Hadamard diagonalizability and cubelike graphs

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Outline

Motivation: Bose and the Quantum Data Bus

- Bose (2003) stresses the importance of transmitting a quantum state from one place to another within a quantum computer.
- This is important, in one example, for linking small quantum processors together for large-scale quantum computing.
- Bose proposes the use of a spin chain as a quantum data bus for short distance quantum communication.
- The probability of state transfer is used to determine the accuracy of quantum state transfer through a quantum data bus between quantum registers and/or processors.

Quantum State Transfer

- The *probability of state transfer* or *fidelity* is a measure of the closeness between two quantum states. It is the quantity $p(t) = |\mathbf{e}_j^T e^{it\mathcal{H}} \mathbf{e}_k|^2$ and is always a number between 0 and 1.
- *Perfect State Transfer* (PST) occurs if the fidelity between two quantum states is equal to 1. That is, a graph exhibits PST between vertices j and k if $\exists t_0 > 0$ such that $p(t_0) = |\mathbf{e}_j^T e^{it_0\mathcal{H}} \mathbf{e}_k|^2 = 1$.
- *Pretty Good State Transfer* (PGST) occurs if the fidelity between two quantum states can be made arbitrarily close to 1: $\forall \epsilon > 0, \exists t_\epsilon > 0$ such that $p(t_0) = |\mathbf{e}_j^T e^{it_\epsilon\mathcal{H}} \mathbf{e}_k|^2 \geq 1 - \epsilon$.

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Quantum State Transfer

Note: Unweighted spin chains only exhibit PST for ≤ 3 vertices.

Questions:

- What graphs other than chains exhibit PST?
- What if we do not require that the state transfer be perfect?
- What about weighted chains?

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Some Graph Theory Basics

The *Cartesian Product* of two graphs, denoted $G_1 \square G_2$ is a graph such that the vertex set of $G_1 \square G_2$ is the Cartesian product $V_1 \times V_2$, and any two vertices (u, u') and (v, v') are adjacent in $G_1 \square G_2$ if and only if either

i) $u = v$ and u' is adjacent to v' in G_2, or

ii) $u' = v'$ and u is adjacent to v in G_1.

The Hypercube

Hypercube: a.k.a. n-cube: the hypercube graph Q_n can be constructed in a number of ways. One way is using 2^n vertices labeled with n-bit binary numbers and connecting two vertices by an edge whenever the Hamming distance of their labels is one.

Another construction is the Cartesian product of K_2 (complete graph on two vertices; that is, a path on two vertices) with itself n times:

$$K_2 \Box K_2 \Box \cdots \Box K_2 = Q_n$$

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More on Graphs

cubelike graph: Take a set $C \subset \mathbb{Z}_2^d = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ (d times), where C does not contain the all-zeros vector. Construct the *cubelike graph* $G(C)$ with vertex set $V = \mathbb{Z}_2^d$ and two elements of V are adjacent if and only if their difference is in C. The set C is called the *connection set* of the graph $G(C)$.

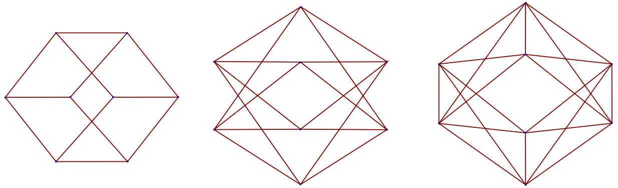

Figure: three nonisomorphic cubelike graphs on eight vertices
(Bernasconi-Godsil-Severini, 2008)

Some results on cubelike graphs

(Bernasconi-Godsil-Severini 2008; Cheung-Godsil 2011)

Theorem

Let C be a subset of $\mathbb{Z}_2^d$ and let σ be the sum of the elements of C. If $\sigma \neq 0$, then PST occurs in $G(C)$ from j to $j + \sigma$ at time $\pi/2$. If $\sigma = 0$, then $G(C)$ is periodic with period $\pi/2$ (every vertex has perfect state transfer with itself at time $t_0 = \pi/2$).

