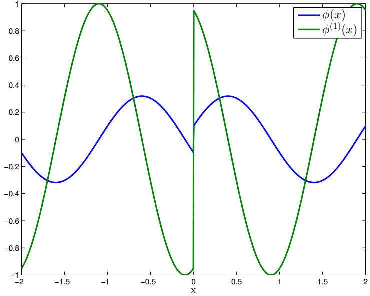
1-D WENO Finite Difference scheme

Smoothness Indicators comparison

The Smooth and Discontinuous Test Functions



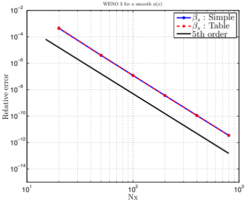
Smooth $\phi(x)$.



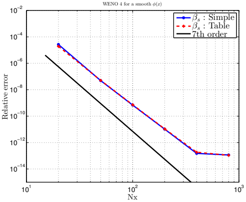
Discontinuous $\phi(x)$.

1-D WENO Finite Difference scheme

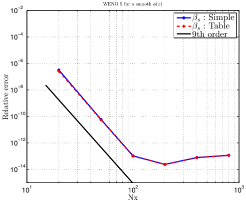
Convergence of the derivative of a Smooth Function



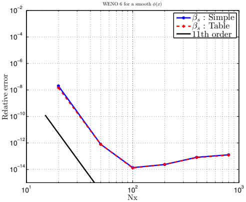
WENO 3.



WENO 4.



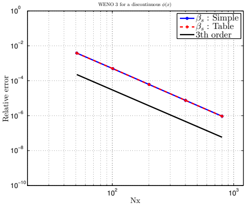
WENO 5.



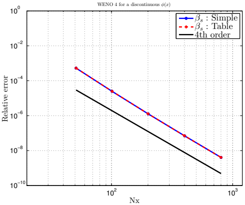
WENO 6.

1-D WENO Finite Difference scheme

Convergence of the derivative of a Discontinuous Function



WENO 3.



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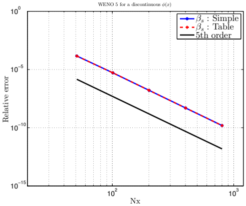

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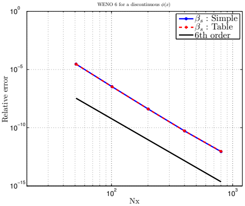


WENO 6.

Hamilton-Jacobi Equations

General form:

$$\phi_t + H(\nabla\phi) = S$$

Spatial discretization ⇒ Numerical Hamiltonian

$$\hat{H}(\phi_x^-, \phi_x^+, \phi_y^-, \phi_y^+, \phi_z^-, \phi_z^+) \approx H(\nabla\phi)$$

- Numerical Hamiltonian approximation applied here:
  - Local Local Lax-Friedrichs Scheme[10]

[10] Shu & Osher *J. Comp. Phys.* **77(2)** (1989)

Hamilton-Jacobi Equations

Lax-Friedrichs Scheme

The Local Local Lax-Friedrichs scheme is given by:

$$\hat{H} = H\left(\frac{\phi_x^- + \phi_x^+}{2}, \frac{\phi_y^- + \phi_y^+}{2}\right) - a^x\left(\frac{\phi_x^+ - \phi_x^-}{2}\right) - a^y\left(\frac{\phi_y^+ - \phi_y^-}{2}\right)$$

where a^x and a^y are dissipation coefficients for controlling the amount of numerical viscosity. They are defined as:

$$a^x = max|H_1(\phi_x, \phi_y)^{\pm}_{i,j}|, \qquad a^y = max|H_2(\phi_x, \phi_y)^{\pm}_{i,j}|$$

H_1 and H_2 are the partial derivatives of H with respect to ϕ_x and ϕ_y, respectively and only the values at grid point i, j are considered.

WENO on the Non-Linear Forward Speed Problem

WENO formulation of the FSBCs:

$$\partial_t \zeta + H_\zeta = \partial_z \tilde{\phi}$$

$$\partial_t \tilde{\phi} + H_\phi = -g\zeta$$

where RHS terms $\Rightarrow$ source terms and

- $H_\zeta = \partial_x \zeta \left(\partial_x \tilde{\phi} - \partial_z \tilde{\phi}\, \partial_x \zeta - U \right)$
- $H_\phi = \partial_x \tilde{\phi} \left(\frac{1}{2} \partial_x \tilde{\phi} - U \right) - \frac{1}{2} (\partial_z \tilde{\phi})^2 \left(1 + \partial_x \zeta\, \partial_x \zeta\right)$

The dissipation coefficients are:

- $a^x_\zeta = max|H_{1,\zeta}(\zeta_x, \tilde{\phi}_x)^\pm_{i,j}| = max|(\tilde{\phi}_x - 2\tilde{\phi}_z\, \zeta_x - U)^\pm_{i,j}|$
- $a^x_\phi = max|H_{1,\phi}(\zeta_x, \tilde{\phi}_x)^\pm_{i,j}| = max|(\tilde{\phi}_x - U)^\pm_{i,j}|$

WENO on the Non-Linear Forward Speed Problem

Representative Test Case

WENO 4 & 6 vs Upwind 6th order FD
Steep Stream Function Wave in Deep Water.

