Coloring Jordan regions and curves

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Packing cycles in planar digraphs

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Theorem (Reed & Shepherd 1996)

For every planar digraph G, $\nu^*(G) \leq 28\,\nu(G)$.

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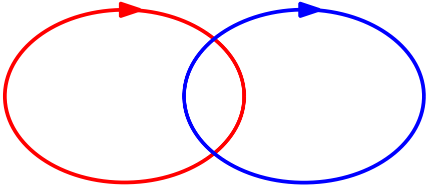

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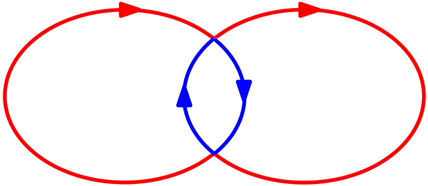

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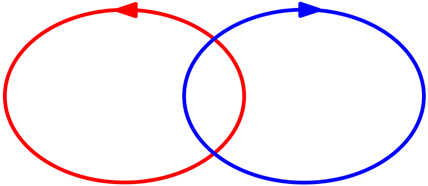

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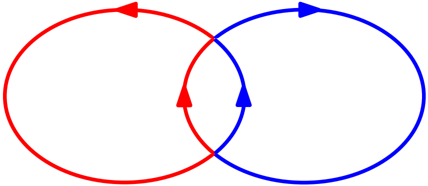

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In particular, if there are n curves, then there are at least $\frac{n}{6ek}$ pairwise disjoint curves. So $\nu(G) \geq \frac{n}{6ek} = \frac{\nu^*(G)}{6e}$, and

$$\nu^*(G) \leq 6e\,\nu(G) \approx 16.3\,\nu(G)$$

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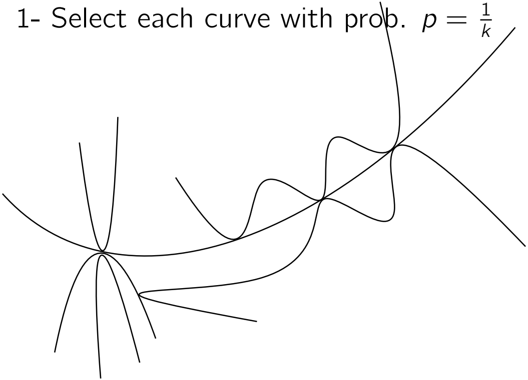

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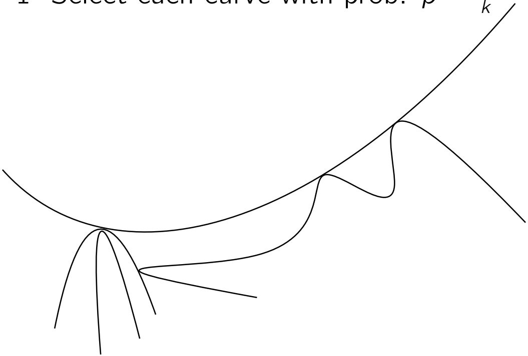

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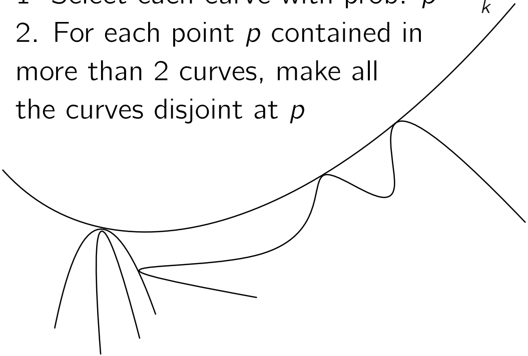

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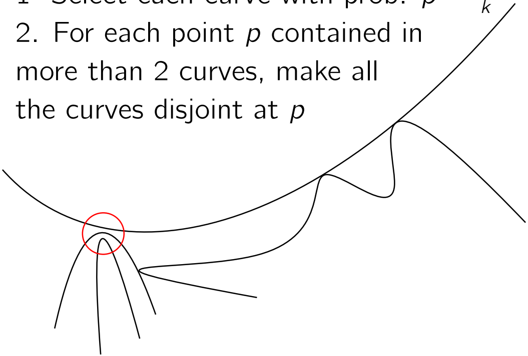

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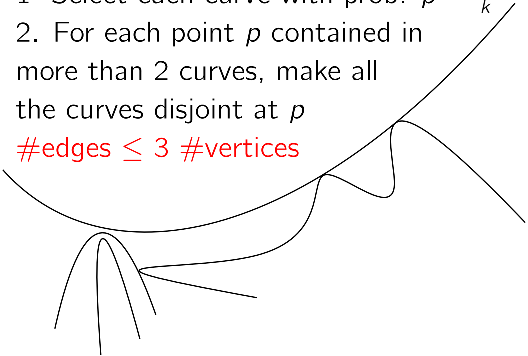
#edges $\leq$ 3 #vertices