The *code* of $G(C)$ is the row space of the $d \times |C|$ matrix M constructed by taking the elements of C as its columns. When the sum of the elements of C is zero, it has been shown that if perfect state transfer occurs on a cubelike graph, then it must take place at time $\pi/(2\,gcd)$, where *gcd* is the greatest common divisor of the (Hamming) weights of the binary strings in the code.

Some Matrices Associated to a Graph

Adjacency Matrix: An $n \times n$ matrix $A = (a_{jk})$ representing a graph G, defined by: $a_{jk} = \begin{cases} w(j,k) & \text{if j and k are adjacent} \\ 0 & \text{otherwise} \end{cases}$

Laplacian Matrix: The Laplacian matrix of a graph is $L = D - A$ where D is the diagonal degree matrix of graph G. (For weighted graphs, the degree of a vertex is the sum of all the weights of the incident edges. For unweighted graphs, just count the number of incident edges.)

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Double Cover of a Graph

If $V_1 = V_2$, let $G_1 \ltimes G_2$ be the graph defined by the adjacency matrix $A(G_1 \ltimes G_2) = \begin{bmatrix} A(G_1) & A(G_2) \\ A(G_2) & A(G_1) \end{bmatrix}$, where $A(\cdot)$ is the adjacency matrix of the given graph. If the edge sets of G_1 and G_2 are disjoint, then $G_1 \ltimes G_2$ is a *double cover* of the graph with adjacency matrix $A(G_1) + A(G_2)$.

Hadamard matrices

A *Hadamard Matrix* H is an $n \times n$ matrix whose entries are either $+1$ or -1 and it satisfies $HH^T = nI$. That is, a $(+1$,-1$)$ matrix is a Hadamard matrix if the inner product of two distinct rows is 0 and the inner product of a row with itself is n.

Ex: The *standard* Hadamards of order 2^n

$$H_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, H_2 = \begin{bmatrix} H_1 & H_1 \\ H_1 & -H_1 \end{bmatrix}, \ldots, H_n = \begin{bmatrix} H_{n-1} & H_{n-1} \\ H_{n-1} & -H_{n-1} \end{bmatrix}$$

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Hadamard Diagonalization

Barik-Fallat-Kirkland (2011)

- It is always possible to permute and sign the rows and columns of a Hadamard matrix so that all entries of the first row and all entries of the first column are all 1's. A Hadamard matrix in this form is said to be *normalized*. —For our results (later), this means we don't need to assume that the graph is connected (i.e. that there is a path between every pair of vertices).
- Given a graph G on n vertices with corresponding Laplacian matrix L, if we can write $L = \frac{1}{n}H\Lambda H^T$ for some Hadamard H and diagonal matrix Λ, then we say that G (or, that L) is *Hadamard diagonalizable*.
- If G is Hadamard diagonalizable by some Hadamard H, then G is also Hadamard diagonalizable by a corresponding normalized Hadamard.

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Hadamard Diagonalization (cont.)

- If G is an integer-weighted graph that is Hadamard diagonalizable, then G is regular (each vertex has the same sum of weights of its incident edges) and all the eigenvalues of its Laplacian are even integers (either $0 \mod 4$ or $2 \mod 4$).

An Eigenvalue Characterisation

—Results below are all from [Johnston-Kirkland-P.-Storey-Zhang, LAA, 2017]

Theorem

Let G be an integer-weighted graph that is Hadamard diagonalizable by a Hadamard of order n. Let $H = (h_{uv})$ be a corresponding normalized Hadamard. Denote the eigenvalues of the Laplacian matrix L corresponding to G by $\lambda_1, \cdots, \lambda_n$, so that $LHe_j = \lambda_j He_j, j = 1, \ldots, n$. Then G has PST from vertex j to vertex k at time $t_0 = \pi/2$ if and only if for each $\ell = 1, \cdots, n$, $\lambda_\ell \equiv 1 - h_{j\ell}h_{k\ell} \mod 4$.

