Two dimensional water waves
- finite depth case -

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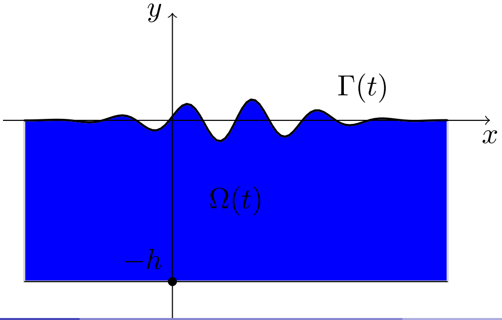
Two dimensional fluids

The setting:

- inviscid incompressible fluid flow (governed by the incompressible Euler equations)
  - irrotational flow
  - with gravity and no surface tension
- fluid is considered in an infinitely wide domain and above a flat, finite bottom at $y = -h < 0$
- free boundary (the interface with air)



The Eulerian formulation

Fluid domain: $\Omega(t)$, free boundary $\Gamma(t)$.

Velocity field u, pressure p, gravity g.

Euler equations in $\Omega(t)$:

$$\begin{cases} u_t + u \cdot \nabla u = \nabla p - g\mathbf{j} \\ \text{div } u = 0 \\ \text{curl } u = 0 \\ u(0,x) = u_0(x) \end{cases}$$

Boundary conditions on $\Gamma(t)$:

$$\begin{cases} \partial_t + u \cdot \nabla \text{ is tangent to } \bigcup \Gamma(t) & \text{(kinematic)} \\ p = p_0 \quad \text{on } \Gamma(t) & \text{(dynamic)} \end{cases}$$

Assume the bottom is impermeable,

$$u \cdot \mathbf{j} = 0 \text{ on } \{y = -h\},$$

Reduction to the boundary for irrotational flows

Velocity potential ϕ which satisfies

$$\begin{cases} u = \nabla\phi, \qquad \Delta\phi = 0 \quad \text{in } \Omega(t) \\ \partial_y \phi = 0, \quad \text{on } y = -h \end{cases}$$

As a consequence ϕ is uniquely determined by its trace on the boundary

$$\psi = \phi|_{\Gamma(t)}$$

- Equations reduced to the boundary in Eulerian formulation in (η, ψ), where η is the elevation and $\psi(t,x) = \phi(t, \eta(t,x)))$:

$$\begin{cases} \partial_t \eta - G(\eta)\psi = 0 \\ \partial_t \psi + g\eta + \frac{1}{2}|\nabla\psi|^2 - \frac{1}{2}\frac{(\nabla\eta\nabla\psi + G(\eta)\psi)^2}{1 + |\nabla\eta|^2} = 0. \end{cases}$$

where G is the Dirichlet to Neuman on the free surface.

Previous work

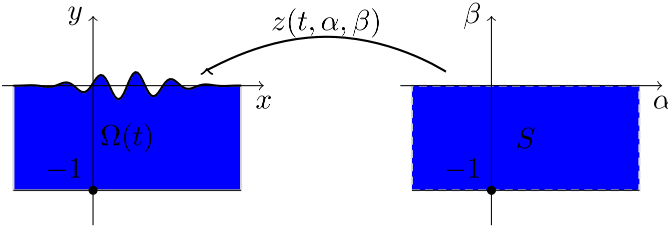
- Local well-posedness (mixture of dimensions and models)
  - Nalimov, Yosihara, Wu, Christodoulou-Lindblad, Lannes, Lindblad, Coutand-Shkoller, Shatah-Zeng, Alazard-Burq-Zuilly, Nguyen...
- Enhanced lifespan (∞ depth)
  - $2d$ **almost global:** Wu, Hunter-I.-Tataru
  - $2d$ **global:** Alazard-Delort, Ionescu-Pusateri, I.-Tataru, Wang
  - **Other** $2d$ **models:** I.-Tataru, Ionescu-Pusateri
  - $3d$ **global:** Wu, Germain-Masmoudi-Shatah, Deng-Ionesu-Pausauder-Pusateri
- Enhanced lifespan (finite depth)
  - $3d$ **enhanced lifespan:** Alvarez-Samaniego-Lannes
  - $3d$ **global:** Wang
  - $2d$**:** Berti-Delort (periodic, gravity-capillary, a.e.)

