Sufficiently strong dispersion removes ill-posedness of truncated series models of water waves

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Collaborators and References

Collaborators:

- Shunlian Liu (Drexel PhD student)
- Jerry Bona (University of Illinois at Chicago)
- David Nicholls (University of Illinois at Chicago)
- Michael Siegel (New Jersey Institute of Technology)

References:

1. D.M. Ambrose, J.L. Bona, and D.P. Nicholls. On ill-posedness of truncated series models of water waves. *Proc. Roy. Soc. A*, **470**:20130849, 2014.

2. D.M. Ambrose, J.L. Bona, and D.P. Nicholls. Well-posedness for a model of water waves with viscosity. *Discrete Cont. Dyn. Syst. Ser. B*, **17**:1113-1137, 2012.

Well-Posedness of the Water Wave Problem

- We consider the initial value problem for the irrotational, incompressible Euler equations, bounded above by a free surface.
- In the absence of the upper fluid, there is no Kelvin-Helmholtz instability.
- The initial value problem has been shown to be well-posed in Sobolev spaces by Wu, Lannes, Ambrose-Masmoudi, and others.
- Wu used a Riemann mapping (2D) and Clifford analysis (3D). Lannes made a change of variables and used a Nash-Moser iteration. Ambrose-Masmoudi used a Hou-Lowengrub-Shelley-type formulation, made a change of variables, and made estimates uniform in surface tension. (These changes of variables are related to the concept of the "good unknown" of Alinhac.)
- Overall, it seems fair to say that the well-posedness theory turned out to be quite subtle.

Truncated Series Models

- We can write the Zakharov-Craig-Sulem formulation of the water wave problem:

$$\partial_t \eta = Y(\eta)[\xi] - (\partial_x \eta) X(\eta)[\xi],$$

$$\partial_t \xi = -g\eta + \frac{1}{2}(Y(\eta)[\xi])^2 - \frac{1}{2}(X(\eta)[\xi])^2 - (\partial_x \eta) X(\eta)[\xi] Y(\eta)[\xi].$$

- Here, $\eta(x,t)$ is the height of the free surface. We have the velocity potential $\varphi(x,y,t)$, and $\xi(x,t) = \varphi(x,\eta(x,t),t)$.
- The operators X and Y are related to the Dirichlet-to-Neumann operator for the fluid domain (which depends on η). [$X = \partial_x v$, $Y = \partial_y v$.]

Truncated Series Models, continued

- The operators $X(\sigma)$ and $Y(\sigma)$ can be expanded as series. Here, σ is a putative surface deformation.
- We let $\sigma = \varepsilon f$, and we write

$$X(\varepsilon f) = \sum_{n=0}^{\infty} \varepsilon^n X_n(f), \qquad Y(\varepsilon f) = \sum_{n=0}^{\infty} \varepsilon^n Y_n(f).$$

- We can work out expressions for the operators X_n and Y_n. For instance, $X_0(f)[\xi] = \partial_x \xi$, $Y_0(f)[\xi] = \Lambda\xi = H\partial_x \xi$, and $X_1(f)[\xi] = (\partial_x f)\Lambda\xi$. ($H$ is the Hilbert transform.)
- The truncated models are formed by truncating these series and substituting the resulting finite sums for X and Y into the evolution equations. (There are other ways to do this, such as truncating in the Hamiltonian instead.)

A Question

- This type of model is widely used for simulation (Craig-Sulem, Guyenne, Nicholls, etc...).
- As we said, the water wave problem is known to be well-posed, and the truncated series models appear to be reasonable approximations to the water wave equations.
- **But, it is also reasonable to ask, Do the truncated series models have well-posed initial value problems?**
- As we said, the estimates needed to establish well-posedness of the water wave problem turned out to be subtle; should we expect the truncations to preserve the structure of the equations necessary to make these estimates?
- Ambrose-Bona-Nicholls showed that a viscous regularization (in the spirit of Dias-Dyachenko-Zakharov) leads to well-posedness; the question of well-posedness without the viscous terms arises if one were to attempt the zero viscosity limit.