How our work relates to cubelike graphs

It is known that the adjacency matrix of any cubelike graph is diagonalized by the standard Hadamard matrix. The following result provides the converse.

Lemma

Suppose that $k \in \mathbb{N}$ and that A is a symmetric $(0, 1)$ matrix that is diagonalizable by the standard Hadamard matrix of order 2^k. Then

1. *A has constant diagonal;*
2. *if A has zero diagonal then it is the adjacency matrix of a cubelike graph;*
3. *if A has all ones on the diagonal, then $A - I$ is the adjacency matrix of a cubelike graph.*

How our work relates to cubelike graphs (cont.)

Corollary

Let G be an unweighted graph with Laplacian matrix L. Then L is diagonalized by the standard Hadamard matrix if and only if G is a cubelike graph.

Recall: Some Graph Operations

- *Graph Complement:* a.k.a. inverse graph of G is a graph G^c on the same vertices such that two distinct vertices of G^c are adjacent if and only if they are not adjacent in G.
- *Graph Join:* $G_1 \vee G_2$ has all the edges that connect the vertices of the first graph, G_1, with the vertices of the second graph, G_2.

Creation of new Hadamard diagonalizable graphs with PST

It is known that the union of a PST graph with itself still exhibits PST.

Proposition

Let G be an integer-weighted graph on $n \geq 4$ vertices that is diagonalizable by a Hadamard matrix H, and that has perfect state transfer from vertex j to vertex k at time $t_0 = \pi/2$. Then its complement G^c is also diagonalizable by H, and has the same PST pairs and PST time as G. Furthermore, the join $G \vee G$ of G with itself is diagonalizable by the Hadamard $\begin{bmatrix} H & H \\ H & -H \end{bmatrix}$, and has PST from vertex j to vertex k at time $t_0 = \pi/2$.

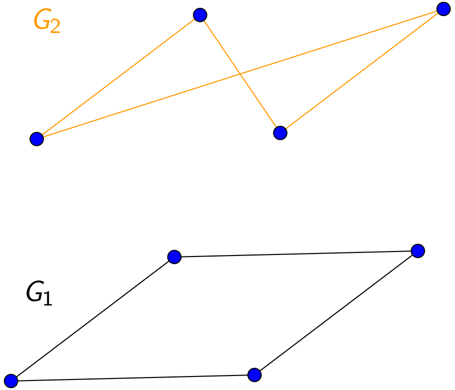
New Operation on Graphs: modified $G_1 \ltimes G_2$

Suppose that G_1 and G_2 are two weighted graphs that are both diagonalizable by a Hadamard matrix H of order n, with Laplacians $L_1 = D_1 - A_1$ and $L_2 = D_2 - A_2$, respectively. Then we define the *merge* of G_1 and G_2 with respect to the weights w_1 and w_2 to be the graph $G_1 \ {}_{w_1}\!\odot_{w_2}\ G_2$ with Laplacian

$$\begin{bmatrix} w_1 L_1 + w_2 D_2 & -w_2 A_2 \\ -w_2 A_2 & w_1 L_1 + w_2 D_2 \end{bmatrix}.$$

In the unweighted case (i.e., when $w_1 = w_2 = 1$), we denote the merge simply by $G_1 \odot G_2$.

Example of Merge Operation



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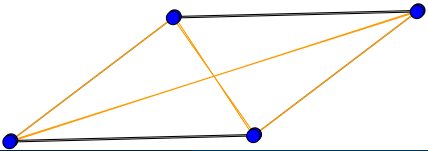

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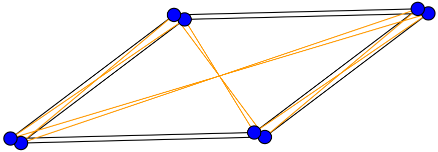

