Numerical range and dilation

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- Alternatively, there is $X : H \to K$ such that

$$X^*X = I_H \quad \text{and} \quad X^*AX = T.$$

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A better question

Identify "good" matrices or operators A such that

$W(T) \subseteq W(A)$ ensures that $T \in B(H)$ has a dilation of the form $I \otimes A$.

Some Dilation Results

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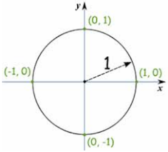

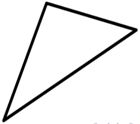

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Let $A \in M_2$ so that $W(A)$ is the elliptical disk with the eigenvalues a_1, a_2 as foci and minor axis of length $\sqrt{\text{tr}\, A^*A - |a_1|^2 - |a_2|^2}$.

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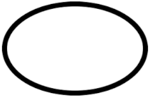

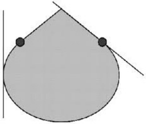

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Suppose $T \in B(H)$ is a contraction with

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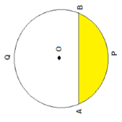

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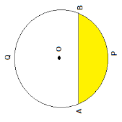

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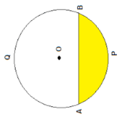

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Corollary

Let $T \in B(H)$ be a contraction. Then

$$\overline{W(T)} = \cap\{\overline{W(U)} : U \in B(H \oplus H) \text{ is a unitary dilation of } T\}.$$

Extension of the result of Mirman

- Let $T_1, \ldots, T_k \in B(H)$ be self-adjoint operators. Define their joint numerical range by

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- Note that one can choose any $v_1, v_2, v_3, v_4 \in \mathbb{R}^3$ as long as

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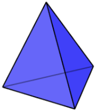

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Joint dilation

Theorem [Binding,Farenick,Li,1995]

Let $T_1, \ldots, T_m \in B(H)$ be self-adjoint such that $W(T_1, \ldots, T_m)$ has non-empty interior in $\mathbb{R}^m$. That is, $\{I, T_1, \ldots, T_m\}$ is linearly independent.

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That is, there is a unitary U such that

$$U^*(I_N \otimes A_j)U = \begin{pmatrix} T_j & * \\ * & * \end{pmatrix}, \quad j = 1, \dots, m.$$

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- Our study is related to the study of quantum channels, whose dual map has some special properties.
- It also allow us to develop techniques in construction unital completely positive maps with some desired properties.

Some recent results [Li & Poon, 2017]

- A direct (constructive) proof is given for the (ellipse-point) result that

  Let $A \in M_2$ *or* $A = [\alpha] \oplus A_2$ *with* $A_2 \in M_2$.

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- Suppose $A \in M_3$ has no reducing eigenvalues, and if A is not unitarily similar to A^t. Then for $T = A^t$ we have

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- This is in contrast to the fact that a positive map from M_3 to M_2 is always co-positive.

Some questions

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- **Conjecture 2.** If $A \in M_n$ satisfies (†), then $A = A_1 \oplus A_2$ such that $W(A) = W(A_1)$, where $A \in M_2$ or $A \in M_3$ with an reducing eigenvalue.

Thank you for your attention!

Hope that you will solve our problems!