GOAL: Initiate study of long time dynamics in $2d$ nonperiodic case:

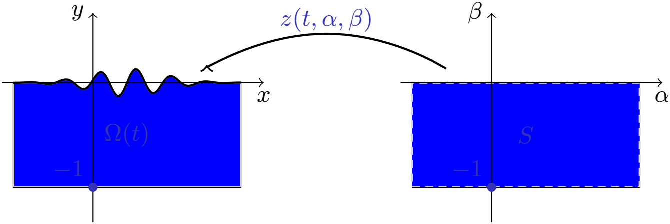
- Rigorously formulate problem in holomorphic coordinates
- Low regularity local well-posedness for large initial data
- Enhanced (cubic) lifespan bounds for small initial data

Holomorphic coordinates

- We use holomorphic coordinates
  - Ovsjannikov, Dyachenko-Zakharov-Kuznetsov, Wu, Choi-Camassa, Li-Hyman-Choi, Hunter-I.-Tataru, I.-Tataru, ...
- We have a conformal map defined as below



Holomorphic coordinates



Properties of our conformal map:

- Key advantage: diagonalizes Dirichlet-to-Neumann map
- The map z is chosen to be holomorphic in S fixing the bottom and to satisfy the asymptotic condition $z \approx \alpha + i\beta$ for $|\alpha| \to \infty$
- The conformal map is then unique up to horizontal translation

Holomorphic variables

- Given the conformal map z we define its trace on the top $\{\beta = 0\}$ to be $Z(t, \alpha) = z(t, \alpha, 0)$
- We then define the holomorphic function

$$W(t, \alpha) = Z(t, \alpha) - \alpha$$

- Taking ϕ to be the velocity potential we define $\psi = \phi \circ z$ and take its harmonic conjugate to be θ
- We then define the holomorphic function (call it holomorphic velocity potential)

$$Q(t, \alpha) = \psi(t, \alpha, 0) + i\theta(t, \alpha, 0)$$

- We may then write the water wave equations as a system for the holomorphic functions (W, Q).

The functional framework

- If U is the trace of a holomorphic function u that satisfies the boundary condition $\Im U = 0$ on the base $\{\beta = -1\}$ then

$$-\mathcal{T}_h \Re U = \Im U$$

  where $\mathcal{T}_h = -i \tanh(hD)$.
- We define the space $\mathfrak{H}^h$ to consist of distributions defined on $\mathbb{R}$ modulo real constants so that

$$\|U\|_{\mathfrak{H}^h}^2 = \|\mathcal{T}_h \Re U\|_{L^2}^2 + \|\Im U\|_{L^2}^2 < \infty,$$

- With respect to the natural inner product we have the orthogonal decomposition

$$\mathfrak{H} = \mathfrak{H}^h \oplus \mathfrak{H}^a,$$

  where $\mathfrak{H}^h$ (resp. $\mathfrak{H}^a$) is the space of (traces of) holomorphic (resp. antiholomorphic) functions
- We define the orthogonal projection onto holomorphic functions

$$\mathbf{P}_h \colon \mathfrak{H} \to \mathfrak{H}^h$$

Water wave equations in holomorphic coords.