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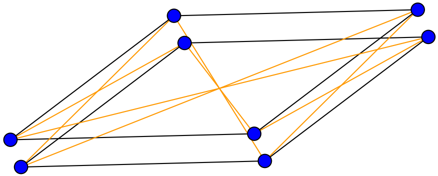

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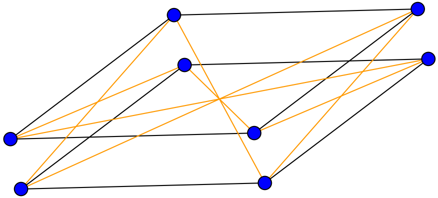

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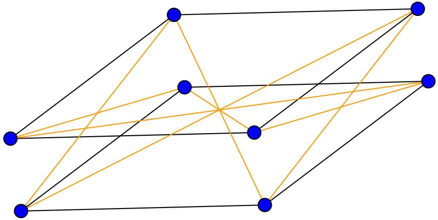

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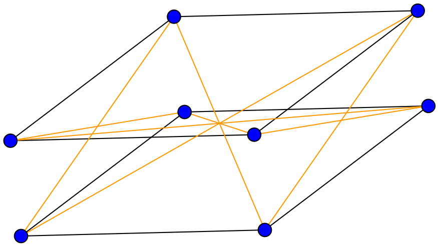

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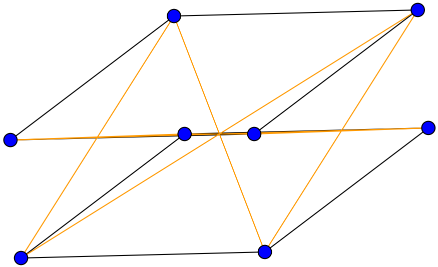

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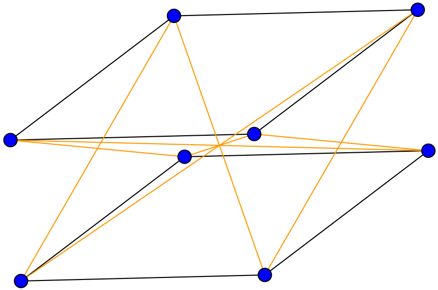

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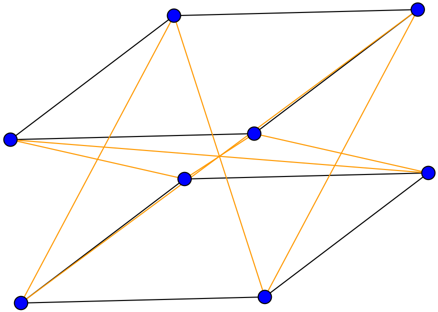

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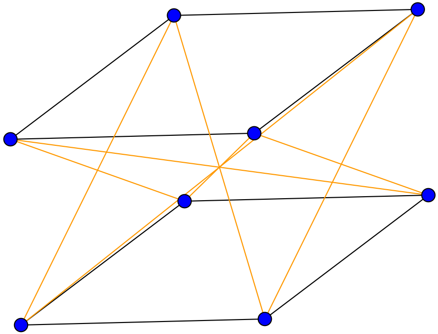

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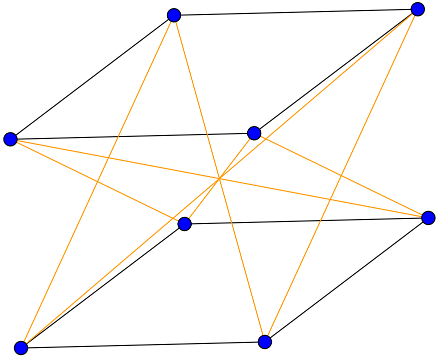

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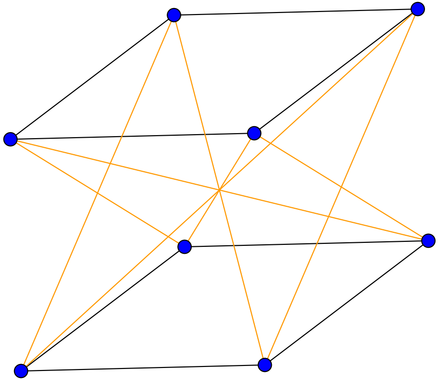

