Integro-difference equations and climate change in a variable environment

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Joint work with M. A. Lewis[1]

Integrodifference Equations in Ecology: 30 years and counting, September 2016

[1] J. Bouhours and M. Lewis, online first in *Bulletin of Mathematical Biology*.

Our problem

Climate change and integro-difference equations

$$u_{t+1}(\xi) = \int_{\mathbb{R}} K(\xi - \eta) g_0(\eta - s_t) f_{r_t}(u_t(\eta)) d\eta, \quad t \in \mathbb{N},\ \xi \in \mathbb{R}.$$

with $(u_t)_t$ density of the population at generation t,

▷ Long time behaviour? Persistence of the population? Critical value for parameters?

1 Climate change in population dynamics

2 The model

3 Persistence of the population

4 Numerical simulations

Climate change and population dynamics[2]

- Required migration

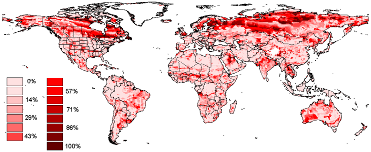


Figure 3. A map showing areas where species might have to achieve unusually high migration rates (≥1,000 metres per year) in order to keep up with 2 × CO_2 global warming in 100 years. Shades of red indicate the percent of 14 models that exhibited unusually high rates.

- Habitat Loss

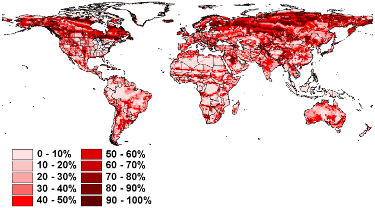


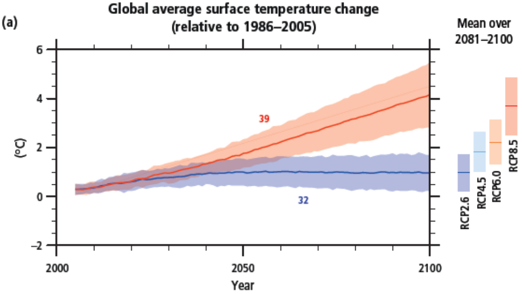
Figure 10. Loss of existing habitat that could occur under a doubling of atmospheric CO_2 concentrations. Shades of red indicate the percent of vegetation models that predicted a change in biome type of the underlying map grid cell.

[2] Global warming and terrestrial biodiversity decline, Malcolm J.R., Markham A., 2000.

Climate change and environmental variability[3]

Environmental variability

- Uncertainty in climate change scenario



- Environmental variability caused by increasing extreme climatic events: temperature extremes, sea levels, precipitation events

[3] IPCC: Climate change, 2007.

Integro-difference equations in heterogeneous environments

Growth and dispersal in heterogeneous environments

$$u_{t+1}(\xi) = \int_{\mathbb{R}} \underbrace{K(\xi,\eta)}_{\text{dispersion}} \underbrace{g_t(\eta)}_{\text{suitability}} \underbrace{f(u_t(\eta))}_{\text{growth}} d\eta, \quad t \in \mathbb{N},\ \xi \in \mathbb{R}$$

▷ Suitability: habitat migration due to climate change

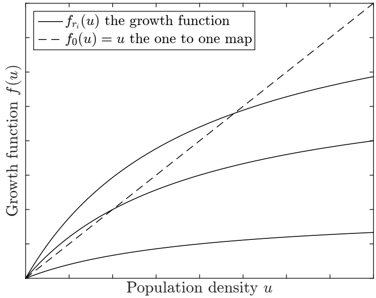
$$g_t(\eta) = g_0(\eta - s_t)$$

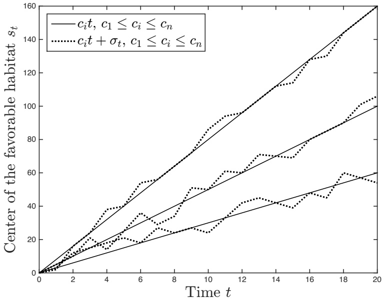
$s_t \in \mathbb{R}$ reference point at time t

Integro-difference equations in variable environments

Variability of the environment

- Variable growth: $f(u) = f_{r_t}(u)$, $(f_{r_t})_t$ sequence of random functions
  ▷ $(r_t)_t$ random per capita growth rate at 0
- Variable reference point: $s_t = ct + \sigma_t$,
  ▷ c uncertain asymptotic migration speed ($c \in \{c_1, \dots, c_n\}$), fixed, $(\sigma_t)_t$ stochastic process, variability of the migration speed