Some Simple Models

- The quadratic model:

$$\eta_t = \Lambda(\xi) - \partial_x \{[H, \eta]\Lambda(\xi)\},$$

$$\xi_t = -g\eta + \boxed{\frac{1}{2}(\Lambda(\xi))^2} - \frac{1}{2}\xi_x^2.$$

- The boxed term is a parabolic term of indefinite sign. This could lead to catastrophic growth (similarly to the backwards heat equation), if there is no way to control it.
- To study this, we begin by introducing a reduced model:

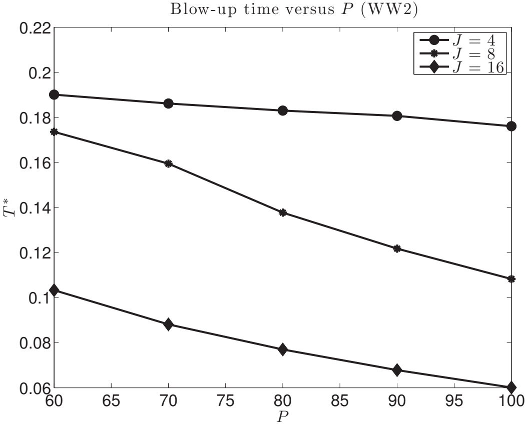
$$\xi_t = (\Lambda(\xi))^2 - \xi_x^2.$$

- This reduced model is in fact (essentially) a complex Burgers equation; this is ill-posed. We prove ill-posedness another way, adapting a method of Bona-Weissler.
- We use what we learn from this ill-posedness proof of the reduced model to provide numerical evidence that the full truncated model is ill-posed.

Reduced Model – Summary

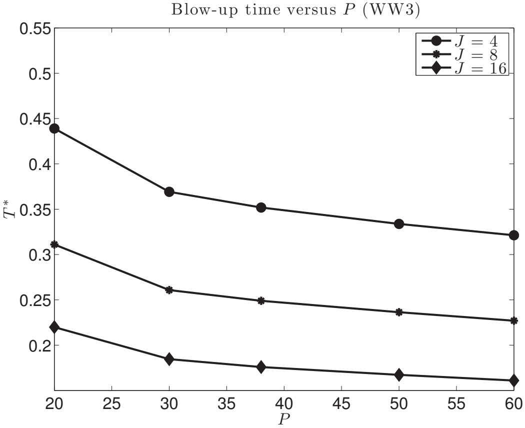
- We find blow-up for certain families of initial data, such as $\xi(x, 0) = \frac{1}{j}\cos(jx)$, with blow-up time proportional to $\frac{1}{j}$.
- This implies ill-posedness of the reduced model – arbitrarily small data yield solutions blowing up arbitrarily quickly.
- While this data is only small in H^s for $s < 1$, we also have versions of this blowup theorem for the reduced model showing ill-posedness in H^s for $s < 3$. Ill-posedness in more regular spaces is certainly true for this reduced model.
- **Can we observe the same behavior in the full truncated model?** The question is, Does the inclusion of the dispersive effects and additional nonlinear terms allow for control of this catastrophic growth?

Numerical Examples



- The data here is $\eta(x,0) = 0, \quad \xi(x,0) = \frac{1}{J^{1/10}}\cos(x) + \frac{2}{J^{5/2}}\cos(Jx)$.

More Numerical Examples



- The data here is $\eta(x,0) = 0, \quad \xi(x,0) = \frac{1}{J^{3/2}} \cos(Jx)$.