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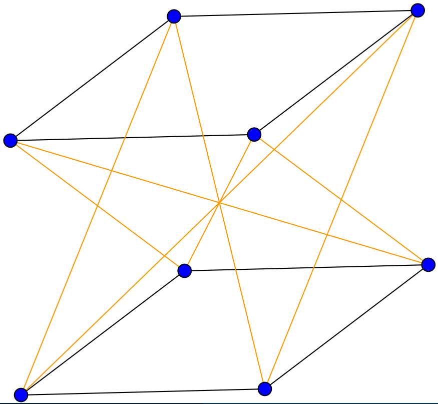

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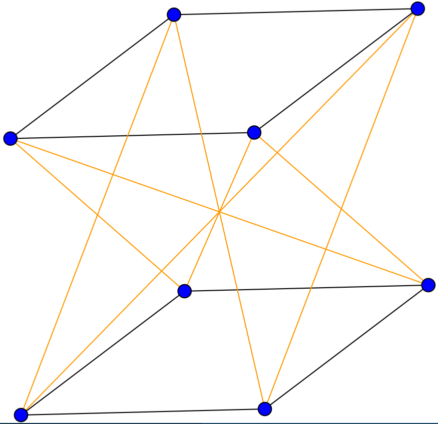

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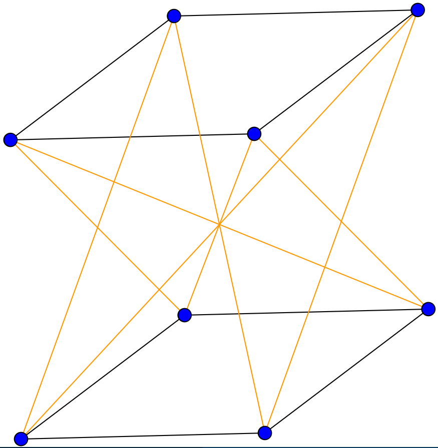

Using Merge to Create Graphs with PST

Theorem

Suppose G_1 and G_2 are integer-weighted graphs on n vertices, both of which are diagonalizable by the same Hadamard matrix H. Fix $w_1, w_2 \in \mathbb{Z}$ and let $L_1 = d_1 I - A_1, L_2 = d_2 I - A_2$ be the Laplacian matrices for G_1, G_2, respectively. Then $G_1 \,{}_{w_1}\!\odot_{w_2} G_2$ has PST from vertex p to q, where $p < q$, at time $t_0 = \pi/2$ if and only if one of the following 8 conditions holds:

The 8 conditions

1. $p, q \in \{1, \ldots, n\}$ and
    1. w_1 is odd, w_2 is even, and G_1 has PST from p to q at $t_0 = \pi/2$, or
    2. w_1 and d_2 are even, w_2 is odd, and G_2 has PST from p to q at $t_0 = \pi/2$, or
    3. w_1 and w_2 are odd, d_2 is even, and the weighted graph with Laplacian $L_1 + L_2$ has PST from p to q at $t_0 = \pi/2$;
2. $p, q \in \{n+1, \ldots, 2n\}$ and
    1. w_1 is odd, w_2 is even, and G_1 has PST from $p - n$ to $q - n$ at $t_0 = \pi/2$, or
    2. w_1 and d_2 are even, w_2 is odd, and G_2 has PST from $p - n$ to $q - n$ at $t_0 = \pi/2$, or
    3. w_1 and w_2 are odd, d_2 is even, and the weighted graph with Laplacian $L_1 + L_2$ has PST from $p - n$ to $q - n$ at $t_0 = \pi/2$;
3. $p \in \{1, \ldots, n\}$, $q \in \{n+1, \ldots, 2n\}$ and
    1. w_1 is even, w_2 and d_2 are odd, and G_2 has PST from p to $q - n$ at $t_0 = \pi/2$, or
    2. w_1, w_2, and d_2 are all odd, and the weighted graph with Laplacian matrix $L_1 + L_2$ has PST from p to $q - n$ at $t_0 = \pi/2$.