- $\mathbf{P}_h$ - projector onto the space of holomorphic functions

Fully nonlinear equations for *holomorphic* variables $(W = Z - \alpha, Q)$:

$$\begin{cases} W_t + F(1 + W_\alpha) = 0 \\ Q_t + FQ_\alpha - g\mathcal{T}_h[W] + \mathbf{P}_h\left[\dfrac{|Q_\alpha|^2}{J}\right] = 0, \end{cases}$$

where

$$F = \mathbf{P}_h\left[\frac{Q_\alpha - \bar{Q}_\alpha}{J}\right], \qquad J = |1 + W_\alpha|^2.$$

Conserved energy (Hamiltonian): $\quad L = (-\mathcal{T}_h^{-1}\partial_\alpha)^{\frac{1}{2}} \approx \langle D\rangle^{\frac{1}{2}}$.

$$E(W, Q) = g\|W\|^2_{\mathfrak{H}_h} + \|L_h Q\|^2_{\mathfrak{H}_h} + 2\langle WW_\alpha, W\rangle_{\mathfrak{H}_h}$$

Symmetries:

- Translations in α and t.
- Scaling $(W(t,x), Q(t,x)) \to (\lambda^{-1}W(t,\lambda x), \lambda^{-1}Q(t,\lambda x))$ (corresponds to $(g,h) \to (\lambda g, \lambda h)$)

Gauge freedom and fixing

- A disadvantage of the holomorphic coordinates is that our system has a gauge freedom

$$(W(t,\alpha), Q(t,\alpha)) \mapsto (W(t,\alpha+\alpha_0(t))+\alpha_0(t), Q(t,\alpha+\alpha_0(t))+q_0(t))$$

  for real-valued functions $\alpha_0(t), q_0(t)$.
    - This corresponds to $F \mapsto F + \alpha_0'(t)$ in the equation for W and a similar choice involving $q_0'(t)$ for the projector in the equation for Q.
    - This is seen in the fact that $\mathbf{P}u$ involves the term $\mathcal{T}^{-1}\Im u$
- To fix the gauge:
    1. At the initial time we must make an arbitrary choice
    2. At later times we fix the choice of α_0 and q_0 by requiring that both F and the projector in the second equation have limit 0 at $-\infty$
- This is allowed because the arguments of $\mathbf{P}$ are either holomorphic, antiholomorphic or in $L^2 \cap L^1$.

The differentiated equations

Self-contained system in (W_α, Q_α): degenerate hyperbolic system with double speed. Alternate quasilinear system for diagonal variables $(\mathbf{W}, R) = (W_\alpha, \frac{Q_\alpha}{1+W_\alpha})$:

$$\begin{cases} \mathbf{W}_t + b\mathbf{W}_\alpha + \dfrac{1+\mathbf{W}}{1+\bar{\mathbf{W}}} R_\alpha = (1+\mathbf{W})M \\ R_t + bR_\alpha = i\dfrac{g\mathbf{W} - a}{1+\mathbf{W}} \end{cases}$$

Physical parameters:

- a is real ($g + a$ is the normal derivative of the pressure)

$$a := g(1+\mathcal{T}^2)\Re\mathbf{W} + 2\Im\mathbf{P}[R\bar{R}_\alpha],$$

- b is real, and plays the role of an advection coefficient

$$b := 2\Re\left[R - \mathbf{P}[R\bar{Y}]\right]$$

Other parameters:

$$M = 2\Re\mathbf{P}[R\bar{Y}_\alpha - \bar{R}_\alpha Y] \quad Y := \mathbf{W}/(1+\mathbf{W}).$$

The Taylor stability condition

Normal derivative of the pressure:

$$\frac{dp}{dn} = g + a, \qquad a = g(1 + \mathcal{T}^2)\Re \mathbf{W} + 2\Im \mathbf{P}[R\bar{R}_\alpha],$$

Taylor stability (necessary for well-posedness)

$$\frac{dp}{dn} > 0$$

Theorem (Lannes,HG-I-T)