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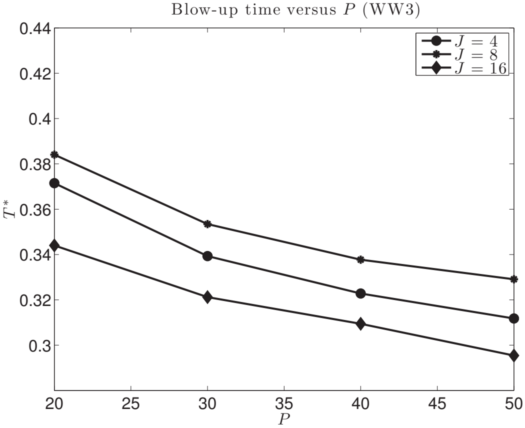

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A full proof of ill-posedness

- In joint work with Michael Siegel (manuscript in preparation), we believe we have found a full proof of ill-posedness of the WW2 model.
- This is based on the Caflisch-Orellana proof of ill-posedness of vortex sheets. The method has been adapted to unstable Darcy flow by Siegel-Caflisch-Howison.
- For those problems, linear ill-posedness was leveraged to get nonlinear ill-posedness: complexify the equations, start with data with a singularity in the complex plane off the real axis, and which will (under the linear evolution) reach the real axis at a certain time.
- Continuing, one shows that the nonlinear terms cannot stop the singularity formation, and that the argument supports arbitrarily small data having real singularities arbitrarily fast.

More on the full proof of ill-posedness

- A main ingredient in the proof is a version of Cauchy-Kowalewski which allows weak singularities (Caflisch).
- A primary difference is that we do not have linear ill-posedness; instead, our singular solutions come from the reduced model, which is nonlinear. (The reduced model is actually a complex Burgers equation.)
- The proof is more complicated than cases for which ill-posedness is driven by linear behavior, but is in many respects similar.
- Using time reversal, we start with a singularity on the real axis:

$$u_+(x_0,0) = ae^{ix_0} + \varepsilon\left(1-\delta e^{ix_0}\right)^{\nu} e^{ix_0}.$$

- The singularity moves approximately with speed $2a$ on the imaginary axis.
- Estimates show that the singularity does move as expected under the nonlinear evolution.

Effect of stronger dispersion

- The problematic term, in a straightforward energy estimate, would contribute terms with 1/2 more derivative than can be bounded.
- Dispersive equations are known to possess smoothing properties. Can we use dispersion to balance this term?
- Dispersion does not automatically control parabolic growth. Example: $u_t = u_{xxx} - u_{xx}$ is ill-posed.
- We prove that sufficiently strong dispersion does regularize the truncated series models; this is part of the thesis work of Drexel PhD student Shunlian Liu.
- We find that with a dispersion relation of order two or higher (i.e., $p \geq 3$), we have short-time well-posedness of the following modification of WW2:

$$\eta_t = \Lambda(\xi) - \partial_x \left\{ [H, \eta] \Lambda(\xi) \right\}, \qquad \xi_t = -g\Lambda^p(\eta) + \frac{1}{2}(\Lambda(\xi))^2 - \frac{1}{2}\xi_x^2.$$

About the dispersion

- We just changed the dispersive term in the ξ_t equation for the purpose of studying the effect of stronger dispersion; this is not strictly motivated by physical concerns.
- The following table summarizes values of p of interest:

Value of p	Power of dispersion relation	Relevance
0	1/2	gravity, ill-posed
2	3/2	surface tension
3	2	threshold for our well-posedness proof
4	5/2	hydroelastic waves

- This leaves an interesting question – can we prove ill-posedness or well-posedness for the surface tension case?

Brief comments about the proof for $p \geq 3$

- The proof uses a change of variables to replace ξ :

$$w = \xi - T_{\Lambda\xi}\eta.$$

- This uses the paradifferential operator $T_{\Lambda\xi}$; we follow the approach of Alazard-Burq-Zuily in defining and using such operators.
- The proof is by the energy method; an approximation is constructed, the Picard theorem is used to prove existence for the approximate solutions, an estimate is made which is uniform in the approximation parameter, and the limit is taken as the approximation parameter vanishes.
- The condition $p \geq 3$ is used in the energy estimate, to bound some terms by the energy.

Closing remarks

- I conjecture that well-posedness or ill-posedness is independent of the order of truncation.
- I conjecture that $p \geq 3$ is sharp for well-posedness.
- I suspect that the method can be reformulated, taking into account the water wave existence theory, to generate other truncated series models which will have well-posed initial value problems in the gravity and surface tension cases.

Thanks for your attention.