- Wave height: 90% of the stable limit, $H/L = 0.1273$
- Periodic lateral boundary conditions
- x-direction: uniform grid
- z-direction: cosine stretched grid
- $-4c \leq U \leq 4c$.
- $Cr = 0.5$.
- $Nx = 64$, $Nz = 9$
- The solution is propagated for ten periods and

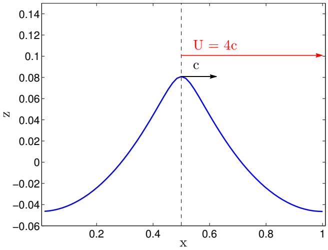
energy conservation is measured:

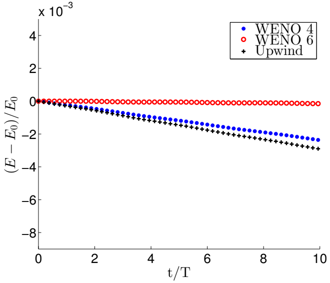
$$E = \frac{\rho}{2} \int_{S_0} (\tilde{\phi}\zeta_t + g\zeta^2) dxdy$$

WENO on the Non-Linear Forward Speed Problem

Test Case Results

$U = 4c$

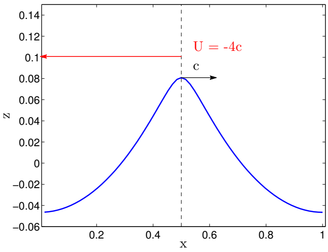


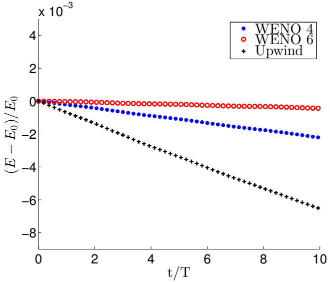


WENO on the Non-Linear Forward Speed Problem

Test Case Results

$U = -4c$

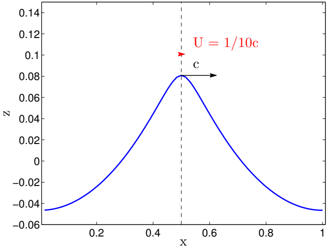


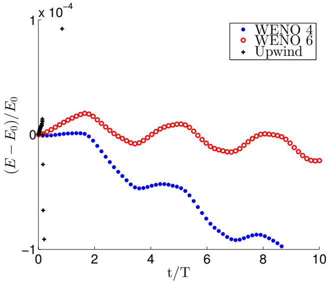


WENO on the Non-Linear Forward Speed Problem

Test Case Results

$U = 1/10c$

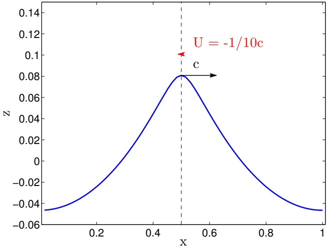


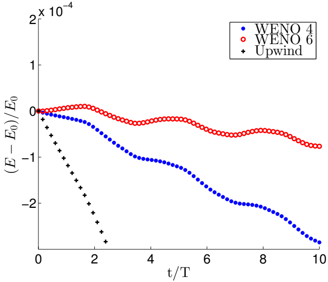


WENO on the Non-Linear Forward Speed Problem

Test Case Results

$U = -1/10c$

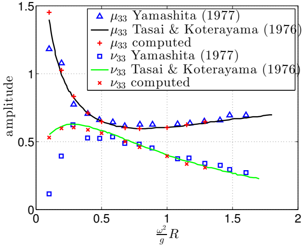




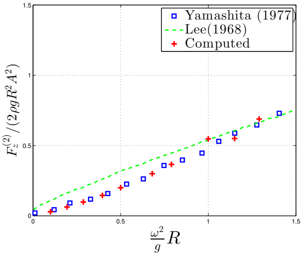
2D Results

A heaving circular cylinder, $U = 0$

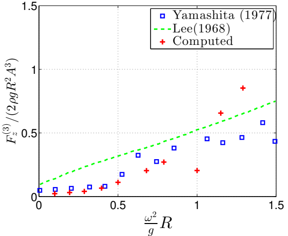
First-Order Forces



Second-Order Forces



Third-Order Forces

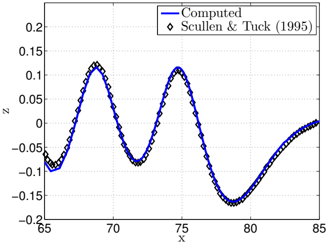


2D Results

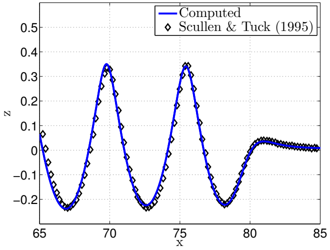
A submerged cylinder at steady forward speed