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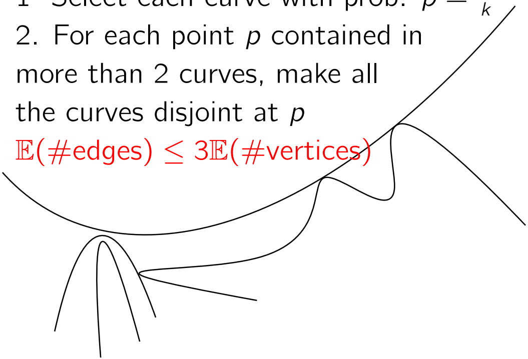

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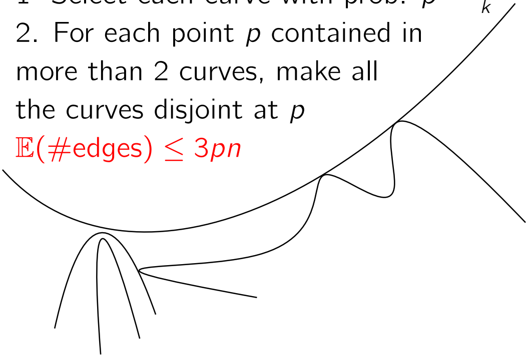

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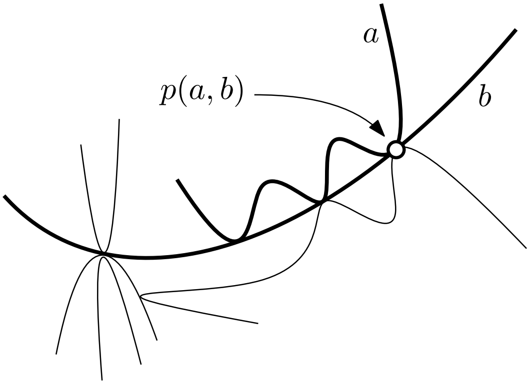

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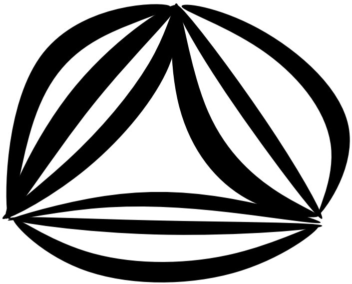

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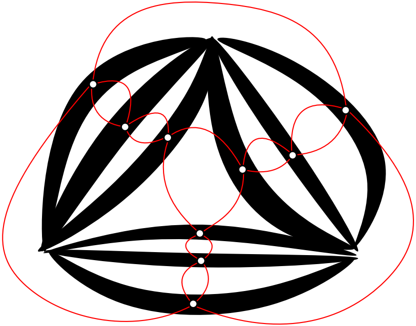

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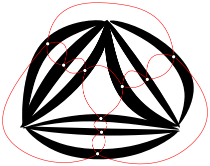

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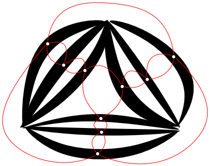

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Consequence

Let $\mathcal{F}$ be a touching family of Jordan regions such that each point lies in at most k regions. Then the intersection graph G of $\mathcal{F}$ satisfies $\chi(G) \leq \frac{3k}{2} + o(k)$.

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Theorem (Cames van Batenburg, E. & Müller 2016)

Let $\mathcal{F}$ be a simple touching family of Jordan regions such that each point lies in at most k regions. Then the intersection graph G of $\mathcal{F}$ satisfies $\chi(G) \leq k+327$ (and $\chi(G) \leq k+1$ if $k \geq 490$).

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Let $\mathcal{S}$ be a one-sided contact system of strings, such that any point of the plane is in at most k strings, and any two strings intersect in at most one point. Is it true that the intersection graph of $\mathcal{S}$ has chromatic number at most $k + o(k)$? (or even $k + c$, for some constant c?)

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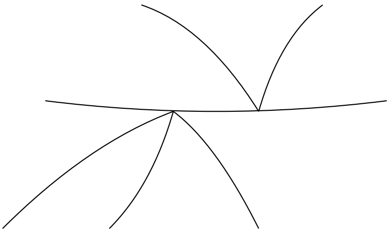

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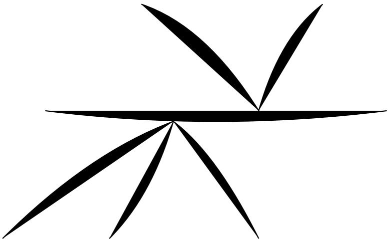

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- We can prove $k + 5$ for segments.
- $2k + c$ is also not hard to derive for closed curves.