A Remark

For a general integer-weighted graph G, assume that gcd is the greatest common divisor of all the edge weights of G and that L is the Laplacian matrix of G. Let G' denote the integer-weighted graph with Laplacian $(1/\, gcd)L$. Since $e^{itL} = e^{it\, gcd(\frac{1}{gcd}L)}$ for all t, we find that G has PST at $\pi/(2\, gcd)$ if and only if G' has PST at $\pi/2$. This allows us to identify more PST graphs: for example, if G has PST at $\pi/2$, and we are given the graph $2G$, we know that $2G$ has PST at $\pi/4$.

Note that when both w_1 and w_2 are even, the graph $G_1 \,{}_{w_1}\!\odot_{w_2} G_2$ does not have PST at time $\pi/2$. Decompose the two integer weights w_j as $w_j = 2^{r_j} \cdot b_j$ (for $j = 1, 2$), where b_j are odd integers. Let $r = \min(r_1, r_2)$. Then the PST property of the graph with Laplacian $\frac{1}{2^r} L_3$ at time $\pi/2$ can be determined according to the above theorem. In the case that PST occurs, the graph $G_1 \,{}_{w_1}\!\odot_{w_2} G_2$ would then have PST at time $\pi/2^{r+1}$.

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Corollaries: weighted hypercube

A number of results follow from this theorem.

Corollary

Suppose $w_1, w_2, \ldots, w_n$ are integers, exactly d of which are odd. Then the weighted hypercube $Q_n := (w_1 K_2)\square(w_2 K_2)\square \cdots \square(w_n K_2)$ exhibits perfect state transfer at time $t_0 = \pi/2$ between every pair of vertices that are a distance of d from each other.

Corollaries: Existence of Hadamard Diagonalizable PST Graph on 2^k Vertices of Almost Every Regularity

Recall: A *bipartite graph* is a graph whose vertex set can be partitioned into two sets V_1 and V_2 such that no edge connects two vertices in the same set.

Theorem

Suppose that $k \in \mathbb{N}$ with $k \geq 3$. For each $d \in \mathbb{N}$ with $k + 1 \leq d \leq 2^k - 2$, there is a connected, unweighted, non-bipartite graph that is

(1) *diagonalizable by the standard Hadamard matrix of order 2^k,*

(2) *d-regular (sum of the weights of incident edges is d), and*

(3) *has PST between distinct vertices at time $t_0 = \pi/2$.*

PST for Graphs With Non-Integer Weights

Proposition

Suppose the graph G_1 with Laplacian L_1 is a rational-weighted Hadamard-diagonalizable graph, and let lcm be the least common multiple of the denominators of its edge weights, and gcd be the greatest common divisor of all the new integer edge weights $lcm \cdot w(j,k)$. Then G_1 has PST at time $t_1 = \frac{lcm}{gcd} \cdot \pi/2$ if and only if the integer-weighted Hadamard-diagonalizable graph G_2 with Laplacian $L_2 = \frac{lcm}{gcd} L_1$ has PST at time $t_0 = \pi/2$ between the same pair of vertices.

Pretty Good State Transfer

While we are not able to extend the previous proposition to the case of irrational weights directly—in general such a graph will neither be Hadamard-diagonalizable nor will it exhibit PST at any time—it is true at least that the resulting graph has pretty good state transfer when exactly one of the two weights in $G_1 \ {}_{w_1}\!\odot_{w_2} G_2$ is irrational. Before giving the theorem, we recall the following result about approximating an irrational real number with rational numbers.