Our model

General problem:

$$u_{t+1}(\xi) = \int_{\mathbb{R}} K(\xi - \eta) g_0(\eta - s_t) f_{r_t}(u_t(\eta)) d\eta, \quad t \in \mathbb{N},\ \xi \in \mathbb{R}.$$

- $x \mapsto K(x)$ continuous, uniformly bounded and positive in $\mathbb{R}$,
- $x \mapsto g_0(x)$ compactly supported in Ω_0, nonnegative, bounded by 1,
- $s_t = ct + \sigma_t$,
- $(\sigma_t, r_t)_t$ bounded, independent, identically distributed random variables,
- $f_r : \mathbb{R}^+ \to \mathbb{R}^+$, continuous, increasing with $f_r(u) = 0$ for all $u \leq 0$,
- $0 < f_r(u) \leq m$ for all positive continuous function u and $r = f_r'(0)$
- if u, v constants such that $0 < v < u$ then $f_r(u)v < f_r(v)u$

Changing the reference frame

Problem in the non moving frame:

$$u_{t+1}(\xi) = \int_{\mathbb{R}} K(\xi - \eta) g_0(\eta - s_t) f_{r_t}(u_t(\eta)) d\eta, \quad t \in \mathbb{N},\ \xi \in \mathbb{R}.$$

- $x = \xi - c(t+1)$, $y = \eta - ct$ and $\bar{u}_t(y) := u_t(y + ct)$

$$\bar{u}_{t+1}(x) = \int_{\mathbb{R}} K(x - y + c) g_0(y - \sigma_t) f_{r_t}(\bar{u}_t(y)) dy.$$

- $\sigma_t \in (\underline{\sigma}, \overline{\sigma}) \implies \Omega := (\inf \Omega_0 + \underline{\sigma}, \sup \Omega_0 + \bar{\sigma})$, "support" of the problem

Dropping the bar

$$u_{t+1}(x) = \int_{\Omega} K(x - y + c) g_0(y - \sigma_t) f_{r_t}(u_t(y)) dy, \quad t \in \mathbb{N},\ x \in \Omega,$$

Previous work:

- Zhou-Kot : $u_{t+1}(\xi) = \int_{\Omega + ct} K(\xi - \eta) f(u_t(\eta)) d\eta$, c fixed, Ω compact,
- Hardin et al, Jacobsen et al: Integro-difference equations in variable environments

Large time behaviour

$$u_{t+1}(x) = \int_\Omega K(x-y+c) g_0(y-\sigma_t) f_{r_t}(u_t(y)) dy, \quad t \in \mathbb{N},\ x \in \Omega,$$

Theorem

Assumptions:

- u_0 *non negative, non trivial, bounded,*
- f_r *KPP, increasing.*

Then u_t *converges in distribution to a random variable* u^* *as* $t \to +\infty$, *independently of the initial condition* u_0, *and* u^* *such that*

$$u^*(x) = \int_\Omega K(x-y+c) g_0(y-\sigma^*) f_{r^*}(u^*(y)) dy.$$

Denoting by μ^* *the distribution associated with* u^**:*

$$\mu^*(\{0\}) = 0 \text{ or } \mu^*(\{0\}) = 1.$$

=> *extinction of the population with probability 0 or 1 only, independently of the initial condition.*

Persistence criterion

What does determine whether $\mu^*(\{0\}) = 0$ or $\mu^*(\{0\}) = 1$?

Define

$$\Lambda_t := \left(\int_\Omega \tilde{u}_t(x) dx \right)^{1/t},$$

where $(\tilde{u}_t)_t$ the solution of the linearised problem around 0:

$$\tilde{u}_{t+1}(x) = \mathcal{L}_{\alpha_t} \tilde{u}_t(x) := \int_\Omega K(x - y + c) g_0(y - \sigma_t) r_t \tilde{u}_t(y) dy.$$

Theorem

$$\lim_{t \to +\infty} \Lambda_t = \Lambda \in [0, +\infty), \text{ with probability 1.}$$

And,

- *If* $\Lambda < 1$, *the population will go extinct, in the sense that* $\mu^*(\{0\}) = 1$,
- *If* $\Lambda > 1$, *the population will persist, in the sense that* $\mu^*(\{0\}) = 0$.