Assume that the fluid stays away from the bottom,

$$\Im W \geq -h_0 > -h$$

Then

$$\frac{dp}{dn} \geq g(h - h_0)$$

Sobolev spaces: Main space:

$$\mathcal{H}_h = \mathfrak{H}_h \times \mathfrak{H}_h^{\frac{1}{2}}$$

Solutions:

$$(W, Q) \in \mathcal{H}_h, \qquad (\mathbf{W}, R) \in \mathcal{H}_h^1$$

High frequency scaling:

$$(\mathbf{W}, R) \in \mathcal{H}_h^{\frac{1}{2}}$$

Control norms:

$$A := \|\mathbf{W}\|_{L^\infty} + \|Y\|_{L^\infty} + g^{-\frac{1}{2}} \|\langle D\rangle^{\frac{1}{2}} R\|_{L^\infty \cap B_2^{0,\infty}}$$

$$B := g^{\frac{1}{2}} \|\langle D\rangle^{\frac{1}{2}} \mathbf{W}\|_{bmo_h} + \|\langle D\rangle R\|_{bmo_h}$$

- the inhomogeneous space bmo is given by the norm

  $$\|f\|_{bmo_h} = \|f_{<h^{-1}}\|_{L^\infty} + \|f_{\geq h^{-1}}\|_{BMO},$$

  where $f = f_{<h^{-1}} + f_{\geq h^{-1}}$
- BMO is the usual space of functions of bounded mean oscillation.

Low regularity local well-posedness:

Theorem

- *The system is locally well-posed for all initial data* (W_0, Q_0) *with regularity*
$$(W_0, Q_0) \in \mathcal{H}_h, \qquad (\mathbf{W}_0, R_0) \in \mathcal{H}_h^1.$$
Further, the solutions can be continued as long as our control parameter $A(t)$ *remains finite, and* $\int B(t)dt$ *remains finite.*
- *This result is uniform with respect to our choice of parameters* $g \lesssim h$ *as follows. If for a large parameter* C *the initial data satisfies*
$$g^{-1}h^{-1}\|(W_0, Q_0)\|_{\mathcal{H}} + g^{-1}\|(\mathbf{W}_0, R_0)\|_{\mathcal{H}} + \|(\mathbf{W}_{0,\alpha}, R_{0,\alpha})\|_{\mathcal{H}} \leq C,$$
then there exists some $T = T(C)$, *independent on* g, h *so that the solution exists on* $[-T, T]$ *with similar bounds.*

A model system

• first order diagonal double speed (b) hyperbolic system written is variables (w, r):

$$\begin{cases} w_t + \mathbf{P}[bw_\alpha] + \mathbf{P}\left[\dfrac{r_\alpha}{1+\bar{\mathbf{W}}}\right] - \mathbf{P}\left[\dfrac{R_\alpha \mathcal{T}^2 w}{1+\bar{\mathbf{W}}}\right] = G \\ r_t + \mathbf{P}[br_\alpha] - \mathbf{P}\left[\dfrac{(g+a)\mathcal{T}[w]}{1+\mathbf{W}}\right] = K \end{cases}$$

• (w, r) and the inhomogeneous terms $(G, K) \in \mathcal{H}$ are holomorphic.

Quasilinear energy:

$$E^{(2)}_{lin}(w,r) = \langle w, w \rangle_{g+a} - \left\langle r, \mathcal{T}^{-1}[r_\alpha] \right\rangle = \langle w, w \rangle_{g+a} + \langle Lr, Lr \rangle$$

Quasilinear weighted energy:

$$E^{(2)}_{\omega,lin}(w,r) = \langle w, w \rangle_{(g+a)\omega} + \langle Lr, Lr \rangle_\omega$$

Energy estimates:

Let I be a time interval where A is bounded and $B \in L^1$. Then in I the following properties hold:
a) The system of equations is well posed in $\mathcal{H}$, and satisfies the estimate

$$\frac{d}{dt} E^{(2)}_{lin}(w,r) = 2\left\langle G, w \right\rangle_{g+a} - 2\left\langle LK, Lr \right\rangle + O_A(B) E^{(2)}_{lin}(w,r).$$

b) Assume in addition that ω is a weight satisfying

$$\|\omega\|_{L^\infty} \leq A, \qquad \|\omega\|_{bmo^{\frac{1}{2}}} \leq B, \qquad \|(\partial_t + b\partial_\alpha)\omega\|_{L^\infty} \leq B$$