Theorem

Let o denote the odd integers and e denote the even integers. Then for every real irrational number w, there are infinitely many relatively prime numbers u, v with $[u, v]$ in each of the three classes $[o, e]$, $[e, o]$, and $[o, o]$, such that the inequality $|w - u/v| < 1/v^2$ holds.

For the graph $G_1 \ {}_{w_1}\!\odot_{w_2} G_2$, we say it has parameters $[w_1, w_2, d_2]$

Pretty Good State Transfer

We will denote the set of irrational numbers by $\overline{\mathbb{Q}}$.

Theorem

Assume that G_1 and G_2 are integer-weighted graphs on n vertices, both of which are diagonalizable by the same Hadamard matrix H. Let d_2 be the degree of G_2. Let L_1 and L_2 denote the Laplacian matrices of G_1 and G_2, respectively. Suppose that one of w_1, w_2 is an integer and the other is irrational, and suppose that $p, q \in \{1, \ldots, n\}$. Then the weighted graph $G_1 \; {}_{w_1}\!\odot_{w_2} \; G_2$ has PGST as stated in the following three cases.

The Three Cases

1. Suppose that G_1 has PST from p to q at time $\pi/2$. Then $G_1 \;{}_{w_1}\!\odot_{w_2}\; G_2$ has PGST from p to q and from $p+n$ to $q+n$.
2. Suppose that G_2 has PST from p to q at time $\pi/2$. If d_2 is even, then $G_1 \;{}_{w_1}\!\odot_{w_2}\; G_2$ has PGST from p to q and from $p+n$ to $q+n$. If d_2 is odd, then $G_1 \;{}_{w_1}\!\odot_{w_2}\; G_2$ has PGST from p to $q+n$ and from q to $p+n$.
3. Suppose that $L_1 + L_2$ has PST from p to q at time $\pi/2$. If d_2 is even, then $G_1 \;{}_{w_1}\!\odot_{w_2}\; G_2$ has PGST from p to q and from $p+n$ to $q+n$. If d_2 is odd, then $G_1 \;{}_{w_1}\!\odot_{w_2}\; G_2$ has PGST from p to $q+n$ and from q to $p+n$.

Proof

Consider the setting where $w_1 \in \overline{\mathbb{Q}}, w_2 \in \mathbb{Z}$.
We approach w_1 with fractions u/v such that $|w_1 - u/v| < 1/v^2$. We denote the graph $G_1\ {}_{w_1}\!\odot_{w_2} G_2$ as G_3. For each such pair of u, v, we denote the graph $G_1\ {}_{u/v}\!\odot_{w_2} G_2$ as G_4, and the graph $G_1\ {}_{w_1-u/v}\!\odot_0 G_2$ as G_5. In particular, the Laplacian of G_3 is the sum of the Laplacian of G_4 with the Laplacian of G_5. Denote the Laplacian matrices of G_3, G_4 and G_5 as L_3, L_4, and L_5, respectively. Now consider the integer-weighted graph $G_4' = G_1\ {}_{u}\!\odot_{vw_2} G_2$, then its Laplacian is vL_4 and has parameters $[u, vw_2, d_2]$.

Cases

i) parameters $[o, e, e]$ or $[o, e, o]$
If G_1 has PST from p to q then (1a) of our main theorem applies
If G_1 has PST from $p+n$ to $q+n$ then (2a) applies
ii) parameters $[e, o, e]$
If G_2 has PST from p to q then (1b) applies
If G_2 has PST from $p+n$ to $q+n$ then (2b) applies
iii) parameters $[o, o, e]$
If $L_1 + L_2$ has PST from p to q then (1c) applies
If $L_1 + L_2$ has PST from $p+n$ to $q+n$ then (2c) applies
iv) parameters $[e, o, o]$
If G_2 has PST from p to $q+n$ then (3a) applies
v) parameters $[o, o, o]$
If $L_1 + L_2$ has PST from p to $q+n$ then (3b) applies
vi) parameters $[e, e, o]$ or $[e, e, e]$ —don't work

The End

Thank you!