Steady wave pattern behind the cylinder

Froude number = 0.4



Froude number=0.8



Conclusions

- High order finite difference methods are a viable alternative to BEM methods for efficient nonlinear potential flow wave-structure interaction
- Nonlinear convective problems require nonlinear discrete convective schemes to achieve stability
- The WENO scheme provides stability while maintaining high-order numerical accuracy

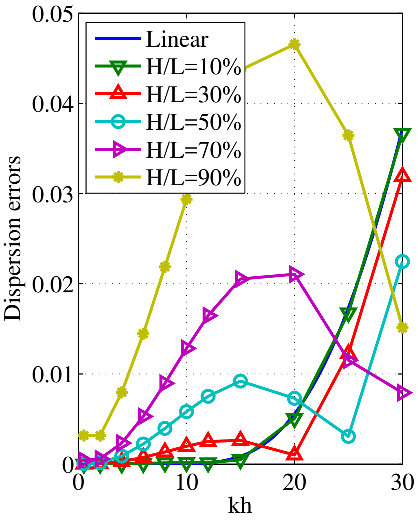
Challenges

- Free-body resonse - couple with the equations of motion
- Capturing the free-surface/body intersection for large motions
- Rational treatment of wave breaking

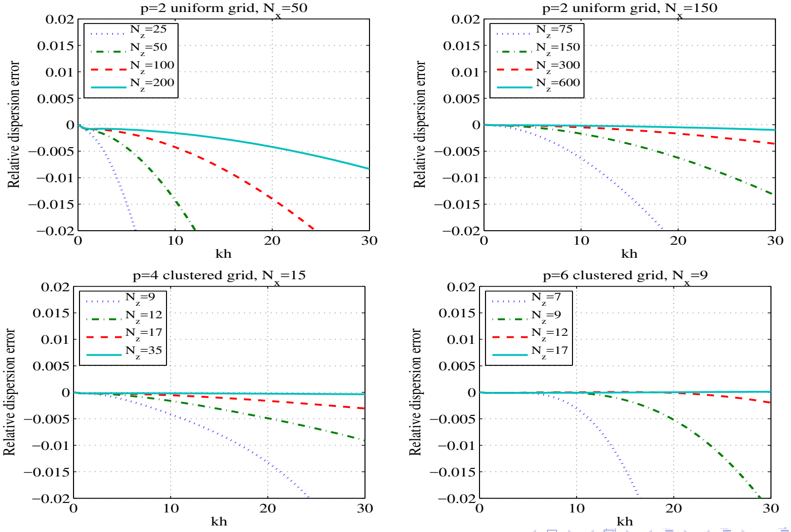
Thanks for your attention!

Error in dispersion for the Padé (4,4) Boussinesq model

(For reference)

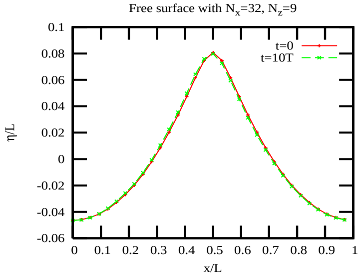


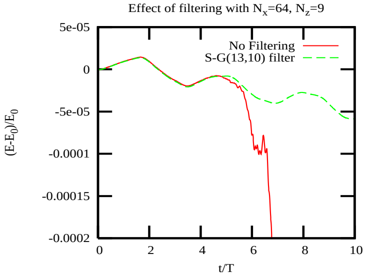
OceanWave3D error in linear dispersion

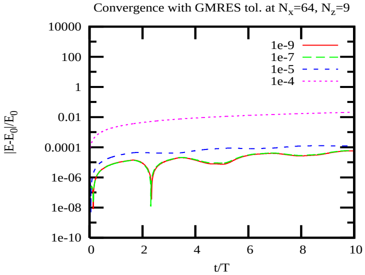


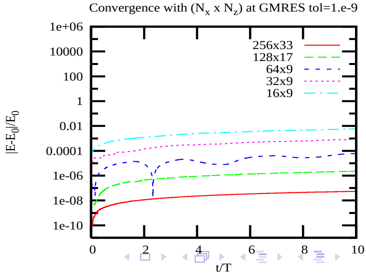
Highly nonlinear deep water waves (stream function theory)

$kh = 2\pi$, H/L =90% of breaking, $p = 6$, $C_r = 0.5$

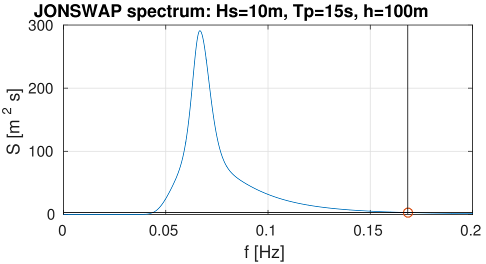








GPU code typical application example



A 50 year storm condition in the North Sea

- Resolve components with $S \geq 0.01 S_{max} \geq 8$ points per λ
- 7 by 7 km domain gives $N \approx 20$ million
- CPU time on one GPU: ≈ 12 min. per peak period
- 1 hour real time $\rightarrow$ 48 hours CPU time

In progress:

- MPI extension to multiple GPU units
- Wave breaking model