Then we also have

$$\frac{d}{dt} E^{(2)}_{\omega,lin}(w,r) = 2\left\langle G, w \right\rangle_{(g+a)\omega} + 2\langle LK, Lr \rangle + O_A(B) E^{(2)}_{lin}(w,r)$$

The linearization at zero

- The linearized system for $(w, q) = (\delta W, \delta Q)$ about zero is given by

$$\begin{cases} w_t + q_\alpha = 0 \\ q_t - g\mathcal{T}[W] = 0, \end{cases}$$

- The dispersion relation is then seen to be

$$\tau^2 = g\xi \tanh \xi$$

- Two branches $\tau = \pm\sqrt{g}\omega(\xi)$ where $\omega(\xi) = \xi\sqrt{\frac{\tanh \xi}{\xi}}$ corresponding to left-moving and right-moving waves
- At high frequency the behavior is similar to the infinite depth case

$$\omega(\xi) \sim \frac{\xi}{\sqrt{|\xi|}}$$

- At low frequency we obtain the KdV like dispersion relation

$$\omega(\xi) \sim \xi - \frac{1}{6}\xi^3$$

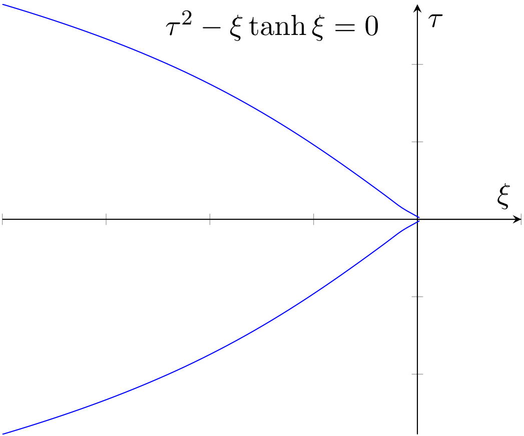


Figure : Dispersion relation

Nonlinear resonances and solitons

- The worst quadratic interactions of linear waves correspond to three-wave resonances: solutions to the system

$$\begin{cases} \omega(\xi_1) \pm \omega(\xi_2) \pm \omega(\xi_3) = 0 \\ \xi_1 + \xi_2 + \xi_3 = 0 \end{cases}$$

- This can only occur when one of ξ_1, ξ_2, ξ_3 vanishes.
- The equation has a null structure that kills these interactions, so it is reasonable to expect that small solutions will exist on longer-than-quadratic timescales
- Small solitons arising from the KdV approximation
- For small localized data solitons emerge at quartic time scales

Cubic lifespan bounds

Theorem

Our fully nonlinear system with small initial data (W_0, Q_0),

$$g^{-1}h^{-1}\|(W,Q)(0)\|_{\mathcal{H}_h} + g^{-1}\|(\mathbf{W},R)(0)\|_{\mathcal{H}_h} + \|(\mathbf{W}_\alpha, R_\alpha)(0)\|_{\mathcal{H}_h} \leq \epsilon.$$

Then the solution (W,Q) *exists and satisfies similar bounds on a time interval* $[-T_\epsilon, T_\epsilon]$ *with* $T_\epsilon \gtrsim \epsilon^{-2}$. *In addition, higher regularity also propagates uniformly on the same scale, i.e. for solutions as above we have*

$$\|(\mathbf{W},R)\|_{C([-T_\epsilon,T_\epsilon];\mathcal{H}_h^k)} \lesssim \|(\mathbf{W},R)(0)\|_{\mathcal{H}_h^k} + \epsilon h^{1-k}$$

whenever the right hand side is finite.

- The regularity of the data is same as in the LWP result.
- Proof idea: *quasilinear modified energy method*
- Bounds for all higher norms propagate on same timescale.
- Result is uniform in the infinite depth limit $h \to \infty$.