Characterisation of Λ

$$\Lambda = e^{E[\ln(r_0)]} \cdot \lim_{t\to+\infty} K_t^{1/t}$$

$$K_t = \underbrace{\int_\Omega \dots \int_\Omega}_{t+1 \text{ terms}} K(x-y_1+c)g_0(y_1-\sigma_{t-1})\cdots K(y_{t-1}-y_t+c)g_0(y_t-\sigma_0)u_0(y_t)dy_t \dots dx$$

- No variability for the shifting speed: $\sigma_t \equiv 0$

$$\implies \Lambda = e^{E[\ln(r_0)]} \cdot \lambda_c$$

with λ_c principal eigenvalue of

$$\mathcal{K}_c[u](x) := \int_{\Omega_0} K(x-y+c)g_0(y)u(y)dy,$$

- The particular case of Gaussian Kernel

$$\lambda_c = e^{-\frac{c^2}{2(\sigma^K)^2}}\lambda_0,$$

Λ decreasing with c $\implies$ existence of a critical speed for persistence:

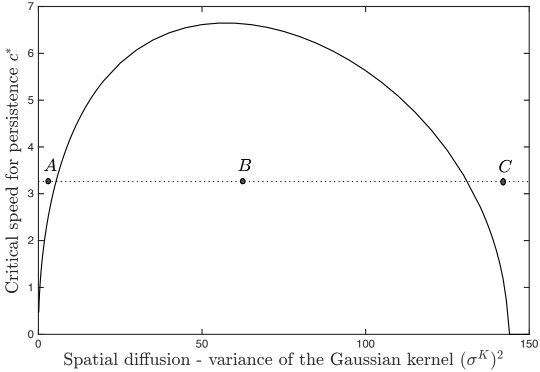
$$c^* = \sqrt{2(\sigma^K)^2\left(\ln(\lambda_0) + E[\ln(r_0)]\right)} > 0$$

Critical speed for Gaussian kernel

2 possible environments: bad $(\overline{\sigma}, \underline{r})$ or good $(\underline{\sigma}, \overline{r})$, with

$$P(\text{Good}) = P(\text{bad}) = 0.5, \quad \underline{\sigma} < 0 < \overline{\sigma}, \quad 0 < \underline{r} < \overline{r}$$

Critical speed as a function of the variance of the dispersal kernel

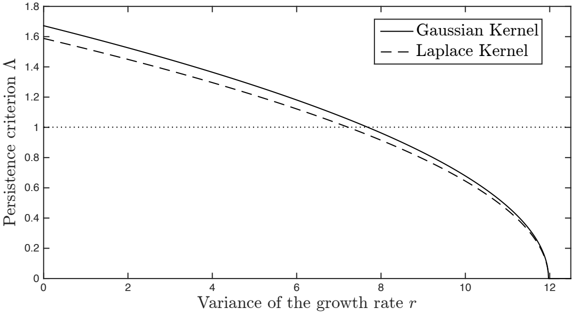


3 different regimes when $c = 3.25$ km/year

Consequence of the variability

Persistence criterion as a function of the variance of the growth rate r

Fixed expectation, increasing the variance

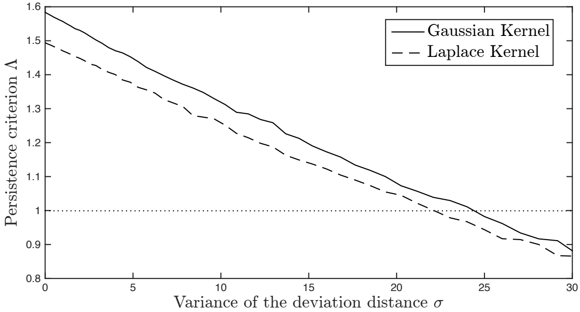


▷ Negative effect of variability on persistence

Consequence of the variability

Persistence criterion as a function of the variance of the deviation speed σ

Fixed expectation, increasing the variance



▷ Negative effect of variability on persistence

Conclusion

=> Long time behaviour of the solution and characterisation of persistence

=> Critical migration speed for Gaussian Kernel

=> Consequences of variability on population persistence

Future investigations:

- Approximation of λ_c (principal eigenvalue)
- Critical migration speed ($\sigma \equiv 0$) for other kernel
- effect of variability on Λ (analysis)

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THANK YOU FOR YOUR ATTENTION!