Normal form for the finite depth water waves

The normal form variables are $(\tilde{W}, \tilde{Q})$:

$$\begin{cases} \tilde{W} = W + B^h[W,W] + \frac{1}{g}C^h[Q,Q] + B^a[W,\bar{W}] + \frac{1}{g}C^a[Q,\bar{Q}] \\ \tilde{Q} = Q + A^h[W,Q] + A^a[W,\bar{Q}] + D^a[Q,\bar{W}], \end{cases}$$

- The symbols have singularites at zero frequency-thus we cannot implement the normal form directly
- When we compute the normal form energies, the repeated symmetrizations lead to cancelations of the singularities - this is what we call a null structure. E.g.:

$$< W, B^h[W,W] >$$

The normal form symbols

$$\Omega(\xi,\eta,\zeta) = J(\xi)^2 + J(\eta)^2 + J(\zeta)^2 - 2J(\xi)J(\eta) - 2J(\eta)J(\zeta) - 2J(\zeta)J(\xi),$$

$$J(\xi) = \omega(\xi)^2 = \xi \tanh \xi$$

$$A^h(\xi,\eta) = \frac{2i\eta J(\xi)\left(J(\xi+\eta) - J(\xi) + J(\eta)\right)}{\Omega(\xi,\eta)},$$

$$B^h(\xi,\eta) = \frac{2i(\xi+\eta)J(\xi)J(\eta)}{\Omega(\xi,\eta)},$$

$$C^h(\xi,\eta) = \frac{i\xi\eta(\xi+\eta)\left(J(\xi+\eta) - J(\xi) - J(\eta)\right)}{\Omega(\xi,\eta)}.$$

$$A^a(\xi,\eta) = \frac{1}{1-e^{2(\xi-\eta)}}\frac{\tanh(\xi-\eta)}{\xi-\eta}$$
$$\Big\{\eta\,(1-\coth\eta)\,B^h(\xi,-\eta) + (1-\tanh\xi)\,C^h(\xi,-\eta)\Big\}$$
$$B^a(\xi,\eta) = -\frac{1}{1-e^{2(\xi-\eta)}}\frac{1}{\xi-\eta}$$
$$\Big\{(J(\xi-\eta)-(\xi+\eta))\,B^h(\xi,-\eta) + (\tanh\xi+\tanh\eta)C^h(\xi,-\eta)\Big\}$$
$$C^a(\xi,\eta) = -\frac{1}{1-e^{2(\xi-\eta)}}\frac{1}{\xi-\eta}$$
$$\Big\{(J(\xi-\eta)-(\xi+\eta))\,C^h(\xi,-\eta) + \xi\eta\,(\coth\eta+\coth\xi)\,B^h(\xi,-\eta)\Big\}$$
$$D^a(\xi,\eta) = -\frac{1}{1-e^{2(\xi-\eta)}}\frac{\tanh(\xi-\eta)}{\xi-\eta}$$
$$\Big\{\xi(1-\coth\xi)B^h(\xi,-\eta) + (1-\tanh\eta)C^h(\xi,-\eta)\Big\}$$

Thank you

The modified energy method

Idea: Modify the energy rather than the equation in order to get cubic energy estimates.

Step 1: Construct a cubic normal form energy

$$E^n_{NF}(W,Q) = (quadratic + cubic)(\|\tilde{W}^{(n)}\|^2_{L^2} + \|\tilde{Q}^{(n)}\|^2_{\dot{H}^{\frac{1}{2}}})$$

Then

$$\frac{d}{dt}E^n_{NF}(W,Q) = quartic + higher$$

Here higher derivatives arise on the right, making it impossible to close.

Step 2: Switch $E^n_{NF}(W,Q)$ to diagonal variables $E^n_{NF}(\mathbf{W},R)$.

Step 3: To account for the fact that the equation is quasilinear, replace the leading order terms in $E^n_{NF}(\mathbf{W},R)$ with their natural quasilinear counterparts to obtain a good cubic quasilinear energy $E^n(\mathbf{W},R)$. Clue: look at the quasilinear energy for the linearized equation.